Research Article

Explicit Formulas Involving $q$-Euler Numbers and Polynomials

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We deal with $q$-Euler numbers and $q$-Bernoulli numbers. We derive some interesting relations for $q$-Euler numbers and polynomials by using their generating function and derivative operator. Also, we derive relations between the $q$-Euler numbers and $q$-Bernoulli numbers via the $p$-adic $q$-integral in the $p$-adic integer ring.

1. Preliminaries

Imagine that $p$ is a fixed odd prime number. Throughout this paper we use the following notations, where $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The $p$-adic absolute value is defined by

$$ |p|_p = \frac{1}{p} $$

In this paper, we will assume that $|q - 1|_p < 1$ as an indeterminate. $[x]_q$ is a $q$-extension of $x$, which is defined by

$$ [x]_q = \frac{1 - q^x}{1 - q}. $$

We note that $\lim_{q \to 1} [x]_q = x$ (see [1–12]).
We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y},
\]
has a limit $f'(a)$ as $(x, y) \to (a, a)$ and denote this by $f \in \text{UD} (\mathbb{Z}_p)$. Let $\text{UD} (\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD} (\mathbb{Z}_p)$, let us start with the expression
\[
\frac{1}{[pN]} \sum_{0 \leq \xi < pN} f(\xi)q^\xi = \sum_{0 \leq \xi < pN} f(\xi)\mu_q (\xi + p^N \mathbb{Z}_p),
\]
which represents $p$-adic $q$-analogue of Riemann sums for $f$. The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit $(N \to \infty)$ of these sums, when it exists. The $p$-adic $q$-integral of function $f \in \text{UD} (\mathbb{Z}_p)$ is defined by Kim
\[
I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q (\xi) = \lim_{N \to \infty} \frac{1}{[pN]} \sum_{\xi = 0}^{pN-1} f(\xi)q^\xi.
\]
The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1(f) = \lim_{q \to 1} I_q(f)$. Similarly, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:
\[
I_{-q}(f) = \lim_{q \to -1} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi)
\]
(for more details, see [9–12]).

In [6], the $q$-Euler polynomials with weight 0 are introduced as
\[
\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-q}(y).
\]
From (1.7), we have
\[
\tilde{E}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \tilde{E}_{n-l,q},
\]
where $\tilde{E}_{n,q}(0) = \tilde{E}_{n,q}$ are called $q$-Euler numbers with weight 0. Then, $q$-Euler numbers are defined as
\[
q (\tilde{E}_q + 1)^n + \tilde{E}_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]
where the usual convention about replacing $(\tilde{E}_q)^n$ by $\tilde{E}_{n,q}$ is used.
Similarly, the $q$-Bernoulli polynomials and numbers with weight 0 are defined, respectively, as

$$\tilde{B}_{n,q}(x) = \lim_{n \to \infty} \frac{1}{[p]^n} \sum_{y=0}^{p^n-1} (x+y)^n q^y = \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y), \quad (1.10)$$

$$\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} y^n d\mu_q(y)$$

(for more information, see [4]).

We, by using the Kim et al. method in [2], will investigate some interesting identities on the $q$-Euler numbers and polynomials arising from their generating function and derivative operator. Consequently, we derive some properties on the $q$-Euler numbers and polynomials and $q$-Bernoulli numbers and polynomials by using $q$-Volkenborn integral and fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$.

2. On the $q$-Euler Numbers and Polynomials

Let us consider Kim’s $q$-Euler polynomials as follows:

$$F^q_x(t) = F^q_x(t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}. \quad (2.1)$$

Here $x$ is a fixed parameter. Thus, by expression of (2.1), we can readily see the following:

$$qe^t F^q_x + F^q_x = [2]_q e^{xt}. \quad (2.2)$$

Last from equality, taking derivative operator $D$ as $D = d/dt$ on the both sides of (2.2). Then, we easily see that

$$qe^t (D + I)^k F^q_x + D^k F^q_x = [2]_q x^k e^{xt}, \quad (2.3)$$

where $k \in \mathbb{N}^*$ and $I$ is identity operator. By multiplying $e^{-t}$ on both sides of (2.3), we get

$$q(D + I)^k F^q_x + e^{-t}D^k F^q_x = [2]_q x^k e^{(x-1)t}. \quad (2.4)$$

Let us take derivative operator $D^m (m \in \mathbb{N})$ on both sides of (2.4). Then we get

$$qe^t D^m (D + I)^k F^q_x + D^m (D - I)^m F^q_x = [2]_q x^k (x-1)^m e^{xt}. \quad (2.5)$$
Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get

$$\sum_{j=0}^{k-j}(\textstyle k \choose j) (q e^{t} D^{k+m-j} F_{x}^{q}(t)) [0] + \sum_{j=0}^{m-j} (\textstyle m \choose j) (-1)^j (D^{k+m-j} F_{x}^{q}(t)) [0] = \text{[2]}_q x^k (x-1)^m. \tag{2.6}$$

By (2.1), we see

$$\left( D^N F_{x}^{q}(t) \right) [0] = \tilde{E}_{N,q}(x), \quad \left( e^t D^N F_{x}^{q}(t) \right) [0] = \tilde{E}_{N,q}(x). \tag{2.7}$$

By expressions of (2.6) and (2.7), we see that

$$\sum_{j=0}^\max\{k,m\} \left[ q \left( \textstyle k \choose j \right) + (-1)^j \left( \textstyle m \choose j \right) \right] \tilde{E}_{k+m-j,q}(x) = \text{[2]}_q x^k (x-1)^m. \tag{2.8}$$

From (2.1), we note that

$$\frac{d}{dx} \left( \tilde{E}_{n,q}(x) \right) = n \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) \tilde{E}_{l,q} x^{n-l} = n \tilde{E}_{n-1,q}(x). \tag{2.9}$$

By (2.9), we easily see

$$\int_{0}^{1} \tilde{E}_{n,q}(x) dx = \frac{\tilde{E}_{n+1,q}(1) - \tilde{E}_{n+1,q}}{n+1} = -\frac{\text{[2]}_q (-1)^{n+1}}{n+1} \tilde{E}_{n+1,q}. \tag{2.10}$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$\left[ \text{[2]}_q (-1)^{n+1} \sum_{j=0}^\max\{k,m\} \left[ q \left( \textstyle k \choose j \right) + (-1)^j \left( \textstyle m \choose j \right) \right] \tilde{E}_{k+m-j+1,q} \right]$$

$$= \left[ \text{[2]}_q (-1)^{m} B(k+1, m+1) \right]$$

$$= \left[ \text{[2]}_q (-1)^{m} \frac{\Gamma(k+1) \Gamma(m+1)}{\Gamma(k+m+2)} \right], \tag{2.11}$$

where $B(m, n)$ is beta function which is defined by

$$B(m, n) = \int_{0}^{1} x^{m-1} (1-x)^{n-1} dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \ n > 0. \tag{2.12}$$

As a result, we obtain the following theorem.
Theorem 2.1. For \( n \in \mathbb{N} \), one has

\[
\sum_{j=1}^{\max\{k,m\}} \left[ q \begin{pmatrix} k \\ j \end{pmatrix} + (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k + m - j + 1} = q \frac{(-1)^{m+1}}{(k + m + 1)(k^{m+1})} - [2]_q \frac{\tilde{E}_{k+m+1,q}}{k + m + 1}.
\]

Substituting \( m = k + 1 \) into Theorem 2.1, we readily get

\[
\sum_{j=1}^{k+1} \left[ q \begin{pmatrix} k \\ j \end{pmatrix} + (-1)^j \begin{pmatrix} k + 1 \\ j \end{pmatrix} \right] \frac{\tilde{E}_{2k+2-j,q}}{2k + 2 - j} = q \frac{(-1)^k}{(2k + 2)(2^{k+1})} - [2]_q \frac{\tilde{E}_{2k+2,q}}{2k + 2}.
\]

By (2.1), it follows that

\[
\sum_{j=0}^{\max\{k,m\}} (k + m - j) \left[ q \begin{pmatrix} k \\ j \end{pmatrix} + (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \right] \tilde{E}_{k+m-j-1,q}(x) = [2]_q x^{k-1}(x - 1)^{m-1}((k + m)x - k).
\]

Let \( m = k \) in (2.1), we see that

\[
\sum_{j=0}^{k} \left[ q \begin{pmatrix} k \\ j \end{pmatrix} + (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \right] \tilde{E}_{2k-j,q}(x) = [2]_q x^k(x - 1)^k.
\]

Last from equality, we discover the following:

\[
[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + (q - 1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j-1,q}(x) = [2]_q x^k(x - 1)^k.
\]

Here \([\cdot]\) is Gauss’ symbol. Then, taking integral from 0 to 1 in both sides of last equality, we get

\[
- [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k - 2j + 1} + [2]_q (1 - q) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k - 2j} = [2]_q (-1)^k B(k + 1, k + 1)
\]

\[
= [2]_q (-1)^k \frac{B(k + 1, k + 1)}{(2k + 1)(\frac{2k}{k})}.
\]

Consequently, we derive the following theorem.
**Theorem 2.2.** The following identity

\[
[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \bar{E}_{2k-2j+1,q} + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \bar{E}_{2k-2j,q} = \frac{q(-1)^{k+1}}{(2k+1) \binom{2k}{k}}
\]

(2.19) is true.

In view of (2.1) and (2.17), we discover the following applications:

\[
\begin{align*}
= & \sum_{j=0}^{k+1} \left[ q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \bar{E}_{2k+1-j,q}(x) \\
= & [2]_q \bar{E}_{2k+1,q}(x) + \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \left[ q \binom{k}{2j} + \binom{k}{2j+1} - \binom{k}{2j} \right] \bar{E}_{2k-2j,q}(x) \\
& + \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j} \bar{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \bar{E}_{2k-2j+1}(x) \\
= & \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \bar{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \bar{E}_{2k-2j+1}(x).
\end{align*}
\]

(2.20)

By expressions (2.17) and (2.20), we have the following theorem.

**Theorem 2.3.** For \( k \in \mathbb{N} \), one has

\[
[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \bar{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \bar{E}_{2k+1-2j,q}(x) \\
+ (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[ \bar{E}_{2k-2j,q}(x) + \frac{1}{1+q} \bar{E}_{2k-2j+1}(x) \right] = x^k(x-1)^k ([2]_q x - q).
\]

(2.21)
3. *p*-adic Integral on $\mathbb{Z}_p$ Associated with Kim’s $q$-Euler Polynomials

In this section, we consider Kim’s $q$-Euler polynomials by means of $p$-adic $q$-integral on $\mathbb{Z}_p$.

Now we start with the following assertion.

Let $m, k \in \mathbb{N}$. Then by (2.8),

$$I_1 = [2]_q \int_{\mathbb{Z}_p} x^k(x-1)^m \, d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} \, d\mu_q(x)$$  \hspace{1cm} (3.1)

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}.$$

On the other hand, in right hand side of (2.8),

$$I_2 = \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l \, d\mu_q(x)$$

$$= \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}.$$  \hspace{1cm} (3.2)

Equating $I_1$ and $I_2$, we get the following theorem.

**Theorem 3.1.** For $m, k \in \mathbb{N}$, one has

$$\sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}.$$  \hspace{1cm} (3.3)

Let us take fermionic *p*-adic $q$-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$I_3 = \int_{\mathbb{Z}_p} x^k(x-1)^k [2]_q x - q \, d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} \, d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} \, d\mu_q(x)$$  \hspace{1cm} (3.4)

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q}.$$
In other words, we consider right hand side of (2.21) as follows:

\[
I_4 = \left[ 2 \right]_q \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k+1-2j-1|q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
+ \sum_{j=1}^{[k/2]} \left( \begin{array}{c} k \cr 2j-1 \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k+1-2j-1|q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
+ \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j + 1 \end{array} \right) \left( q-1 \right) \sum_{l=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \cr l \end{array} \right) \bar{E}_{2k-2j-1|q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
+ \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j + 1 \end{array} \right) \left( q-1 \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k-2j-1|q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)
\right]
\]

(3.5)

Equating \( I_3 \) and \( I_4 \), we get the following theorem.

**Theorem 3.2.** For \( k \in \mathbb{N} \), one has

\[
\sum_{j=0}^{k} \left( \begin{array}{c} k \cr j \end{array} \right) (-1)^{k-j} \left[ 2 \right]_q \bar{E}_{k+1|l,q} - q \bar{E}_{k|l,q}
\]

\[
= \left[ 2 \right]_q \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k+1-2j-1|q} \bar{E}_{l,q} \\
+ \sum_{j=1}^{[k/2]} \left( \begin{array}{c} k \cr 2j-1 \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k+1-2j-1|q} \bar{E}_{l,q} \\
+ \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j + 1 \end{array} \right) \left( q-1 \right) \sum_{l=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \cr l \end{array} \right) \bar{E}_{2k-2j-1|q} \bar{E}_{l,q} \\
+ \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \cr 2j + 1 \end{array} \right) \left( q-1 \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \cr l \end{array} \right) \bar{E}_{2k-2j-1|q} \bar{E}_{l,q}
\right]
\]

(3.6)
Now, we consider (2.8) and (2.1) by means of \( q \)-Volkenborn integral. Then, by (2.8), we see

\[
[2]_q \int_{\mathbb{Z}_p} x^k(x - 1)^m d\mu_q(x) = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^l d\mu_q(x)
\]

(3.7)

On the other hand,

\[
\sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k + m - j}{l} \bar{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)
\]

(3.8)

Therefore, we get the following theorem.

**Theorem 3.3.** For \( m, k \in \mathbb{N} \), one has

\[
[2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \bar{B}_{l+k,q} = \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k + m - j}{l} \bar{E}_{k+m-j-l,q} \bar{B}_{l,q}.
\]

(3.9)

By using fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) in left hand side of (2.21), we get

\[
I_5 = [2]_q \int_{\mathbb{Z}_p} x^k(x - 1)^k ([2] x - q) d\mu_q(x)
\]

(3.10)

\[
= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \bar{B}_{k+l,q}.
\]
Also, we consider right hand side of (2.21) as follows:

\[
I_6 = [2]_q \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \\ l \end{array} \right) \bar{E}_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x) + \sum_{j=1}^{[k/2]} \left( \begin{array}{c} k \\ 2j-1 \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \\ l \end{array} \right) \bar{E}_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x) + \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j+1 \end{array} \right) \left[ (q-1) \sum_{j=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \\ l \end{array} \right) \bar{E}_{2k-2j-l,q} \int_{Z_p} x^l d\mu_q(x) \right] + \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j+1 \end{array} \right) \left[ \sum_{j=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \\ l \end{array} \right) \bar{E}_{2k-2j-l+1,q} \int_{Z_p} x^l d\mu_q(x) \right]
\]

(3.11)

Equating \( I_5 \) and \( I_6 \), we get the following corollary.

**Corollary 3.4.** For \( k \in \mathbb{N} \), one gets

\[
\sum_{l=0}^{k} \left( \begin{array}{c} k \\ l \end{array} \right) (-1)^{k-l} \left\{ [2]_q \bar{B}_{k+l+1,q} - q\bar{B}_{k+1,q} \right\} = [2]_q \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \\ l \end{array} \right) \bar{E}_{2k+1-2j-l,q} \bar{B}_{l,q} + \sum_{j=1}^{[k/2]} \left( \begin{array}{c} k \\ 2j-1 \end{array} \right) \sum_{l=0}^{2k-2j+1} \left( \begin{array}{c} 2k-2j+1 \\ l \end{array} \right) \bar{E}_{2k+1-2j-l,q} \bar{B}_{l,q} + \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j+1 \end{array} \right) \left\{ (q-1) \sum_{j=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \\ l \end{array} \right) \bar{E}_{2k-2j-l,q} \bar{B}_{l,q} \right\} + \sum_{j=0}^{[k/2]} \left( \begin{array}{c} k \\ 2j+1 \end{array} \right) \left\{ \sum_{j=0}^{2k-2j} \left( \begin{array}{c} 2k-2j \\ l \end{array} \right) \bar{E}_{2k-2j-l+1,q} \bar{B}_{l,q} \right\}
\]

(3.12)
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