Research Article
The Zeros of Orthogonal Polynomials for Jacobi-Exponential Weights

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This paper gives the estimates of the zeros of orthogonal polynomials for Jacobi-exponential weights.

1. Introduction and Results

This paper deals with the zeros of orthogonal polynomials for Jacobi-exponential weights. Let \( w \) be a weight in \( I := (a, b) \), \( -\infty \leq a < 0 < b \leq \infty \), for which the moment problem possesses a unique solution. Denote by \( N \) the set of positive integers. \( P_n \) stands for the set of polynomials of degree at most \( n \).

Assume that \( W = e^{-Q} \) where \( Q : I \rightarrow [0, \infty) \) is continuous. Also, let \( 0 < p < \infty \),

\[
a \leq t_r < t_{r-1} < \cdots < t_2 < t_1 \leq b,
\]

\[
p_i > \frac{-1}{p}, \quad i = 1, 2, \ldots, r,
\]

(1.1)

\[
U(x) = \prod_{i=1}^{r}|x-t_i|^{p_i}.
\]

The letters \( c, C_0, C_1, \ldots \) stand for positive constants independent of variables and indices, unless otherwise indicated, and their values may be different at different occurrences, even in subsequent formulas. Moreover, \( C_n \sim D_n \) means that there are two constants \( c_1 \) and
$c_2$ such that $c_1 \leq C_n/D_n \leq c_2$ for the relevant range of $n$. We write $c = c(\lambda)$ or $c \neq c(\lambda)$ to indicate dependence on or independence of a parameter $\lambda$.

**Definition 1.1** (see [1, Definition 1.7, page 14]). Given $c, t \geq 0$ and a nonnegative Borel measure $\nu$ with compact support in $C$ and total mass $\leq t$, one says that

$$P(z) := c \exp\left(\int \ln|z - t|d\nu(t)\right)$$  \hspace{1cm} (1.2)

is an exponential of a potential of mass $\leq t$. One denotes the set of all such $P$ by $P_t$.

One notes that, for $P \in P_n$,

$$|P| \in P_t, \quad t \geq n.$$  \hspace{1cm} (1.3)

**Definition 1.2** (see [1, page 19]). Let $w$ be a weight in $I$. For $0 < p < \infty$, generalized Christoffel functions with respect to $w$ for $z \in C$ are defined by

$$\lambda_{p,n}(w; z) = \inf_{P \in P_n} \left(\frac{\|Pw\|_{L^p(I)}}{|P(z)|}\right)^p.$$  \hspace{1cm} (1.4)

For $p = \infty$, generalized Christoffel functions with respect to $w$ for $z \in C$ are defined by

$$\lambda_{\infty,n}(w; z) = \inf_{P \in P_n} \frac{\|Pw\|_{L^\infty(I)}}{|P(z)|}.$$  \hspace{1cm} (1.5)

Obviously, for the classical Christoffel function $\lambda_n(w^2; x)$ with respect to $w^2$, we have

$$\lambda_n\left(w^2; x\right) = \lambda_{2,n-1}(w; x).$$  \hspace{1cm} (1.6)

A function $f : (c, d) \to (0, \infty)$ is said to be *quasi-increasing* (or *quasi-decreasing*) if there exists $C > 0$ such that

$$f(x) \leq (\text{or } \geq) C f(y), \quad c < x \leq y < d.$$  \hspace{1cm} (1.7)

**Definition 1.3** (see [1, pages 10–12]). Let $a < 0 < b$. Assume that $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties

(a) $Q' \in C(I)$ and $Q(0) = 0$.
(b) $Q'$ is nondecreasing in $I$.
(c) We have

$$\lim_{t \to a^+} Q(t) = \lim_{t \to b^-} Q(t) = \infty.$$  \hspace{1cm} (1.8)
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(d) The function

\[ T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0, \quad (1.9) \]

is quasi-decreasing in \((a, 0)\) and quasi-increasing in \((0, b)\), respectively. Moreover

\[ T(t) \geq \Lambda > 1, \quad t \in I \setminus \{0\}. \quad (1.10) \]

(e) There exists \(\epsilon_0 \in (0, 1)\) such that, for \(y \in I \setminus \{0\}\),

\[ T(y) \sim T \left( y \left[ 1 - \frac{\epsilon_0}{T(y)} \right] \right). \quad (1.11) \]

Then we write \(W \in \mathcal{F}\).

(f) In addition, assume that there exist \(C, \epsilon_1 > 0\) such that, for all \(x \in I \setminus \{0\}\),

\[ \int_{x-e_1|x/T(x)}^{x} \frac{|Q'(t) - Q'(x)|}{|t-x|^{3/2}} dt \leq C |Q'(x)| \left[ \frac{T(x)}{|x|} \right]^{1/2}. \quad (1.12) \]

Then we write \(W \in \mathcal{F}(\text{Lip}(1/2))\).

For \(W \in \mathcal{F}\) and \(t > 0\), the Mhaskar-Rahmanov-Saff numbers \(a_{-} := a_{-}(Q) < 0 < a_{+} := a_{+}(Q)\) are defined by the equations

\[ t = \frac{1}{\pi} \int_{a_{-}}^{a_{+}} \frac{xQ'(x)}{[(x - a_{-})(a_{+} - x)]^{1/2}} dx, \quad (1.13) \]

\[ 0 = \frac{1}{\pi} \int_{a_{-}}^{a_{+}} \frac{Q'(x)}{[(x - a_{-})(a_{+} - x)]^{1/2}} dx. \]

Put for \(t > 0\),
\[ \Delta_t := \Delta_t(Q) := [a_{-t}, a_t], \]
\[ \delta_t := \delta_t(Q) := \frac{1}{2}(a_t + |a_{-t}|), \quad \eta_{2t} := \eta_{2t}(Q) := \frac{tT(a_{2t})\sqrt{|\frac{a_{2t}}{\delta_t}|}}{2^{2/3}}, \]
\[ \varphi_t(x) := \varphi_t(Q; x) := \begin{cases} \frac{|x - a_{-2t}|x - a_{2t}|}{t\sqrt{[|x - a_{-2t}| + |a_{-2t}|][|x - a_{2t}| + a_{2t}\eta_{2t}]}}, & x \in [a_{-1}, a_1], \\ \varphi_t(a_t), & x \in (a_t, b), \\ \varphi_t(a_{-t}), & x \in (a_{-t}, a_{t}), \end{cases} \quad (1.14) \]
\[ J_{L,t} := J_{L,t}(Q) := [a_{-t}(1 + L\eta_{-t}), a_t(1 + L\eta_t)], \quad L > 0, \]
\[ K_{L,t} := K_{L,t}(Q) := [-1 + L(1 + a_{-t}), 1 - L(1 - a_t)], \quad L > 1. \]

Let
\[ U_t(x) := \prod_{i=1}^{r} \left( |x - t_i| + \delta_t \right)^{p_i}, \quad \rho := \rho(U) := \sum_{i=1}^{r} \max\{p_i, 0\}. \quad (1.15) \]

In 1994 and 2001, Levin and Lubinsky [1, 2] published their monographs on orthogonal polynomials for exponential weights $W^2$. Then they [3, 4] discussed orthogonal polynomials for exponential weights $x^{2a}W(x)^2, a > -1/2$, in $[0, b)$, since the results of [1, 2] cannot be applied to such weights. Kasuga and Sakai [5] considered generalized Freud weights $|x|^{2a}W(x)^2$ in $(-\infty, \infty)$. Recently the second author [6] obtained the $L_p$ Christoffel functions for Jacobi-exponential weights $IW$, which are the combination of the two best important weights: Jacobi weight and the exponential weight, and restricted range inequalities.

**Theorem 1.4** (see [6, Theorem 1.1]). Let $W \in \mathcal{F}(\text{Lip} (1/2)), \quad L > 0, \quad 0 < p < \infty$. Assume that
\[ \lim_{t \to \infty} \frac{|a_t|}{a_t} = \gamma, \quad 0 < \gamma < \infty. \quad (1.16) \]
Then there exists $n_0 > 0$ such that, for $n \geq n_0$ and $x \in J_{L,n}$, the relation
\[ \lambda_{p,n}(IW; x) - \varphi_{n}(x)U_{n}(x)pW(x)^p \quad (1.17) \]
uniformly holds.

**Theorem 1.5** (see [6, Theorem 1.2]). Let $W = e^{-Q(x)}$, where $Q : I \to [0, \infty)$ is convex with $Q(a+) = Q(b-) = \infty$ and $Q(x) > Q(0) = 0, x \in I \setminus \{0\}$. Let $0 < p \leq \infty$. Assume that relation (1.16) is valid. Then there exist $C, t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_{t-2/p}$,
\[ ||PUIW||_{L_p(I)} \leq C||PUIW||_{L_p(\Delta_t)}, \]
\[ ||PUI_tW||_{L_p(I)} \leq C||PUI_tW||_{L_p(\Delta_t)}. \quad (1.18) \]
Theorem 1.6 (see [6, Theorem 1.3]). Let \( W \in \mathcal{F}(\text{Lip}(1/2)) \), \( L > 0 \), and \( 0 < p < \infty \). Assume that relation (1.16) is valid. Then there exist \( C, t_0 > 0 \) such that, for \( t \geq t_0 \) and \( P \in \mathcal{P}_t \),

\[
\|PUW\|_{L^p(I(\Delta))} \leq C\|PUW\|_{L^p([a, \cdot\cdot\cdot, \cdot\cdot\cdot, a, 1-L\eta, a(1-L\eta)))}.
\] (1.19)

In this paper we discuss the zeros of orthogonal polynomials for Jacobi-exponential weights \( UW \) and restricted range inequalities.

**Theorem 1.7.** Let \( W \in \mathcal{F}(\text{Lip}(1/2)) \). Assume that (1.16) is valid, and

\[
a < t_r < \cdots < t_1 < b,
\]

(1.20)

\[
\varphi_t(x) = O(1), \quad t \to \infty.
\]

(1.21)

Then

\[
x_{kn} - x_{k+1,n} \leq c\varphi_n(x_{kn}), \quad k = 1, 2, \ldots, n - 1.
\] (1.22)

**Theorem 1.8.** Let \( W = e^{-Q(x)} \), where \( Q : I \to [0, \infty) \) is convex with \( Q(a+) = Q(b-) = \infty \) and \( Q(x) > Q(0) = 0 \), \( x \in I \setminus \{0\} \). Let \( 0 < p \leq \infty \). Assume that all \( p_i \) are positive and relation (1.16) is valid. Then there exist \( t_0 > 0 \) such that, for \( t \geq t_0 \) and \( P \in \mathcal{P}_{t-p-2/p} \),

\[
\|PUW\|_{L^p(I(\Delta))} \leq \|PUW\|_{L^p(\Delta)}.
\] (1.23)

**Theorem 1.9.** Let the assumptions of Theorem 1.8 prevail. Then

\[
x_{1n} < a_{n+p+1/2},
\]

(1.24)

\[
x_{nn} > a_{-n-p-1/2}.
\]

(1.25)

**Theorem 1.10.** Let \( W \in \mathcal{F}(\text{Lip}(1/2)) \). Then

\[
x_{1n} \geq a_n(1 - c\eta_n),
\]

(1.26)

\[
x_{nn} \leq a_{-n}(1 - c\eta_{-n}).
\]

(1.27)

If all \( p_i \geq 0 \), then

\[
1 - \frac{x_{1n}}{a_n} \sim \eta_n,
\]

(1.28)

\[
1 - \frac{x_{nn}}{a_{-n}} \sim \eta_{-n}.
\]

(1.29)

Here we should point out that our main result (Theorem 1.7) cannot follow from [7] given by Mastroianni and Totik, because in general Jacobi-exponential weights \( UW \) are
not doubling weights, although Jacobi weights \( U \) are doubling weights. A doubling weight means that the measure of a twice enlarged interval is less than a constant times the measure of the original interval. For example, for \( W(t) = \exp(-t^2) \), by L’Hospital rule

\[
\lim_{d \to \infty} \int_{d/2}^{5d/2} \frac{\exp(-t^2)}{\int_{d/2}^{2d} \exp(-t^2) dt} dt = \lim_{d \to \infty} \frac{\exp\left(-\frac{5d^2}{2}\right) - \exp\left(-\frac{d^2}{2}\right)}{\exp\left(-\frac{2d^2}{2}\right) - \exp(-d^2)} = \infty,
\]

which implies that \( W(t) = \exp(-t^2) \) is not a doubling weight.

We will give some auxiliary lemmas in Section 2 and the proofs of Theorems 1.7–1.10 in Section 3, respectively.

2. Auxiliary Lemmas

**Lemma 2.1** (Levin and Lubinsky [1, Lemma 3.5, pages 71-72]). Let \( W \in \mathcal{F} \). Then for fixed \( L > 1 \) and uniformly for \( t > 0 \),

\[
a_{L,t} \sim a_t. \tag{2.1}
\]

Moreover, there exists \( \tau_0 > 0 \) such that, for \( t \geq \tau \geq \tau_0 \), the inequalities

\[
1 \leq \frac{\delta_t}{\delta_\tau} \leq c \left( \frac{t}{\tau} \right)^{1/\lambda} \tag{2.2}
\]

hold.

**Lemma 2.2** (Shi [6]). Let \( W \in \mathcal{F} \). Then, for large enough \( t \),

\[
a_{2t} \geq a_t (1 + \eta_t). \tag{2.3}
\]

**Lemma 2.3.** Let \( I = (-1,1) \), \( W \in \mathcal{F} \), and \( L > 1 \). Then, for \( x \in K_{L,t} \),

\[
\varphi_t(x) \sim \frac{1}{t} \left[ (a_{2t} - x)(x - a_{-2t}) \right]^{1/2}, \tag{2.4}
\]

\[
\varphi_t(x) \leq c \delta_t \tag{2.5}
\]

**Proof.** By the same argument as that of [8, (2.25)] we can prove (2.4). By (2.4) and (2.1) for \( x \in K_{L,t} \),

\[
\varphi_t(x) \leq c \cdot \frac{1}{t} \frac{1}{2} (a_{2t} - a_{-2t}) \leq c \delta_t \tag{2.6}
\]

\( \square \)
Lemma 2.4. Let \( W \in \mathcal{F} \). Then, for \( x \in I \),

\[
\varphi_t(x) \leq \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. 
\tag{2.7}
\]

Proof. By the definition of \( \varphi_t \) it is enough to prove (2.7) for \( x \in \Delta_t \). Without loss of generality we can assume that \( 0 \leq x \leq a_t \). By Lemma 3.11(b) in [1, page 81] for \( t > 0 \),

\[
\left| \frac{a_{2t}}{a_t} - 1 \right| \sim \frac{1}{T(a_t)}. 
\tag{2.8}
\]

By Lemma 2.12 in [8], (2.3), (2.1), and (2.8),

\[
S(x) = \frac{a_{2t} - x}{a_t(1 + \eta_t) - x} \cdot \frac{a_{-2t} - x}{a_{-t}(1 + \eta_{-t}) - x} 
\leq \frac{a_{2t} - a_t}{a_t \eta_t} \cdot \frac{a_{-2t}}{a_{-t}(1 + \eta_{-t})} \leq c \frac{a_{2t}/a_t - 1}{\eta_t} \leq \frac{c}{\eta_t T(a_t)}. 
\tag{2.9}
\]

By (1.63) in [1, page 15],

\[
\eta_t T(a_t) \geq t^{-2/3}T(a_t)^{1/3} 
\tag{2.10}
\]

and hence

\[
S(x) \leq c t^{2/3}T(a_t)^{-1/3}. 
\tag{2.11}
\]

Thus

\[
\varphi_t(x) = \frac{[(a_{2t} - x)(x - a_{-2t})]^{1/2}}{t^{1/2}} S(x)^{1/2} \leq \frac{c\delta_t}{t^{1/2}} \left[ t^{2/3}T(a_t)^{-1/3} \right]^{1/2} = \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. 
\tag{2.12}
\]

Let \( I_k = [x_{k+1,n}, x_{kn}] \), \( d_k = x_{kn} - x_{k+1,n}, k = 1, 2, \ldots, n - 1 \). Let, for \( n \geq n_0 \) and \( d := \min_{1 \leq i \leq r}(t_i - t_{i+1}) \),

\[
\max_{1 \leq k \leq n-1} d_k \leq \frac{d}{4}. 
\tag{2.13}
\]

Lemma 2.5. For fixed index \( k, 1 \leq k \leq n - 1 \), let \( j, 1 \leq j \leq r \), satisfy

\[
\min_{x \in I_k} |x - t_j| = \min_{1 \leq i \leq r} \min_{x \in I_k} |x - t_i|. 
\tag{2.14}
\]
Then
\[
\prod_{i \neq j} |x_{kn} - t_j|^p_i \sim \prod_{i \neq j} \left( |x_{kn} - t_i| + \frac{\delta_n}{n} \right)^{p_i} \sim \prod_{i \neq j} |x - t_i|^p_i, \quad x \in I_k, \quad \kappa = k, k + 1.
\] (2.15)

Proof. We give the proof of (2.15) for \( \kappa = k \) only, the proof of (2.15) for \( \kappa = k + 1 \) being similar.

We claim that, for \( i \neq j \),
\[
|x_{kn} - t_i| \geq \frac{3}{8} d.
\] (2.16)

In fact, suppose without loss of generality that \( x_{kn} \geq t_j \). It is enough to show (2.16) for \( i = j - 1 \).

If \( t_j \in I_k \) then by (2.13)
\[
|x_{kn} - x_{k+1,n}| \leq \frac{d}{4} \leq t_{j-1} - t_j \leq |t_{j-1} - x_{kn}| + |x_{kn} - t_j|
\] (2.17)
and hence
\[
|x_{k+1,n} - t_j| \leq |x_{kn} - t_{j-1}|;
\] (2.18)
if \( t_j \not\in I_k \) then by (2.14)
\[
|x_{k+1,n} - t_j| = \min_{x \in I_k} |x - t_j| \leq \min_{x \in I_k} |x - t_{j-1}| = t_{j-1} - x_{kn},
\] (2.19)
which again implies (2.18). Then by (2.18)
\[
d \leq |t_{j-1} - t_j| \leq |t_{j-1} - x_{kn}| + |x_{kn} - x_{k+1,n}| + |x_{k+1,n} - t_j|
\leq 2|x_{kn} - t_{j-1}| + d_k
\leq 2|x_{kn} - t_{j-1}| + \frac{1}{4} d
\] (2.20)
and hence \( |x_{kn} - t_{j-1}| \geq 3d/8 \). This proves (2.16).

With the help of (2.16) for \( x \in I_k \) and \( i \neq j \),
\[
|x - t_i| \leq |x_{kn} - t_i| + |x - x_{kn}| \leq |x_{kn} - t_i| + \frac{d}{4} \leq \frac{5}{3} |x_{kn} - t_i|,
\] (2.21)
\[
|x - t_i| \geq |x_{kn} - t_i| - |x - x_{kn}| \geq |x_{kn} - t_i| - \frac{d}{4} \geq \frac{1}{3} |x_{kn} - t_i|.
\]

Hence
\[
|x - t_i| \sim |x_{kn} - t_i|.
\] (2.22)
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Furthermore, by (2.2) with \( \tau = 1 \)

\[
\frac{\delta_i}{t} \leq c\delta_t^{1/\Lambda-1} = o(1), \quad t \to \infty.
\] (2.23)

So for \( i \neq j \),

\[
|x_{kn} - t_i| - |x_{kn} - t_i| + \frac{\delta_n}{n}.
\] (2.24)

This proves (2.15).

By the same argument as that of Lemma 7.2.7 in [9, page 157] replacing \( 1/n \) by \( C_n \), we can get its extension.

**Lemma 2.6.** Let \( p \geq 0, B_n \geq A_n \geq 0, C_n \geq 0, \sigma = \pm 1, \) and

\[
B_n^{p+1} + \sigma A_n^{p+1} \leq CC_n[(B_n + C_n)^p + (A_n + C_n)^p].
\] (2.25)

Then

\[
B_n + \sigma A_n \leq cC_n.
\] (2.26)

**Lemma 2.7.** Let \( W \in \mathcal{F}. \) Let (1.16), (1.20), and (1.21) prevail. Then there exists \( t_0 > 0 \) such that, for \( t \geq t_0 \) and for each index \( j, \) \( 1 \leq j \leq r \),

\[
|x - t_j| + \frac{\delta_i}{t} \sim |x - t_j| + \varphi_t(x)
\] (2.27)

holds uniformly for \( x \in I \).

**Proof.** Let \( 0 < \epsilon < \min\{b - t_1, t_r - a\} \) and \( \Delta = [t_r - \epsilon, t_1 + \epsilon] \). We separate two cases. \( \text{Case 1} (x \in \Delta) \). In this case by (2.3) and (2.1),

\[
\varphi_t(x) \geq \frac{1}{t} [(a_{2i} - x)(x - a_{2i})]^{1/2} \geq \frac{1}{t} [(a_{2i} - t_1 - \epsilon)(t_r - \epsilon - a_{2i})]^{1/2} \geq \frac{c\delta_i}{t}
\] (2.28)

which coupled with (2.5) gives

\[
\varphi_t(x) \sim \frac{\delta_i}{t}.
\] (2.29)

Hence (2.27) follows. \( \text{Case 2} (x \notin \Delta) \). In this case by (2.23),

\[
|x - t_j| \geq \epsilon \geq \frac{c\delta_i}{t}
\] (2.30)
and by (1.21)
\[ |x - t| \geq e \geq c\varphi_t(x). \quad (2.31) \]

Again (2.27) follows. \(\square\)

**Corollary 2.8.** Let \( W \in \mathcal{F} \). Let (1.16) and (1.20) prevail. If
\[
\frac{\delta_i}{t^{2/3}T(a_t)^{1/6}} = O(1), \quad t \to \infty
\]
then (2.27) holds. In particular, if \( \Lambda \geq 3/2 \) then (2.32), (1.21), and (2.27) hold.

**Proof.** By (2.7) relation (2.32) implies (1.21). Then by Lemma 2.7 relation (2.27) is valid. In particular, if \( \Lambda \geq 3/2 \) then by (2.2) with \( \tau = \tau_0 \) relation (2.32) is valid and hence (1.21) and (2.27) hold. \(\square\)

### 3. Proof of Theorems

#### 3.1. Proof of Theorem 1.7

Denote by \( \ell_{kn} \)'s the fundamental polynomials based on the zeros \( x_{kn} \)'s. By Theorem 1.4 and Lemma 11.8 in [8, pages 320–321]

\[
\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2}
\]
\[
= \int_{x_{k+1,n}}^{x_{kn}} \left[ \ell_{kn}(t)^2W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2W(x_{k+1,n})^{-2} \right] W(t)^2U(t)^2 dt
\]
\[
\geq \int_{x_{kn}}^{x_{k+1,n}} \left[ \ell_{kn}(t)^2W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2W(x_{k+1,n})^{-2} \right] W(t)^2U(t)^2 dt
\]
\[
\geq \frac{1}{2} \int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt.
\]

On the other hand, by Theorem 1.4,
\[
\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2}
\]
\[
\leq c \left[ \varphi_n(x_{kn})U_n(x_{kn})^2 + \varphi_n(x_{k+1,n})U_n(x_{k+1,n})^2 \right].
\]

Then for \( \overline{\varphi}_n(x_{kn}) := \max\{\varphi_n(x_{kn}), \varphi_n(x_{k+1,n})\} \),
\[
\int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt \leq c\overline{\varphi}_n(x_{kn}) \left[ U_n(x_{kn})^2 + U_n(x_{k+1,n})^2 \right].
\]
Let \( j \) be defined by (2.14). Using Lemma 2.5 it follows from (3.3) that

\[
\int_{x_{k+1,n}}^{x_{kn}} |t - t_j|^{2p_j} dt \leq c \overline{p}_n(x_{kn}) \left[ \left( \left| x_{kn} - t_j \right| + \frac{\delta_n}{n} \right)^{2p_j} + \left( \left| x_{k+1,n} - t_j \right| + \frac{\delta_n}{n} \right)^{2p_j} \right].
\] (3.4)

Further, by (2.27),

\[
\int_{x_{k+1,n}}^{x_k} |t - t_j|^{2p_j} dt \leq c \overline{p}_n(x_{kn}) \left\{ \left[ \left| x_{kn} - t_j \right| + \overline{p}_n(x_{kn}) \right]^{2p_j} + \left[ \left| x_{k+1,n} - t_j \right| + \overline{p}_n(x_{kn}) \right]^{2p_j} \right\}.
\] (3.5)

By calculation from (3.5) we get

\[
\frac{1}{2p_j + 1} \left[ \left| x_{kn} - t_j \right|^{2p_j+1} + \sigma \left| x_{k+1,n} - t_j \right|^{2p_j+1} \right]
\]
\[
= \int_{x_{k+1,n}}^{x_k} |t - t_j|^{2p_j} dt \leq c \overline{p}_n(x_{kn}) \left\{ \left[ \left| x_{kn} - t_j \right| + \overline{p}_n(x_{kn}) \right]^{2p_j} + \left[ \left| x_{k+1,n} - t_j \right| + \overline{p}_n(x_{kn}) \right]^{2p_j} \right\},
\] (3.6)

where

\[
\sigma = \begin{cases} 
1, & t_j \in I_k, \\
-1, & t_j \notin I_k.
\end{cases}
\] (3.7)

We separate two cases.

**Case 1** \((p_j \geq 0)\). Using Lemma 2.6 it follows from (3.6) that

\[
x_{kn} - x_{k+1,n} \leq c \overline{p}_n(x_{kn}).
\] (3.8)

**Case 2** \((p_j < 0)\). Suppose without loss of generality that \(x_{k+1,n} > t_j\) for the case when \(t_j \notin I_k\). By (3.6),

\[
\frac{1}{2p_j + 1} \left[ \left| x_{kn} - t_j \right|^{2p_j+1} + \sigma \left| x_{k+1,n} - t_j \right|^{2p_j+1} \right]
\]
\[
= \int_{x_{k+1,n}}^{x_k} |t - t_j|^{2p_j} dt \leq c_0 \overline{p}_n(x_{kn}) \min \left\{ \overline{p}_n(x_{kn})^{2p_j}, \left| x_{k+1,n} - t_j \right|^{2p_j} \right\}.
\] (3.9)

**Subcase 2.1** \((t_j \in I_k)\). Inequality (3.9) gives

\[
\left| x_{kn} - t_j \right|^{2p_j+1} \leq c \overline{p}_n(x_{kn})^{2p_j+1}, \quad \kappa = k, k + 1
\] (3.10)

which yields (3.8).
Subcase 2.2 \( (t_j \notin I_k) \). In this case we distinguish two subcases.

1. \( |x_{kn} - t_j| \geq 2c_0\varphi_n(x_{kn}) \), where \( c_0 \) is given by (3.9). In this case

\[
\int_{x_{kn} - t_j}^{x_{kn}} (t - t_j)^{2p} dt = \int_{x_{kn} - t_j}^{x_{kn}} (t - t_j)(t - t_j)^{2p-1} dt \\
\geq (x_{kn} - t_j) \int_{x_{kn} - t_j}^{x_{kn}} (t - t_j)^{2p-1} dt \\
= (x_{kn} - t_j) \frac{1}{2|p_j|} \left[ (x_{kn} - t_j)^{2p} - (x_{kn} - t_j)^{2p} \right] \\
\geq \frac{c_0\varphi_n(x_{kn})}{|p_j|} \left[ (x_{kn} - t_j)^{2p} - (x_{kn} - t_j)^{2p} \right],
\]

which by (3.9) gives

\[
(x_{kn} - t_j)^{2p} \leq (1 - |p_j|)^{-1}(x_{kn} - t_j)^{2p} \leq 2(x_{kn} - t_j)^{2p}.
\]

On the other hand, by (3.9) and (3.12),

\[
c_0\varphi_n(x_{kn}) (x_{kn} - t_j)^{2p} \geq \int_{x_{kn} - t_j}^{x_{kn}} (t - t_j)^{2p} dt \geq \frac{1}{2} (x_{kn} - t_j)^{2p} (x_{kn} - x_{kn+1,n})
\]

and hence (3.8) follows.

2. \( |x_{kn} - t_j| < 2c_0\varphi_n(x_{kn}) \). By (3.9),

\[
c_0\varphi_n(x_{kn})^{2p+1} \geq \frac{1}{2p_j + 1} \left[ (x_{kn} - t_j)^{2p+1} - (x_{kn} - t_j)^{2p+1} \right]
\]

So \( x_{kn} - t_j \leq c_0\varphi_n(x_{kn}) \) and (3.8) follows.

Finally, applying Theorem 5.7(b) in [1, page 125] we conclude \( \varphi_n(x_{kn}) \sim \varphi_n(x_{kn}) \) and hence (1.22) follows from (3.8).

### 3.2. Proof of Theorem 1.8

For \( P \in \rho_{l-p-2/p} \), we have \( PL \in \rho_{l/2} \) and hence apply Theorem 1.8 in [1, page 15] to obtain (1.23).
3.3. Proof of Theorem 1.9

Use the same argument as that of Theorem 11.1 in [1, page 313].

3.4. Proof of Theorem 1.10

We give the proofs of (1.26) and (1.28) only, the proofs of (1.27) and (1.29) being similar.

First let us prove (1.26). Choose \( \alpha, \beta > 1 \) so that

\[
\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad 2\beta p_i > -1, \quad i = 1, \ldots, r. \tag{3.15}
\]

Let \( L_n \) denote the linear map of \( \Delta_n \) onto \([-1, 1]\). By Lemma 11.7 in [1, page 318] there exists \( y_n \in \Delta_n \) such that

\[
L_n(y_n) = \cos \frac{2\pi}{m}, \quad m = m(n), \tag{3.16}
\]

and for large enough \( n \) and \( R_n \in P_{n-2m} \) such that

\[
R_n(x)W(x)^{1/n} \geq C_1, \quad x \in [0, y_n], \quad \left\| R_n W^{1/n} \right\|_{L_\infty(I)} \leq C_2. \tag{3.17}
\]

Using (11.7) in [1, page 318] in the form

\[
1 - \frac{x_{1n}}{a_n} = \min_{p \in P_{n-1}} \frac{\int (1-x/a_n)(PUW)^2(x)dx}{\int (PUW)^2(x)dx}. \tag{3.19}
\]

Again choose [1, page 319]

\[
P(x) = R_n(x)V_{m, \cos(2\pi/m)}(L_n(x))^2 \in P_{n-2}. \tag{3.20}
\]
Applying Theorem 1.5 and (3.18), and using the same argument as that in [1, pages 319-320], we can get

\[
\int_1 \left(1 - \frac{x}{a_n}\right)(PUW)^2(x)dx \\
\leq c \int_{\Delta_n} \left(1 - \frac{x}{a_n}\right)(PUW)^2(x)dx \\
= c \int_{\Delta_n} \left[\left(1 - \frac{x}{a_n}\right)P(x)\right] \left[U(x)W(x)^{1/\beta}\right]^2 dx \\
\leq c \left\{ \int_{\Delta_n} \left[\left(1 - \frac{x}{a_n}\right)P(x)\right]^2 dx \right\}^{1/\alpha} \left\{ \int_{\Delta_n} \left[U(x)W(x)^{1/\beta}\right]^2 dx \right\}^{1/\beta} \\
\leq c \left\{ \int_{\Delta_n} \left[\left(1 - \frac{x}{a_n}\right)V_{m,\cos(2\pi/m)}(L_n(x))^{1/4}\right]^a dx \right\}^{1/\alpha} \\
= \frac{c\delta_n}{a_n} \left\{ \int_{\Delta_n} \left[\left(1 - L_n(x)\right)V_{m,\cos(2\pi/m)}(L_n(x))^{1/4}\right]^a dx \right\}^{1/\alpha} \\
= \frac{c\delta_n^2}{a_n} \left\{ \int_{-1}^{1} \left[\left(1 - t\right)V_{m,\cos(2\pi/m)}(t)^{1/4}\right]^a dt \right\}^{1/\alpha} \\
\leq \frac{c\delta_n^2}{a_n m^4} \left\{ \int_{-\infty}^{\infty} \left[\left(1 + |v|\right)\min\left\{1, \frac{c}{|v|}\right\}\right]^a dv \right\}^{1/\alpha} \\
\leq c a_n \eta_n^2. 
\tag{3.21}
\]

On the other hand, by (3.17),

\[
\int_1 (PUW)^2(x)dx \geq \int_{y_n(1 - C_1\eta_n)}^{y_n} (PUW)^2(x)dx \\
\geq \int_{y_n(1 - C_1\eta_n)}^{y_n} V_{m,\cos(2\pi/m)}(L_n(x))^{4}U(x)^2 dx. 
\tag{3.22}
\]

By (1.20) for large enough \(n\), we have

\[
U(x) \geq c > 0, \quad x \in [y_n(1 - C_1\eta_n), y_n]. 
\tag{3.23}
\]
Hence (3.22) implies
\[
\int I (PUW)^2(x) dx \geq c \int_{y_0(x-1+C_1\eta_n)}^{y_a} V_m, \cos(2\pi/m) (L_n(x))^4 dx
\]
\[
= c \delta_n \int_{\cos(2\pi/m) - C_1 y_a \eta_n / \delta_n}^{\cos(2\pi/m)} V_m, \cos(2\pi/m) (t)^4 dt.
\]

But in [1, page 320] the following estimate is given:
\[
\delta_n \int_{\cos(2\pi/m) - C_1 y_u \eta_n / \delta_n}^{\cos(2\pi/m)} V_m, \cos(2\pi/m) (t)^4 dt \geq c a_n \eta_n.
\]

Substituting this estimate into (3.24) gives
\[
\int I (PUW)^2(x) dx \geq c a_n \eta_n
\]
which coupled with (3.21) yields (1.26).

Next let us prove (1.28). We already know that
\[
a_n (1 - c \eta_n) \leq x_{1n} < a_{n+\rho+1/2} = a_n (1 + o(\eta_n)),
\]
by (1.26) and (1.24). We must prove that, for some $c_1 > 0$, and $n$ large enough, we have
\[
x_{1n} < a_n (1 - c_1 \eta_n).
\]

We use the idea for the proof of Corollary 13.4(b) in [1, pages 380-381] with modification. By the same argument as that proof with $A = a_{n+\rho+1/2} (1 - \epsilon \eta_n)$ instead, applying Theorem 1.8 we obtain
\[
1 - \frac{x_{1n}}{A} = \lambda_n \left( (UW)^2, x_{1n} \right)^{-1} \int I \left( 1 - \frac{x}{A} \right) (\hat{e}_{1n} UW)(x)^2 dx,
\]
\[
\int I \left( 1 - \frac{x}{A} \right) (\hat{e}_{1n} UW)(x)^2 dx
\]
\[
= \int_a^A \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx - \int_a^{b} \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx
\]
\[
\geq \int_a^A \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx
\]
\[
- \int_a^{a_{n+\rho+1/2}} \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx - \int_{a_{n+\rho+1/2}}^{A} \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx
\]
\[
\geq -2 \int_a^{a_{n+\rho+1/2}} \left| 1 - \frac{x}{A} \right| (\hat{e}_{1n} UW)(x)^2 dx,
\]
where \( \ell_{1n} \) denotes the fundamental polynomial of Lagrange interpolation based on the zeros of the \( n \)th orthogonal polynomial with respect to the weight \( (UW)^2 \).

But
\[
\int_A^{a_{n+p+1/2}} \left| 1 - \frac{x}{A} \right| (\ell_{1n}UW)(x)^2 \, dx \\
\leq \left( \frac{a_{n+p+1/2}}{A} - 1 \right) \int_1^A (\ell_{1n}UW)(x)^2 \, dx = \left( \frac{a_{n+p+1/2}}{A} - 1 \right) \lambda_n ((UW)^2, x_{1n})
\]
(3.31)

which, coupled with (3.30) and (3.29), gives
\[
1 - \frac{x_{1n}}{A} \geq -c\epsilon\eta_n.
\]
(3.32)

Thus
\[
\frac{x_{1n}}{a_n} = \frac{x_{1n}}{A} \frac{A}{a_{n+p+1/2}} \frac{a_{n+p+1/2}}{a_n} \\
\leq (1 + c\epsilon\eta_n) (1 - \epsilon\eta_n) (1 + o(\eta_n)) \\
< 1 - c_1\eta_n,
\]
(3.33)

for \( n \) large enough, provided \( \epsilon > 0 \) is small enough.

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**References**


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