Research Article

Characteristic Functions and Borel Exceptional Values of $E$-Valued Meromorphic Functions

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The main purpose of this paper is to investigate the characteristic functions and Borel exceptional values of $E$-valued meromorphic functions from the $\mathbb{C}_R = \{ z : |z| < R \}, 0 < R \leq +\infty$ to an infinite-dimensional complex Banach space $E$ with a Schauder basis. Results obtained extend the relative results by Xuan, Wu and Yang, Bhoosnurmath, and Pujari.

1. Introduction and Preliminaries


In [4], Xuan and Wu also proved Chuang’s inequality (see, e.g., [5]) of $E$-valued meromorphic mapping $f(z)$ in the whole complex plane, which compares the relationship between $T(r, f)$ and $T(r, f')$, and also obtained that the order and the lower order of $E$-valued meromorphic mapping $f(z)$ and those of its derivative $f'(z)$ are the same. In Section 2, we
shall prove that Chuang’s inequality is valid for \(E\)-valued meromorphic mapping \(f(z)\) in the unit disc and prove that for any infinite-order \(E\)-valued meromorphic function \(f(z)\) defined in the unit disc has the same Xiong’s proximate order as its derivative \(f'(z)\).

In [5], Yang obtained much stronger results than those of Gopalakrishna and Bhoosnurmath [6] for the Borel exceptional values of meromorphic functions dealing with multiple values. In Section 3, we shall extend Le Yang’s result to \(E\)-valued meromorphic functions of finite and infinite orders in

\[
\mathbb{C}_R := \{z : |z| < R\}, \quad 0 < R \leq +\infty.
\] (1.1)

Let \(E_n\) be an \(n\)-dimensional projective space of \(E\) with a basis \(\{e_j\}_1^n\). The projective operator \(P_n : E \to E_n\) is a realization of \(E_n\) associated with basis.

The elements of \(E\) are called vectors and are usually denoted by letters from the alphabet: \(a, b, c, \ldots\). The symbol \(0\) denotes the zero vector of \(E\). We denote vector infinity, complex number infinity, and the norm infinity by \(\infty, \infty, \text{ and } +\infty\), respectively. A vector-valued mappings is called holomorphic (meromorphic) if all \(f_j(z)\) are holomorphic (some of \(f_j(z)\) are meromorphic). The \(j\)th derivative \(j = 1, 2, \ldots\) of \(f(z)\) is defined by

\[
f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_k^{(j)}(z), \ldots).
\] (1.2)

A point \(z_0 \in \mathbb{C}_r\) is called a “pole” (or \(\infty\) point) of

\[
f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots)
\] (1.4)

if \(z_0\) is a pole (or \(\infty\) point) of at least one of the component functions \(f_k(z)\) \((k = 1, 2, \ldots)\). A point \(z_0 \in \mathbb{C}_r\) is called a “zero” of \(f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots)\) if \(z_0\) is a zero of all the component functions \(f_k(z)\) \((k = 1, 2, \ldots)\). A point \(z_0 \in \mathbb{C}_r\) is called a pole or an \(\infty\)-point of \(f(z)\) of multiplicity \(q \in \mathbb{N}^+\), meaning that in such a point \(z_0\) at least one of the meromorphic component functions \(f_j(z)\) has a pole of this multiplicity in the ordinary sense of function theory. A point \(z_0 \in \mathbb{C}_r\) is called a zero of \(f(z)\) of multiplicity \(q \in \mathbb{N}^+\), meaning that in such a point \(z_0\) all component functions \(f_j(z)\) vanish, each with at least this multiplicity.
Let \( n(r, f) \) or \( n(r, \infty) \) denote the number of poles of \( f(z) \) in \( |z| \leq r \) and let \( n(r, a, f) \) denote the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), counting with multiplicities. Define the volume function associated with \( E \)-valued meromorphic function \( f(z) \) by

\[
V(r, \infty, f) = V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi)\| \, dx \wedge dy, \quad \xi = x + iy,
\]

and the counting function of finite or infinite \( a \)-points by

\[
N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt, \tag{1.6}
\]

\[
N(r, \infty) = n(0, \infty) \log r + \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} \, dt, \tag{1.7}
\]

\[
N(r, a, f) = n(0, a, f) \log r + \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} \, dt, \tag{1.8}
\]

respectively. Next, we define

\[
m(r, f) = m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f(re^{i\theta}) \right\| \, d\theta,
\]

\[
m(r, a) = m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\left\| f(re^{i\theta}) - a \right\|} \, d\theta,
\]

\[
T(r, f) = m(r, f) + N(r, f). \tag{1.9}
\]

Let \( \overline{n}(r, f) \) or \( \overline{n}(r, \infty) \) denote the number of poles of \( f(z) \) in \( |z| \leq r \), and let \( \overline{n}(r, a, f) \) denote the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), ignoring multiplicities. Similarly, we can define the counting functions \( \overline{N}(r, f), \overline{N}(r, \infty), \) and \( \overline{N}(r, a, f) \) of \( \overline{n}(r, f), \overline{n}(r, \infty), \) and \( \overline{n}(r, a, f) \).

If \( f(z) \) is an \( E \)-valued meromorphic function in the whole complex plane, then the order and the lower order of \( f(z) \) are defined by

\[
\lambda(f) = \limsup_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r}, \tag{1.10}
\]

\[
\mu(f) = \liminf_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r}.
\]
If \( f(z) \) is an \( E \)-valued meromorphic function in \( \mathbb{C}_R, 0 < R < +\infty \), then the order and the lower order of \( f(z) \) are defined by

\[
\lambda(f) = \limsup_{r \to R} \frac{\log T(r, f)}{\log^*(1/(R-r))},
\]
\[
\mu(f) = \liminf_{r \to R} \frac{\log T(r, f)}{\log^*(1/(R-r))}.
\]

**Lemma 1.1.** Let \( B(x) \) be a positive and continuous function in \([0, +\infty)\) which satisfies \( \limsup_{x \to +\infty} (\log B(x)/\log x) = \infty \). Then there exists a continuously differentiable function \( \rho(x) \), which satisfies the following conditions.

(i) \( \rho(x) \) is continuous and nondecreasing for \( x \geq x_0 (x_0 > 0) \) and tends to \( +\infty \) as \( x \to +\infty \).

(ii) The function \( U(x) = x^{\rho(x)} (x \geq x_0) \) satisfies the following:

\[
\lim_{x \to +\infty} \frac{\log U(X)}{\log U(x)} = 1, \quad X = x + \frac{x}{\log U(x)}.
\]

(iii) \( \limsup_{x \to +\infty} (\log B(x)/\log U(x)) = 1 \).

Lemma 1.1 is due to K. L. Hiong (also Qinglai Xiong) and \( \rho(x) \) is called the proximate order of Hiong. A simple proof of the existence of \( \rho(r) \) was given by Chuang [7]. Suppose that \( f(z) \) is an \( E \)-valued meromorphic function of infinite order in the unit disk \( \mathbb{C}_1 \). Let \( x = 1/(1-r) \) and \( X = 1/(1-R) \). From (ii) and (iii) in Lemma 1.1, we have

\[
\lim_{r \to 1^-} \frac{\log U(1/(1-r))}{\log U(1/(1-r))} = 1, \quad R = \frac{r \log U(1/(1-r)) + 1}{\log U(1/(1-r)) + 1},
\]

\[
\limsup_{x \to 1^+} \frac{\log T(r, f)}{\log U(1/(1-r))} = 1.
\]

Here, the functions \( \rho(1/(1-r)) \) and \( U(1/(1-r)) \) are called the proximate order and type function of \( f(z) \), respectively.

**Definition 1.2.** An \( E \)-valued meromorphic function \( f(z) \) in \( \mathbb{C}_R, 0 < R \leq +\infty \) is of compact projection, if for any given \( \epsilon > 0 \), \( \|P_n(f(z)) - f(z)\| < \epsilon \) has sufficiently large \( n \) in any fixed compact subset \( D \subset \mathbb{C}_R \).

Throughout this paper, we say that \( f(z) \) is an \( E \)-valued meromorphic function meaning that \( f(z) \) is of compact projection. C.-G. Hu and Q. Hu [3] established the following Nevanlinna’s first and second main theorems of \( E \)-valued meromorphic functions.
Theorem 1.3. Let \( f(z) \) be a nonconstant \( \mathbb{E} \)-valued meromorphic function in \( \mathbb{C}_R, 0 < R \leq +\infty \). Then for \( 0 < r < R \), \( a \in \mathbb{E} \), \( f(z) \equiv a \),

\[
T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log^+ \|c_q(a)\| + \varepsilon(r, a). 
\]

(1.14)

Here, \( \varepsilon(r, a) \) is a function satisfying that

\[
|\varepsilon(r, a)| \leq \log^+ \|a\| + \log 2, \quad \varepsilon(r, 0) \equiv 0,
\]

(1.15)

and \( c_q(a) \in \mathbb{E} \) is the coefficient of the first term in the Laurent series at the point \( a \).

Theorem 1.4. Let \( f(z) \) be a nonconstant \( \mathbb{E} \)-valued meromorphic function in \( \mathbb{C}_R, 0 < R \leq +\infty \) and \( a^{[k]} \in \mathbb{E} \cup \{\infty\} \) \( (k = 1, 2, \ldots, q) \) be \( q \geq 3 \) distinct points. Then for \( 0 < r < R \),

\[
(q - 2)T(r, f) \leq \sum_{k=1}^{q} \left[ V(r, a^{[k]}) + N(r, a^{[k]}) \right] + S(r, f).
\]

(1.16)

If \( R = +\infty \), then

\[
S(r, f) = O(\log T(r, f) + \log r)
\]

(1.17)

holds as \( r \to +\infty \) without exception if \( f(z) \) has finite order and otherwise as \( r \to +\infty \) outside a set \( J \) of exceptional intervals of finite measure \( \int_J dr < +\infty \). If the order of \( f(z) \) is infinite and \( \rho(r) \) is the proximate order of \( f(z) \), then

\[
S(r, f) = O(\log \rho(r))
\]

(1.18)

holds as \( r \to +\infty \) without exception.

If \( 0 < R < +\infty \), then

\[
S(r, f) = O\left( \log T(r, f) + \log \frac{1}{R - r} \right)
\]

(1.19)

holds as \( r \to R \) without exception if \( f(z) \) has finite order and otherwise as \( r \to R \) outside a set \( J \) of exceptional intervals of finite measure \( \int_J d((r/(R - r)) < +\infty \).

In all cases, the exceptional set \( J \) is independent of the choice of \( a^{[k]} \).
2. Characteristic Function of $E$-Valued Meromorphic Functions in the Unit Disc $\mathbb{C}_1$

In [4], Xuan and Wu proved the following.

**Theorem A.** Let $f(z)$ ($z \in \mathbb{C}$) be a nonconstant $E$-valued meromorphic function and $f(0) \neq \hat{\infty}$. Then for $\tau > 1$ and $0 < r < R$, one has

$$T(r, f) < C_\tau T(\tau r, f') + \log^+ \tau r + 4\log \|f(0)\|, \quad (2.1)$$

where $C_\tau$ is a positive constant.

**Theorem B.** Let $f(z)$ ($z \in \mathbb{C}$) be a nonconstant $E$-valued meromorphic function. Then we have

$$T(r, f') < 2T(r, f) + O(\log r + \log^+ T(r, f)). \quad (2.2)$$

**Theorem C.** For a nonconstant $E$-valued meromorphic function $f(z)$ ($z \in \mathbb{C}$) of order $\lambda(f) < +\infty$, one has $\lambda(f) = \lambda(f')$, $\mu(f) = \mu(f')$.

In this section, we shall prove that Theorems A, B, and C are valid for $E$-valued meromorphic function in the unit disc $\mathbb{C}_1$.

**Lemma 2.1.** Let $f(z)$ be an $E$-valued meromorphic function defined in the unit disc, and $f(0) \neq \hat{\infty}$. If $0 < R < R' < 1$, then there exists a $\theta_0 \in [0, 2\pi)$, such that for any $0 \leq r \leq R$, one has

$$\log^+ \|f(re^{i\theta_0})\| \leq \frac{R' + R}{R' - R} m(R', f) + n(R', f) \log 4 + N(R', f). \quad (2.3)$$

**Lemma 2.2.** Let $f(z)$ be an $E$-valued meromorphic function defined in the unit disc, and let $0 < R < R' < R'' < 1$. Then there exists a positive number $R \leq \rho \leq R'$, such that for $|z| = \rho$, one has

$$\log^+ \|f(re^{i\theta_0})\| \leq \frac{R'' + R'}{R'' - R'} m(R'', f) + n(R'', f) \log \frac{8eR''}{R'' - R'}. \quad (2.4)$$

Lemmas 2.1 and 2.2 are due to Xuan and Wu [4] for the $E$-valued meromorphic function defined in the whole complex plane. From the proof of Xuan and Wu [4], we know that Lemmas 2.1 and 2.2 are also valid for the $E$-valued meromorphic function defined in the unit disc $\mathbb{C}_1$.

**Lemma 2.3.** Let $f(z)$ ($z \in \mathbb{C}_1$) be a nonconstant $E$-valued meromorphic function and $f(0) \neq \hat{\infty}$. Suppose that $h(r) \geq 1$, $R = (1 + rh(r))/(1 + h(r))$, then when $r$ sufficiently tends to 1, one has

$$n(r, f) \leq \frac{6h(r)}{1 - r} N(R, f). \quad (2.5)$$
Abstract and Applied Analysis

Proof.

\[ N(R, f) = n(0, f) \log r + \int_0^R \frac{n(t, f) - n(0, f)}{t} dt = \int_0^R \frac{n(t, f)}{t} dt \]

\[ \geq \int_r^R \frac{n(t, f)}{t} dt \geq n(r, f) \log \frac{R}{r} \]

\[ = n(r, f) \log \left( 1 + \frac{1 - r}{r(1 + h(r))} \right) \geq n(r, f) \left( \frac{1 - r}{r(1 + h(r))} - \frac{(1 - r)/r + h(r)}{2} \right) \]

\[ \geq n(r, f) \left( \frac{(1 - r)/r(1 + h(r))}{2} \right) \geq n(r, f) \frac{1 - r}{6h(r)}. \]

(2.6)

Lemma 2.4 (see [4]). Let \( f(z) \ (z \in \mathbb{C}, 0 < R \leq +\infty) \) be a nonconstant \( E \)-valued meromorphic function and \( f(0) \neq \infty \), and \( L \) a curve from the origin along the segment \( \arg z = \theta_0 \) to \( pe^{i\theta_0} \), and along \( \{|z| = \rho < r\} \) turn a rotation to \( pe^{i\theta_0} \). Then for any \( \{|z| = \rho \leq r\} \), one has

\[ \log ||f(z)|| \leq \log^* M + O(1), \]

where \( M = \max \{ ||f'(z)||, z \in L \} \).

Lemma 2.5 (see [3]). Let \( f(z) \) be a nonconstant \( E \)-valued meromorphic function in \( \mathbb{C}_1 \). Then for \( 0 < r < 1 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{||f'(re^{i\theta})||}{||f(re^{i\theta})||} \right) d\theta < K \left( \log T(r, f) + \log \frac{1}{1 - r} \right). \]

(2.8)

where \( K \) is a sufficiently large constant.

We are now in the position to establish the main results of this section.

Theorem 2.6. Let \( f(z) \ (z \in \mathbb{C}_1) \) be a nonconstant \( E \)-valued meromorphic function and \( f(0) \neq \infty \). Then for \( e > 1 \) and any real function \( h(x) \geq 1 \), when \( r \) sufficiently tend to \( 1 \), one has

\[ T(r, f) < \frac{ch^{1+e}(r)}{(1 - r)^{1+e}} T(R, f'), \quad R = \frac{1 + rh(r)}{1 + h(r)}. \]

(2.9)
Proof. Denote $R_1 = (R + 2r)/3, R_2 = (r + 2R)/3$, we can get

$$ r < R_1 < R_2 < R, \quad R_1 - r = R_2 - R_1 = R - R_2 = \frac{R-r}{3}, $$

$$ R = \frac{1 - 3R_2 h(r)}{1 + 3h(r)}, \quad R_2 + R_1 = r + R < 2, \quad 1 - R_2 = \frac{(1 - r)(1 + 3h(r))}{3(1 + h(r))} \geq 1 - \frac{r}{2}; \quad (2.10) $$

$$ R - r = \frac{1 - r}{1 + h(r)} \geq \frac{1 - r}{2h(r)}. $$

Applying Lemma 2.1 to $f'(z)$ and combining Lemma 2.3, we can find a real number $\theta_0 \in [0, 2\pi)$ such that for any $0 \leq t \leq R_1$, one has

$$ \log^+ \|f'(te^{i\theta_0})\| \leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log 4 + N(R_2, f') $$

$$ \leq \left( \frac{6}{R - r} + \frac{6h(r)}{1 - R_2} \log 4 + 1 \right) T(R_2, f') $$

$$ \leq \left( \frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \log 4 + \frac{1 - r}{1 - r} \right) T(R, f') $$

$$ \leq \frac{6 + 6h(r) + 24h(r) + 1 - r}{1 - r} T(R, f') \leq \frac{40h(r)}{1 - r} T(R, f'). \quad (2.11) $$

In view of Lemma 2.2, there is a $\rho \in [r, R_1]$ such that for any $z \in ||z| = \rho||$, one has

$$ \log^+ \|f'(z)\| \leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log \frac{8eR_2}{R_3 - R} $$

$$ \leq \left( \frac{6}{R - r} + \frac{6h(r)}{1 - R_2} \frac{48eh(r)}{1 - r} \log \frac{1 - r}{1 - r} \right) T(R_2, f') $$

$$ \leq \left( \frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \frac{144h(r)}{1 - r} \log \frac{1 - r}{1 - r} \right) T(R, f') $$

$$ \leq \left( \frac{12h(r)}{1 - r} \left( 9 + \log \frac{h(r)}{1 - r} \right) \right) T(R, f') $$

$$ \leq \left( \frac{12h(r)}{1 - r} \left( 9 + \left( \frac{h(r)}{1 - r} \right) \right) \right) T(R, f') $$

$$ \leq 120 \left( \frac{h(r)}{1 - r} \right) T(R, f'). \quad (2.12) $$

From the origin along the segment $\arg z = \theta_0$ to $pe^{i\theta_0}$ and along $||z| = \rho||$, turn a rotation to $pe^{i\theta_0}$. We denote this curve by $L$. In virtue of Lemma 2.4, we have

$$ \log^+ \|f(z)\| \leq \log^+ \|M + O(1)\| \quad (2.13) $$
Abstract and Applied Analysis holds for any \(|z| = r \leq \rho\), where \(M = \max\{\|f'(z)\|, z \in L\}\). In virtue of (2.11), (2.12), and (2.13), we have

\[
m(r, f) \leq m(\rho, f) \leq m(\rho, f') \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ M d\theta \leq 121 \left( \frac{h(r)}{1-r} \right)^{1+\epsilon} T(R, f').
\]  

Hence,

\[
T(r, f) = m(r, f) + N(r, f) \leq m(r, f) + 2N(r, f') \leq 123 \left( \frac{h(r)}{1-r} \right)^{1+\epsilon} T(R, f').
\]  

**Theorem 2.7.** Let \(f(z) (z \in \mathbb{C}_1)\) be a nonconstant \(E\)-valued meromorphic function and \(f(0) \neq 0, \infty\). Then for any \(0 < r < R < 1\), one has

\[
T(r, f') < 2T(r, f) + O \left( \log^+ \frac{1}{1-r} + \log^+ T(r, f) \right).
\]  

**Proof.** By Lemma 2.5, we have

\[
T(r, f') = m(r, f') + N(r, f') \leq m(r, f) + m \left( r, \frac{f'}{f} \right) + 2N(r, f) \leq 2T(r, f) + m \left( r, \frac{f'}{f} \right).
\]  

**Theorem 2.8.** For a nonconstant \(E\)-valued meromorphic function \(f(z) (z \in \mathbb{C}_1)\) of order \(\lambda(f) < +\infty\), one has \(\lambda(f) = \lambda(f')\), \(\mu(f) = \mu(f')\).

Theorem 2.8 only discussed the \(E\)-valued meromorphic function of finite order. In fact, for any \(E\)-valued meromorphic function of infinite order, we have the following.

**Theorem 2.9.** If \(f(z) (z \in \mathbb{C}_1)\) is a nonconstant \(E\)-valued meromorphic function of order \(\lambda(f) = +\infty\), then the proximate orders of \(f(z)\) and \(f'(z)\) are the same.

**Proof.** Let \(h(r) = \log U(1/(1-r))\), in view of Theorems 2.6 and 2.7, we can easily derive Theorem 2.9.
3. \textit{E-Valued Borel Exceptional Values of Meromorphic Functions in $\mathbb{C}_R$}

Some definitions in this section can be found in [8].

\textbf{Definition 3.1.} Let $f(z)$ ($z \in \mathbb{C}_R, 0 < R \leq +\infty$) be an $E$-valued meromorphic function and $a \in E \cup \{\overline{\infty}\}$, if $k$ is a positive integer, let $\overline{n}_k(r,f)$ or $\overline{n}_k(r,\overline{\infty})$ denote the number of distinct poles of $f(z)$ of order $\leq k$ in $|z| \leq r$, and let $\overline{n}_k(r,a)$ denote the number of distinct $a$-points of $f(z)$ of order $\leq k$ in $|z| \leq r$. Similarly, we can define the counting functions $\overline{N}_k(r,f)$, $\overline{N}_k(r,\overline{\infty})$, and $\overline{N}_k(r,a)$ of $\overline{n}_k(r,f)$, $\overline{n}_k(r,\overline{\infty})$, and $\overline{n}_k(r,a)$.

\textbf{Definition 3.2.} Let $f(z)$ ($z \in \mathbb{C}_R, 0 < R \leq +\infty$) be an $E$-valued meromorphic function and $a \in E \cup \{\overline{\infty}\}$. If $R = +\infty$, we define

$$
\overline{p}_k(a,f) = \limsup_{r \to +\infty} \frac{\log^+ \left[ V(a,f) + \overline{N}_k(r,a) \right]}{\log r},
$$

$$
\overline{p}(a,f) = \limsup_{r \to +\infty} \frac{\log^+ \left[ V(a,f) + \overline{N}(r,a) \right]}{\log r},
$$

$$
\rho(a,f) = \limsup_{r \to +\infty} \frac{\log^+ \left[ V(a,f) + N(r,a) \right]}{\log r}.
$$

(3.1)

If $R < +\infty$, we define

$$
\overline{p}_k(a,f) = \limsup_{r \to R} \frac{\log^+ \left[ V(a,f) + \overline{N}_k(r,a) \right]}{\log(1/(R-r))},
$$

$$
\overline{p}(a,f) = \limsup_{r \to R} \frac{\log^+ \left[ V(a,f) + \overline{N}(r,a) \right]}{\log(1/(R-r))},
$$

$$
\rho(a,f) = \limsup_{r \to R} \frac{\log^+ \left[ V(a,f) + N(r,a) \right]}{\log(1/(R-r))}.
$$

(3.2)

\textbf{Definition 3.3.} Let $f(z)$ ($z \in \mathbb{C}_R, 0 < R \leq +\infty$) be an $E$-valued meromorphic function and $a \in E \cup \{\overline{\infty}\}$ and $k$ is a positive integer, we say that $a$ is an

(i) \textit{E-valued evB (exceptional value in the sense of Borel) for $f$ for distinct zeros of order $\leq k$ if $\overline{p}_k(a,f) < \lambda(f)$}.

(ii) \textit{E-valued evB for $f$ for distinct zeros if $\overline{p}(a,f) < \lambda(f)$}.

(iii) \textit{E-valued evB for $f$ (for the whole aggregate of zeros) if $\rho(a,f) < \lambda(f)$}.

In [5], Yang proved the following result.
Abstract and Applied Analysis

Theorem D. Let \( f(z) \) \((z \in \mathbb{C}, R = +\infty)\) be a meromorphic function with finite order \( \lambda > 0 \) and \( k_j \) \((j = 1, 2, \ldots, q)\) be \( q \) positive integers. \( a \) is called a pseudo-Borel exceptional value of \( f(z) \) of order \( k \) if

\[
\limsup_{r \to +\infty} \frac{\log^j V(r,a) }{ \log r } < \lambda(f) .
\]  

(3.3)

If \( f(z) \) has \( q \) distinct pseudo-Borel exceptional values \( a_j \) of order \( k_j \) \((j = 1, 2, \ldots, q)\), then

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.
\]  

(3.4)

It is natural to consider whether there exists a similar result, if meromorphic function \( f \) is replaced by \( E \)-valued meromorphic function \( f \). In this section, we extend the above theorem to \( E \)-valued meromorphic function in \( \mathbb{C}_R, 0 < R \leq +\infty \).

Theorem 3.4. Let \( f(z) \) \((z \in \mathbb{C}_R, 0 < R \leq +\infty)\) be an \( E \)-valued meromorphic function with finite order \( \lambda > 0 \), \( a^{[j]} \) \((j = 1, 2, \ldots, q)\) any system of distinct elements in \( E \cup \{ \infty \} \), and \( k_j \) \((j = 1, 2, \ldots, q)\) any system such that \( k_j \) is a positive integer or \( +\infty \). If \( a^{[j]} \) is an \( E \)-valued cvB for \( f \) for distinct zeros of order \( \leq k_j \) \((j = 1, 2, \ldots, q)\), then

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.
\]  

(3.5)

Proof. By Theorem 1.4, we have

\[
(q - 2)T(r,f) \leq \sum_{j=1}^{q} \left[ V \left( r, a^{[j]} \right) + N \left( r, a^{[j]} \right) \right] + S(r,f)
\]  

(3.6)

holds for \( 0 < r < R \). For any \( j = 1, 2, \ldots, q \), we have

\[
N \left( r, a^{[j]} \right) \leq \frac{1}{k_j + 1} \left\{ k_j N_{k_j} \left( r, a^{[j]} \right) + N \left( r, a^{[j]} \right) \right\},
\]  

(3.7)

\[
N \left( r, a^{[j]} \right) \leq T(r,f) - V \left( r, a^{[j]} \right) + O(1).
\]
Using (3.7) and (7) in (3.6), we get

\[
(q - 2)T(r, f) \leq \sum_{j=1}^{q} \left( V(r, a^{[j]}) + \frac{1}{k_j + 1} \left( k_j N_{k_j}(r, a^{[j]}) + N(r, a^{[j]}) \right) \right) + S(r, f) \\
= \sum_{j=1}^{q} \left( V(r, a^{[j]}) + \frac{k_j}{k_j + 1} N_{k_j}(r, a^{[j]}) + \frac{1}{k_j + 1} N(r, a^{[j]}) \right) + S(r, f) \\
\leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V(r, a^{[j]}) + N_{k_j}(r, a^{[j]}) \right) + \sum_{j=1}^{q} \frac{1}{k_j + 1} T(r, f) + S(r, f).
\]

(3.8)

Therefore, we have

\[
\left[ \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V(r, a^{[j]}) + N_{k_j}(r, a^{[j]}) \right) + S(r, f).
\]

(3.9)

By hypothesis, we have

\[
\bar{p}_{k_j}(a^{[j]}, f) < \lambda, \quad j = 1, 2, \ldots, q.
\]

(3.10)

If \( R = +\infty \), then there is a positive number \( \rho < \lambda \), such that for \( j = 1, 2, \ldots, q \), we can get

\[
V(r, a^{[j]}) + N_{k_j}(r, a^{[j]} \leq r^\rho).
\]

(3.11)

Using (3.11) to (3.9), we have

\[
\left[ \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} r^\rho + S(r, f).
\]

(3.12)

If \( \sum_{j=1}^{q} (1 - (1/(k_j + 1))) > 2 \), then by Theorem 1.4 and (3.12), we can get a contradiction \( \lambda \leq \rho \). So

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.
\]

(3.13)

If \( R = +\infty \), then there is a positive number \( \rho < \lambda \), such that for \( j = 1, 2, \ldots, q \), we can get

\[
V(r, a^{[j]}) + N_{k_j}(r, a^{[j]}) \leq \left( \frac{1}{R - r} \right)^\rho.
\]

(3.14)
Using (3.14) to (3.9), we have

\[
\left[ \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( \frac{1}{R - r} \right)^\rho + S(r, f). \tag{3.15}
\]

If \( \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) > 2 \), then by Theorem 1.4 and (3.15), we can get a contradiction \( \lambda \leq \rho \). So

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2. \tag{3.16}
\]

From the proof of Theorem 3.4, we can get the following.

**Corollary 3.5.** Let \( f(z) \) (\( z \in \mathbb{C}_R, 0 < R \leq +\infty \)) be a nonconstant \( E \)-valued meromorphic function. Then for any system \( a_j^{[i]} (j = 1, 2, \ldots, t) \) of distinct elements in \( E \cup \{ \infty \} \) and any system \( k_j (j = 1, 2, \ldots, t) \) such that \( k_j \) is a positive integer or \( +\infty \), we have the following:

1. if all of \( a_j^{[i]} (j = 1, 2, \ldots, q) \) in \( E \), then

\[
\left( q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2 \right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V(r, a_j^{[i]}, f) + N_k(r, a_j^{[i]}, f) \right) + S(r, f), \tag{3.17}
\]

2. if one of \( a_j^{[i]} (j = 1, 2, \ldots, q) \) is \( \infty \), say \( a_j^{[q]} = \infty \). Then,

\[
\left( q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2 \right) T(r, f) \leq \sum_{j=1}^{q-1} \frac{k_j}{k_j + 1} \left( V(r, a_j^{[i]}, f) + N_k(r, a_j^{[i]}, f) \right) + \frac{k_q}{k_q + 1} N_k(r, f) + S(r, f). \tag{3.18}
\]

**Remark 3.6.** If \( R = +\infty \), let \( q = r + t + s \) and \( k_j \equiv k (j = 1, 2, \ldots, r) \), \( k_j \equiv l (j = r + 1, \ldots, r + t) \) and \( k_j \equiv m (j = r + t + 1, \ldots, r + t + s) \) in Theorem 3.4. We can get the following result by Bhoosnurmath and Pujari [8].

**Theorem E.** Let \( f(z) \) (\( z \in \mathbb{C}_R, 0 < R \leq +\infty \)) be an \( E \)-valued meromorphic function of order \( \lambda(f), 0 < \lambda(f) \leq +\infty \). If there exist distinct elements

\[
a_j^{[1]}, a_j^{[2]}, \ldots, a_j^{[r]}; \quad b_j^{[1]}, b_j^{[2]}, \ldots, b_j^{[l]}; \quad c_j^{[1]}, c_j^{[2]}, \ldots, c_j^{[s]} \tag{3.19}
\]
in $E \cup \{\infty\}$ such that $a^{[1]}, a^{[2]}, \ldots, a^{[r]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq k$, $b^{[1]}, b^{[2]}, \ldots, b^{[l]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq l$, $c^{[1]}, c^{[2]}, \ldots, c^{[s]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq m$, where $k, l,$ and $m$ are positive integers, then

$$\frac{rk}{k + 1} + \frac{tl}{l + 1} + \frac{sm}{m + 1} \leq 2. \quad (3.20)$$

Bhoosnurmath and Pujari [8] pointed out that Theorem E is valid for $0 \leq \lambda(f) \leq +\infty$. In fact, Definition 3.3 is not well in the case of $\lambda(f) = 0$. In the case of $\lambda(f) = +\infty$, $a$ is an $E$-valued evB for $f$ if and only if $\rho_k(a, f)$ is finite. When $\rho_k(a, f)$ is infinite, we shall give the following definitions.

**Definition 3.7.** Let $f(z)$ ($z \in \mathbb{C}$) be an $E$-valued meromorphic function of infinite order and $\rho(r)$ is a proximate order of $f$ and $a \in E \cup \{\infty\}$. We say that $a$ is an

(i) $E$-valued evB (exceptional value in the sense of Borel) for $f$ for distinct zeros of order $\leq k$ if

$$\limsup_{r \to +\infty} \frac{\log^+ [V(a, f) + N_k(r, a)]}{\log U(r)} < 1; \quad (3.21)$$

(ii) $E$-valued evB for $f$ for distinct zeros if

$$\limsup_{r \to +\infty} \frac{\log^+ [V(a, f) + \bar{N}(r, a)]}{\log U(r)} < 1; \quad (3.22)$$

(iii) $E$-valued evB for $f$ (for the whole aggregate of zeros) if

$$\limsup_{r \to +\infty} \frac{\log^+ [V(a, f) + N(r, a)]}{\log U(r)} < 1. \quad (3.23)$$

**Theorem 3.8.** Let $f(z)$ ($z \in \mathbb{C}$) be an $E$-valued meromorphic function of infinite order and $\rho(r)$ is a proximate order of $f$, $a^{[j]}$ ($j = 1, 2, \ldots, q$) any system of distinct elements in $E \cup \{\infty\}$, and $k_j (j = 1, 2, \ldots, q)$ any system such that $k_j$ is a positive integer or $+\infty$. If $a^{[j]}$ is an $E$-valued evB for $f$ for distinct zeros of order $\leq k_j (j = 1, 2, \ldots, q)$, then

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2. \quad (3.24)$$

**Proof.** By Corollary 3.5, we have

$$\left(q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2\right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left(V\left(r, a^{[j]}\right) + \bar{N}_{k_j}\left(r, a^{[j]}\right)\right) + S(r, f). \quad (3.25)$$
By hypothesis, there exists a positive number $\eta < 1$ such that

$$V(r, a^{[l]}) + N_k(r, a^{[l]}) < U^q(r), \quad j = 1, 2, \ldots, q.$$  \hfill (3.26)

Using (3.25) to (3.26), we have

$$\left\lceil \sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) - 2 \right\rceil T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} U^q(r) + S(r, f).$$  \hfill (3.27)

If $\sum_{j=1}^{q} (1 - 1/(k_j + 1)) > 2$, then by Theorem 1.4 and (3.27), we can get a contradiction. So

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2.$$  \hfill (3.28)

\hfill \Box

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**References**


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