Research Article

More on \((\alpha, \beta)\)-Normal Operators in Hilbert Spaces

Rasoul Eskandari, Farzollah Mirzapour, and Ali Morassaei

Department of Mathematics, Faculty of Sciences, University of Zanjan, Zanjan 45195-313, Iran

Correspondence should be addressed to Ali Morassaei, morassaei@znu.ac.ir

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We study some properties of \((\alpha, \beta)\)-normal operators and we present various inequalities between the operator norm and the numerical radius of \((\alpha, \beta)\)-normal operators on Banach algebra \(B(\mathcal{H})\) of all bounded linear operators \(T : \mathcal{H} \to \mathcal{H}\), where \(\mathcal{H}\) is Hilbert space.

1. Introduction

Throughout the paper, let \(B(\mathcal{H})\) denote the algebra of all bounded linear operators acting on a complex Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\), \(B_0(\mathcal{H})\) denote the algebra of all self-adjoint operators in \(B(\mathcal{H})\), and \(I\) is the identity operator. In case of \(\dim \mathcal{H} = n\), we identify \(B(\mathcal{H})\) with the full matrix algebra \(M_n(\mathbb{C})\) of all \(n \times n\) matrices with entries in the complex field. An operator \(A \in B_0(\mathcal{H})\) is called positive if \(\langle Ax, x \rangle \geq 0\) is valid for any \(x \in \mathcal{H}\), and then we write \(A \geq 0\).

Moreover, by \(A > 0\) we mean \(\langle Ax, x \rangle > 0\) for any \(x \in \mathcal{H}\). An operator \(B - A \geq 0\). An operator \(A\) is majorized by \(B\), if there exists a constant \(\lambda\) such that \(\|Ax\| \leq \lambda\|Bx\|\) for all \(x \in \mathcal{H}\) or equivalently \(A^*A \leq \lambda^2B^*B\) [1].

For real numbers \(\alpha\) and \(\beta\) with \(0 \leq \alpha \leq 1 \leq \beta\), an operator \(T\) acting on a Hilbert space \(\mathcal{H}\) is called \((\alpha, \beta)\)-normal [2, 3] if

\[ \alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T. \]  

An immediate consequence of above definition is

\[ \alpha^2 \langle T^*x, x \rangle \leq \langle TT^*x, x \rangle \leq \beta^2 \langle T^*x, x \rangle, \]  

where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(\mathcal{H}\).
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from which we obtain

\[ \alpha \| Tx \| \leq \| T^*x \| \leq \beta \| Tx \|, \]  

(1.3)

for all \( x \in \mathcal{X} \).

Notice that, according to (1.1), if \( T \) is \((\alpha, \beta)\)-normal operator, then \( T \) and \( T^* \) majorize each other.

In [3], Moslehian posed two problems about \((\alpha, \beta)\)-normal operators as follows.

For fixed \( \alpha > 0 \) and \( \beta = 1 \),

(i) give an example of an \((\alpha, \beta)\)-normal operator which is neither normal nor hyponormal;

(ii) is there any nice relation between norm, numerical radius, and spectral radius of an \((\alpha, \beta)\)-normal operator?

Dragomir and Moslehian answered these problems in [2], as more as, they propounded a nice example of \((\alpha, \beta)\)-normal operator that is neither normal nor hyponormal, as follows.

The matrix \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) in \( \mathcal{B}(\mathbb{C}^2) \) is an \((\alpha, \beta)\)-normal with \( \alpha = \sqrt{3 - \sqrt{5}}/2 \) and \( \beta = \sqrt{3 + \sqrt{5}}/2 \).

The numerical radius \( w(T) \) of an operator \( T \) on \( \mathcal{X} \) is defined by

\[ w(T) = \sup \{ |\langle Tx, x \rangle| : \| x \| = 1 \}. \]  

(1.4)

Obviously, by (1.4), for any \( x \in \mathcal{X} \) we have

\[ |\langle Tx, x \rangle| \leq w(T) \| x \|^2. \]  

(1.5)

It is well known that \( w(\cdot) \) is a norm on the Banach algebra \( \mathcal{B}(\mathcal{X}) \) of all bounded linear operators. Moreover, we have

\[ w(T) \leq \| T \| \leq 2w(T) \quad (T \in \mathcal{B}(\mathcal{X})). \]  

(1.6)

For other results and historical comments on the numerical radius see [4].

The antieigenvalue of an operator \( T \in \mathcal{B}(\mathcal{X}) \) defined by

\[ \mu_1(T) := \inf_{Tx \neq 0} \frac{\text{Re} \langle Tx, x \rangle}{\| Tx \| \| x \|}. \]  

(1.7)

The vector \( x \in \mathcal{X} \) which takes \( \mu_1(T) \) is called an antieigenvector of \( T \). We refer more study on this matter to [4].

In this paper, we prove some properties of \((\alpha, \beta)\)-normal operators and state various inequalities between the operator norm and the numerical radius of \((\alpha, \beta)\)-normal operators in Hilbert spaces.
2. Some Properties of \((\alpha, \beta)\)-Normal Operators

In this section, we establish some properties of \((\alpha, \beta)\)-normal operators. It is easy to see that if \(T\) is an \((\alpha, \beta)\)-normal \((\alpha > 0)\) then \(T^*\) is \((1/\beta, 1/\alpha)\)-normal. We find numbers \(z \in \mathbb{C}\) such that \(z + T\) is \((\alpha, \beta)\)-normal where \(T\) is \((\alpha, \beta)\)-normal.

We know by the Cauchy-Schwartz inequality that \(-1 \leq \mu_1(T) \leq 1\). Also we can write

\[
\mu_1(T) = \inf_{\|x\|=1} \frac{\text{Re}(Tx, x)}{\|Tx\|}. \tag{2.1}
\]

We define

\[
\mu_2(T) := \sup_{\|x\|=1} \frac{\text{Re}(Tx, x)}{\|Tx\|}. \tag{2.2}
\]

We know that if \(T\) is normal operator then \(z + T\) is also normal.

**Theorem 2.1.** Let \(T\) be an \((\alpha, \beta)\)-normal operator on a Hilbert space such that \(0 \leq \alpha < 1 < \beta\) and \(z \in \mathbb{C}\). Then \(z + T\) is \((\alpha, \beta)\)-normal, if provided one of the following conditions holds:

(i) \(\mu_1(\overline{z}T) \geq 0\),

(ii) \(\mu_1(\overline{z}T) < 0, |z|^2 \geq -2\|z\|\|T\|\mu_1(\overline{z}T)\).

**Proof.** In both of above cases, we show that

\[
|z|^2 + 2 \text{Re}(\overline{z}Tx, x) \geq 0, \quad \forall x \in \mathcal{H} \text{ with } \|x\| = 1, Tx \neq 0. \tag{2.3}
\]

By the assumption (i), \(\mu_1(\overline{z}T) \geq 0\), we have \(\text{Re}(\overline{z}Tx, x)/\|z\|\|Tx\| \geq 0\) for every \(x \in \mathcal{H}\) with \(\|x\| = 1\) and \(Tx \neq 0\), consequently we get \(\text{Re}(\overline{z}Tx, x) \geq 0\), and therefore (2.3) is valid. On the other hand, if (ii) holds and we set \(B := \mu_1(\overline{z}T)\) then we get \(B \leq \text{Re}(\overline{z}Tx, x)/\|z\|\|Tx\|\) for every \(x \in \mathcal{H}\) with \(\|x\| = 1\) and \(Tx \neq 0\), consequently:

\[
\inf \{B\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf \left\{\|Tx\| \frac{\text{Re}(\overline{z}Tx, x)}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0\right\}. \tag{2.4}
\]

Since \(B < 0\), we obtain

\[
-B \inf \{-\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf \left\{\|Tx\| \frac{\text{Re}(\overline{z}Tx, x)}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0\right\}, \tag{2.5}
\]

and so

\[
B \sup \{\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf \left\{\|Tx\| \frac{\text{Re}(\overline{z}Tx, x)}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0\right\}. \tag{2.6}
\]
Now, by using the last inequality, we have

\[
|z|^2 + 2|z||T||\mu_1(\mathcal{Z}T)| = |z|^2 + 2|z| \left( \sup_{\|x\|=1} \|Tx\| \right) \left( \inf_{\|x\|=1} \left\{ \text{Re}(\mathcal{Z}Tx, x) / |z||Tx| \right\} \right) \\
\leq |z|^2 + 2|z| \inf_{\|x\|=1} \left\{ \|Tx\| / |z||Tx| \right\} \\
= |z|^2 + 2 \inf_{\|x\|=1} \{ \text{Re}(\mathcal{Z}Tx, x) \}.
\]

This shows that (2.3) holds for (ii), too. Thus, for any \( x \in \mathcal{H} \) with \( \|x\| = 1 \) we have

\[
\alpha^2 \langle (z + T)^* (z + T)x, x \rangle = \alpha^2 \left\{ \langle |z|^2 x, x \rangle + \langle \mathcal{Z}Tx, x \rangle + \langle zT^*x, x \rangle \right\} + \alpha^2 \langle T^*Tx, x \rangle \\
\leq \langle |z|^2 x, x \rangle + \langle \mathcal{Z}Tx, x \rangle + \langle zT^*x, x \rangle + \langle TT^*x, x \rangle \\
= \langle (z + T)(z + T)^*x, x \rangle \\
\leq \beta^2 \left\{ \langle |z|^2 x, x \rangle + \langle \mathcal{Z}Tx, x \rangle + \langle zT^*x, x \rangle \right\} + \beta^2 \langle TT^*x, x \rangle \\
= \beta^2 \langle (z + T)^* (z + T)x, x \rangle
\]

and this completes the proof. \( \square \)

**Corollary 2.2.** Let \( T \) be an \((\alpha, \beta)\)-normal operator. We have the following.

(i) If \( \mu_1(T) \geq 0 \) then \( z + T \) is \((\alpha, \beta)\)-normal operator for any \( z > 0 \).

(ii) If \( \mu_2(T) \leq 0 \) then \( z + T \) is \((\alpha, \beta)\)-normal operator for any \( z < 0 \).

**Proof.** (i) By the definition of the first antieigenvalue of \( T \), for all \( z > 0 \) we have

\[
\mu_1(\mathcal{Z}T) = \mu_1(zT) = \mu_1(T) \geq 0.
\]

By using Theorem 2.1(i) we imply that \( z + T \) is an \((\alpha, \beta)\)-normal.

(ii) If \( z < 0 \), then

\[
\mu_1(\mathcal{Z}T) = -\mu_2(T) \geq 0.
\]

By using Theorem 2.1(i) we imply that \( z + T \) is an \((\alpha, \beta)\)-normal. \( \square \)

**Corollary 2.3.** Let \( T \) be an injective and \((\alpha, \beta)\)-normal operator with \( \alpha > 0 \). Then

(i) \( \mathcal{R}(T) \) is dense,

(ii) \( T^* \) is injective,

(iii) if \( T \) is surjective then \( T^{-1} \) is also \((\alpha, \beta)\)-normal.
Proof. Since the inequality (1.3) is valid, we obtain $\mathcal{A}(T^*) = \mathcal{A}(T)$, and therefore $\mathcal{R}(T)^{\perp} = \mathcal{R}(T) = \mathcal{R}(T) = 0$, thus $\mathcal{R}(T)$ is a dense subspace of $\mathcal{A}$ and $T^*$ is injective. This proves (i) and (ii).

To prove (iii), we note that since $T$ is surjective, we imply that $T$ is invertible. On the other hand, we have $(T^*)^{-1} = (T^{-1})^*$. Also, we know that if $A$ and $B$ are two positive and invertible operators with $0 < A \leq B$ then $B^{-1} \leq A^{-1}$. Since $T$ is $(\alpha, \beta)$-normal, by taking inverse from all sides of (1.1), we get

$$\frac{1}{\beta^2}T^{-1}(T^*)^{-1} \leq (T^*)^{-1}T^{-1} \leq \frac{1}{\alpha^2}T^{-1}(T^*)^{-1}.$$  \hfill (2.11)

This means that $(T^{-1})^*$ is $(1/\beta, 1/\alpha)$-normal, thus $T^{-1}$ is $(\alpha, \beta)$-normal. \hfill \Box

Example 2.4. Consider the following matrix $T$ in $\mathcal{B}(\mathbb{C}^2)$:

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \hfill (2.12)$$

$T$ is an $(\alpha, \beta)$-normal operator, with parameters $\alpha = \sqrt{(3 - \sqrt{5})/2}$ and $\beta = \sqrt{(3 + \sqrt{5})/2}$. Then $T^{-1} = \left( \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} \right)$ is $(\alpha, \beta)$-normal.

For $T \in \mathcal{B}(\mathcal{A})$ we call

$$r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \} \hfill (2.13)$$

the spectral radius of $T$, where $\sigma(T)$ is the spectrum of $T$ and it is known that $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$ [5, page 102].

Theorem 2.5. Let $T$ be an $(\alpha, \beta)$-normal operator such that $T^{2n}$ is $(\alpha, \beta)$-normal operator for every $n \in \mathbb{N}$, too. Then, we have

$$\frac{1}{\beta} \|T\| \leq r(T) \leq \|T\|. \hfill (2.14)$$

Proof. For any $T \in \mathcal{B}(\mathcal{A})$ we have

$$\|T^*T\| = \|T\|^2. \hfill (2.15)$$

In particular, if $T$ is a self-adjoint operator then $\|T^2\| = \|T\|^2$. Thus, by the definition of $(\alpha, \beta)$-normal operator, we have

$$\|T^2T^2\| \geq \frac{1}{\beta^2} \|(T^*T)^2\| = \frac{1}{\beta^2} \|T\|^4. \hfill (2.16)$$
By induction on $n$, we imply that
\[ \left\| T^{*2^n}T^{2^n} \right\| \geq \frac{1}{\beta^{2^{n+1}-2}} \| T \|^{2^{n+1}}, \]  
(2.17)
from which we obtain
\[ r(T)^2 = r(T^*)r(T) = \lim_{n \to \infty} \left( \left\| T^{*2^n}T^{2^n} \right\| \right)^{1/2^n} \]
\[ \geq \lim_{n \to \infty} \left\| T^{*2^n}T^{2^n} \right\|^{1/2^n} \]
\[ \geq \lim_{n \to \infty} \left( \frac{1}{\beta^{2^{n+1}-2}} \| T \|^{2^{n+1}} \right)^{1/2^n} \]
\[ = \frac{1}{\beta^2} \| T \| \lim_{n \to \infty} \frac{1}{\beta^{2-2/2^n}} = \frac{1}{\beta^2} \| T \|^2. \]
(2.18)

Therefore, we get \((1/\beta)\| T \| \leq r(T) \leq \| T \|\). This completes the proof. \(\square\)

Below, we give an example of \((\alpha, \beta)\)-normal operator such that it satisfies in Theorem 2.5.

**Example 2.6.** Assume that $\mathcal{H}$ is a separable Hilbert space and \(\{e_n : n \in \mathbb{Z}\}\) is an orthonormal basis for $\mathcal{H}$. We define the operator $T \in \mathcal{B}(\mathcal{H})$ as follows:

\[ Te_n = \begin{cases} 
    e_{n-1}, & n \equiv 0 \pmod{3}, \\
    \frac{1}{2}e_{n-1}, & n \equiv 1 \pmod{3}, \\
    2e_{n-1}, & n \equiv 2 \pmod{3},
\end{cases} \]
(2.19)
so

\[ T^*e_n = \begin{cases} 
    \frac{1}{2}e_{n+1}, & n \equiv 0 \pmod{3}, \\
    2e_{n+1}, & n \equiv 1 \pmod{3}, \\
    e_{n+1}, & n \equiv 2 \pmod{3},
\end{cases} \]
(2.20)
and by simple computation we get

\[ TT^*e_n = \begin{cases} 
    \frac{1}{4}e_n, & n \equiv 0 \pmod{3}, \\
    4e_n, & n \equiv 1 \pmod{3}, \\
    e_n, & n \equiv 2 \pmod{3},
\end{cases} \]
\[ T^*e_n = \begin{cases} 
    e_n, & n \equiv 0 \pmod{3}, \\
    4e_n, & n \equiv 1 \pmod{3}, \\
    4e_n, & n \equiv 2 \pmod{3}.
\end{cases} \]
(2.21)

Consequently, $T$ is \((1/4, 4)\)-normal operator and also $T^n$ is \((1/4, 4)\)-normal operator, for any integer $n \geq 0$. Thus we have \(\| T \| = 2\) and \(r(T) = 1\), hence (2.14) is valid.
3. Inequalities Involving Norms and Numerical Radius

In this section we state some inequalities involving norms and numerical radius.

**Theorem 3.1.** Let $T \in \mathcal{B}(\mathcal{H})$ be an $(\alpha, \beta)$-normal operator.

(i) For positive real numbers $p$ and $q$ with $p \geq 2$ and $(1/p) + (1/q) = 1$ we have

$$\|T + T^*\|^p + \|T - T^*\|^p \geq 2(1 + \alpha^{2q - 1})\|T\|^p.$$  

(ii) If $0 \leq p \leq 1$ or $p \geq 2$, then we have

$$\left(\|T + T^*\|^2 + \|T - T^*\|^2\right)^{p/2} \geq \|T\|^p \varphi(\alpha, p),$$

where $\varphi(\alpha, p) = 2^p[1 + \alpha^2]^{p/2}.$

(iii) If $\mathcal{N}(T) = 0$ and for any $x \in \mathcal{H}$ with $\|x\| = 1$ we have

$$\left\|\frac{Tx}{\|Tx\|} - \frac{T^*x}{\|T^*x\|}\right\| \leq \rho,$$

then, we obtain

$$\alpha\|T\|^2 \leq \omega\left(T^2\right) + \frac{\rho^2}{2}\beta\|T\|^2.$$  

**Proof.** (i) We use the following known inequality:

$$\|a + b\|^p + \|a - b\|^p \geq 2(||a||^q + ||b||^q)^{p-1},$$

which is valid for any $a, b \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space.

Now, if we take $a = Tx$ and $b = T^*x$ in (3.5), then for any $x \in \mathcal{H}$ we get

$$\|Tx + T^*x\|^p + \|Tx - T^*x\|^p \geq 2(||Tx||^q + ||T^*x||^q)^{p-1}$$

$$\geq 2(||Tx||^q + \alpha^q||Tx||^q)^{p-1}$$

$$= 2(1 + \alpha^{2q-1})||Tx||^{q(p-1)}$$

$$= 2(1 + \alpha^{2q-1})||T||^p.$$  

Taking the supremum in (3.6) over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result (3.1).

(ii) We use the following inequality [6, Theorem 8, page 551]:

$$\left(\|a + b\|^2 + \|a - b\|^2\right)^{p/2} \geq 2^p\left(||a||^p + ||b||^p\right)^{p} + \left(2^{p-2}2^2\right)||a||^p||b||^p,$$

where $a$ and $b$ are two vectors in a Hilbert space and $0 \leq p \leq 1$ or $p \geq 2.$
Now, if we put \( a = Tx \) and \( b = T^*x \) in (3.7), then we obtain
\[
\left( \|Tx + T^*x\|^2 + \|Tx - T^*x\|^2 \right)^p \geq 2p \left( \|Tx\|^p + \|T^*x\|^p \right)^2 + \left( 2p - 2^2 \right) \|Tx\|^p \|T^*x\|^p,
\]
\[
\geq 2p \bigg( \|Tx\|^{2p}(1 + a^p)^2 + \left( 2p - 2^2 \right) a^p \|Tx\|^{2p} \bigg) = 2p \|Tx\|^{2p} \left( (1 + a^p)^2 + \left( 2p - 2^2 \right) a^p \right)
\]
\[
= \|Tx\|^{2p} \varphi(a, p).
\]

Now, taking the supremum over \( \|x\| = 1 \) in (3.8), we get the desired result (3.2).

(iii) We use the following reverse of Schwarz’s inequality:
\[
(0 \leq) |a||b| - |\langle a, b \rangle| \leq |a||b| - \Re \langle a, b \rangle \leq \frac{1}{2} \rho^2 |a||b|,
\]
which is valid for \( a, b \in \mathcal{H} \setminus \{0\} \) and \( \rho > 0 \), with \( \|(a/\|b\|) - (b/\|a\|)\| \leq \rho \) (see [7]). We take \( a = Tx \) and \( b = T^*x \) in (3.9) to get
\[
\|Tx\|\|T^*x\| \leq |\langle Tx, T^*x \rangle| + \frac{1}{2} \rho^2 \|Tx\|\|T^*x\|.
\]
Thus, we obtain
\[
\alpha \|Tx\|^2 \leq |\langle Tx, T^*x \rangle| + \frac{1}{2} \rho^2 \beta \|Tx\|^2.
\]

Now, taking the supremum over \( \|x\| = 1 \) in recent inequality, we get the desired result (3.4).

**Theorem 3.2.** Assume that \( T \) is an \((\alpha, \beta)\)-normal operator. Then, we have
\[
\left( 1 + a^2 \right) \|T\|^2 \leq \frac{1}{2} \|T - T^*\|^2 + \omega(T^2).
\]

**Proof.** By [2, Theorem 3.1], we have
\[
2(1 + a^p)\|T\|^p \leq \frac{1}{2} \left[ \|T + T^*\|^p + \|T - T^*\|^p \right],
\]
and also
\[
\left\| \frac{T^*T + TT^*}{2} \right\|^{p/2} \leq \frac{1}{4} \left[ \|T + T^*\|^p + \|T - T^*\|^p \right].
\]
On the other hand, it is known \(8\) that for \(A, B \in \mathcal{B}(\mathcal{H})\) we have
\[
\left\| \frac{A + B}{2} \right\|^2 \leq \frac{1}{2} \left\| \frac{A^* A + B^* B}{2} \right\| + \omega(B^* A) \] \tag{3.15}
\]

By using this inequality we get
\[
\left\| \frac{T + T^*}{2} \right\|^2 \leq \frac{1}{2} \left[ \left\| \frac{T^* T + TT^*}{2} \right\| + \omega(T^2) \right]. \tag{3.16}
\]

If we put \(p = 2\) in (3.14), we obtain
\[
\left\| \frac{T + T^*}{2} \right\|^2 \leq \frac{1}{2} \left[ \frac{1}{4} \left( \left\| T + T^* \right\|^2 + \left\| T - T^* \right\|^2 \right) + \omega(T^2) \right] \\
= \frac{1}{2} \left[ \left\| \frac{T + T^*}{2} \right\|^2 + \left\| \frac{T - T^*}{2} \right\|^2 + \omega(T^2) \right]. \tag{3.17}
\]

Thus we get
\[
\frac{1}{2} \left\| \frac{T + T^*}{2} \right\|^2 \leq \frac{1}{2} \left\| \frac{T - T^*}{2} \right\|^2 + \frac{\omega(T^2)}{2}. \tag{3.18}
\]

Now, we take \(p = 2\) in (3.13) to obtain
\[
(1 + a^2) \left\| T \right\|^2 \leq \left\| \frac{T - T^*}{2} \right\|^2 + \left\| \frac{T - T^*}{2} \right\|^2 + \omega(T^2) = \frac{1}{2} \left\| T - T^* \right\|^2 + \omega(T^2). \tag{3.19}
\]

This completes the proof. \(\square\)

**Theorem 3.3.** Assume that \(T\) is an \((\alpha, \beta)\)-normal operator. Then for any real \(s\) with \(0 \leq s \leq 1\), we have
\[
\left( (1 - s) \frac{1}{\beta^2} + s \right) \left( (1 - s) + s \frac{1}{\beta^2} \right) \left\| T \right\|^4 \leq \left[ 1 - s + s \beta^2 \right] \left\| T - T^* \right\|^2 + \omega(T^2)^2. \tag{3.20}
\]

**Proof.** By [9, Theorem 2.6] (see also [10, Theorem 2.4]), we have
\[
\left[ (1 - s) \left\| a \right\|^2 + s \left\| b \right\|^2 \right] \left[ (1 - s) \left\| b \right\|^2 + s \left\| a \right\|^2 \right] - |\langle a, b \rangle|^2 \leq \left[ (1 - s) \left\| a \right\|^2 + s \left\| b \right\|^2 \right] \left[ (1 - s) \left\| b - ta \right\|^2 + s \left\| tb - a \right\|^2 \right], \tag{3.21}
\]

where $0 \leq s \leq 1$, $t \in \mathbb{R}$ and $a, b \in \mathcal{A}$. By taking $t = 1$, $a = Tx$, and $b = T^*x$ in (3.21), we get

$$\left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right] \left[\|(1-s)T^*x\|^2 + s\|Tx\|^2\right] - |\langle Tx, T^*x \rangle|^2 \leq \left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right] \left[(1-s)\|T^*x - Tx\|^2 + s\|T^*x - T^2x\|^2\right],$$

(3.22)

thus, we have

$$\left[(1-s)\|T^*x\|^2 + s\|T^*x\|^2\right] \left[(1-s)\|T^*x\|^2 + \frac{s}{\beta^2}\|T^*x\|^2\right] - \left|\langle T^2x, x \rangle\right|^2 \leq \left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right] \left[(1-s)\|T^*x - Tx\|^2 + \frac{s}{\beta^2}\|T^*x - T^2x\|^2\right],$$

(3.23)

Finally, we take supremum over $\|x\| = 1$ from both sides of

$$\left(\frac{(1-s)}{\beta^2} + \frac{s}{\beta^2}\right)\|T^*x\|^4 \leq \left[(1-s)\|Tx\|^2 + s\beta^2\|T^*x\|^2\right]\|T^*x - Tx\|^2 \leq \left[(1-s)\|Tx\|^2 + s\beta^2\|T^*x\|^2\right]\|T^*x - T^2x\|^2 + \left|\langle T^2x, x \rangle\right|^2,$$

(3.24)

and we use triangle inequality for suprema to complete the proof.

**Corollary 3.4.** Let $T$ be an $(\alpha, \beta)$-normal operator. Then, we have

$$\frac{1}{\beta} \|T\|^2 \leq \|T\|\|T - T^*\| + o(T^2).$$

(3.25)

**Proof.** By using the inequality (3.21) we get

$$\left(1 - s + sa^2\right)\left(1 - s\right) \|T\|^4 \leq \left[1 - s + sa^2\right]\|T\|^2\|T - T^*\|^2 + \left|\langle T^2x, x \rangle\right|^2,$$

(3.26)

We take $s = 0$ in inequalities (3.20) and (3.26) to imply

$$\max\left\{\frac{1}{\beta^2} \|T\|^4 \|\langle T^2x, x \rangle\|^2 \leq \|Tx\|^2\|T - T^*\|^2 + \omega(T^2)\right\}.$$

(3.27)

Thus, $\max\{1/\beta, \alpha\} \|Tx\|^2 \leq \|Tx\|\|T - T^*x\| + o(T^2)$. Now, taking supremum overall $x$ with $\|x\| = 1$, the desired inequality is obtained
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References

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