Research Article

Neutral Operator and Neutral Differential Equation

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In this paper, we discuss the properties of the neutral operator \( Ax(t) = x(t) - cx(t - \delta(t)) \), and by applying coincidence degree theory and fixed point index theory, we obtain sufficient conditions for the existence, multiplicity, and nonexistence of (positive) periodic solutions to two kinds of second-order differential equations with the prescribed neutral operator.

1. Introduction

In [1], Zhang discussed the properties of the neutral operator \( A_1x(t) = x(t) - cx(t - \delta(t)) \), which became an effective tool for the research on differential equations with this prescribed neutral operator, see, for example, [2–5]. Lu and Ge [6] investigated an extension of \( A_1 \), namely, the neutral operator \( A_2x(t) = x(t) - \sum_{i=1}^{n} c_{ix}(t - \delta_i) \) and obtained the existence of periodic solutions for a corresponding neutral differential equation.

In this paper, we consider the neutral operator \( (Ax)(t) = x(t) - cx(t - \delta(t)) \), where \( c \) is constant and \( |c| \neq 1, \delta \in C^1(\mathbb{R}, \mathbb{R}) \), and \( \delta \) is an \( \omega \)-periodic function for some \( \omega > 0 \). Although \( A \) is a natural generalization of the operator \( A_1 \), the class of neutral differential equation with \( A \) typically possesses a more complicated nonlinearity than neutral differential equation with \( A_1 \) or \( A_2 \). For example, the neutral operators \( A_1 \) and \( A_2 \) are homogeneous in the following sense \( (A_1x)'(t) = (A_2x')(t) \) for \( i = 1, 2 \), whereas the neutral operator \( A \) in general is inhomogeneous. As a consequence many of the new results for differential equations with the neutral operator \( A \) will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows: in Section 2, we first analyze qualitative properties of the neutral operator \( A \) which will be helpful for further studies of differential equations.
with this neutral operator; in Section 3, by Mawhin’s continuation theorem, we obtain the existence of periodic solutions for a second-order Rayleigh-type neutral differential equation; in Section 4, by an application of the fixed point index theorem we obtain sufficient conditions for the existence, multiplicity, and nonexistence of positive periodic solutions to second-order neutral differential equation. Several examples are also given to illustrate our results. Our results improve and extend the results in [1, 2, 4, 7].

2. Analysis of the Generalized Neutral Operator

Let \( C_\omega = \{ x \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t), \ t \in \mathbb{R} \} \) with norm \( \|x\|_\omega = \max_{t \in [0,\omega]} |x(t)| \). Then \( (C_\omega, \|\cdot\|) \) is a Banach space. A cone \( K \) in \( C_\omega \) is defined by \( K = \{ x \in C_\omega : x(t) \geq \alpha \|x\|, \forall t \in \mathbb{R} \} \), where \( \alpha \) is a fixed positive number with \( \alpha < 1 \). Moreover, define operators \( A, B : C_\omega \to C_\omega \) by

\[
(Ax)(t) = x(t) - cx(t - \delta(t)), \quad (Bx)(t) = cx(t - \delta(t)).
\]  

Lemma 2.1. If \( |c| \neq 1 \), then the operator \( A \) has a continuous inverse \( A^{-1} \) on \( C_\omega \), satisfying

1. 
\[
(A^{-1} f)(t) = \begin{cases} 
  f(t) + \sum_{j=1}^{\infty} c^j f \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right), & \text{for } |c| < 1, \ \forall f \in C_\omega, \\
  -\frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} \left(1/c^{j+1}\right) f \left( s + \delta(t) + \sum_{i=1}^{j-1} \delta(D_i) \right), & \text{for } |c| > 1, \ \forall f \in C_\omega.
\end{cases}
\]

2. \( |(A^{-1} f)(t)| \leq \|f\|/|1-|c||, \text{ for all } f \in C_\omega.\)

3. \( \int_0^{\omega} |(A^{-1} f)(t)| dt \leq 1/|1-|c|| \int_0^{\omega} |f(t)| dt, \text{ for all } f \in C_\omega.\)

Proof. We have the following cases

Case 1 (\( |c| < 1 \)). Let \( t - \delta(t) = s \) and \( D_j = s - \sum_{i=1}^{j-1} \delta(D_i), j = 1, 2, \ldots \). Therefore,

\[
B^j x(t) = c^j x \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right),
\]

\[
\sum_{j=0}^{\infty} (B^j f)(t) = f(t) + \sum_{j=1}^{\infty} c^j f \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right). \tag{2.3}
\]
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Since $A = I - B$, we get from $\|B\| \leq |c| < 1$ that $A$ has a continuous inverse $A^{-1} : C_\omega \to C_\omega$ with

$$A^{-1} = (I - B)^{-1} = I + \sum_{j=1}^{\infty} B^j = \sum_{j=0}^{\infty} B^j,$$  \hspace{1cm} (2.4)

where $B^0 = I$. Then

$$\left( A^{-1} f(t) \right) = \sum_{j=0}^{\infty} \left[ B^j f \right](t) = \sum_{j=0}^{\infty} c^j f \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right),$$  \hspace{1cm} (2.5)

and consequently

$$\left| \left( A^{-1} f \right)(t) \right| = \left| \sum_{j=0}^{\infty} \left[ B^j f \right](t) \right| = \left| \sum_{j=0}^{\infty} c^j f \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right) \right| \leq \frac{\|f\|}{1 - |c|},$$  \hspace{1cm} (2.6)

Moreover,

$$\int_0^{\omega} \left| \left( A^{-1} f \right)(t) \right| dt = \int_0^{\omega} \left| \sum_{j=0}^{\infty} \left[ B^j f \right](t) \right| dt$$

$$\leq \sum_{j=0}^{\infty} \int_0^{\omega} \left| \left[ B^j f \right](t) \right| dt$$

$$= \sum_{j=0}^{\infty} \int_0^{\omega} c^j f \left( s - \sum_{i=1}^{j-1} \delta(D_i) \right) dt$$

$$\leq \frac{1}{1 - |c|} \int_0^{\omega} |f(t)| dt.$$  \hspace{1cm} (2.7)

Case 2 ($|c| > 1$). Let

$$E : C_\omega \to C_\omega, \hspace{0.5cm} (Ex)(t) = x(t) - \frac{1}{c} x(t + \delta(t)),$$  \hspace{1cm} (2.8)

$$B_1 : C_\omega \to C_\omega, \hspace{0.5cm} (B_1 x)(t) = \frac{1}{c} x(t + \delta(t)).$$

By definition of the linear operator $B_1$, we have

$$\left( B_1^j f \right)(t) = \frac{1}{c^j} f \left( s + \sum_{i=1}^{j-1} \delta(D_i) \right),$$  \hspace{1cm} (2.9)
where $D_i$ is defined as in Case 1. Summing over $j$ yields

$$\sum_{j=0}^{\infty} \left( B_j^i f \right)(t) = f(t) + \sum_{j=1}^{\infty} \frac{1}{c_j} f \left( s + \sum_{i=1}^{j-1} \delta(D_i) \right). \tag{2.10}$$

Since $\|B_1\| < 1$, we obtain that the operator $E$ has a bounded inverse $E^{-1}$,

$$E^{-1} : C_\omega \rightarrow C_\omega, \quad E^{-1} = (I - B_1)^{-1} = I + \sum_{j=1}^{\infty} B_j^i, \tag{2.11}$$

and for all $f \in C_\omega$ we get

$$\left( E^{-1} f \right)(t) = f(t) + \sum_{j=1}^{\infty} \left( B_j^i f \right)(t). \tag{2.12}$$

On the other hand, from $(Ax)(t) = x(t) - cx(t - \delta(t))$, we have

$$(Ax)(t) = x(t) - cx(t - \delta(t)) = -c \left[ x(t - \delta(t)) - \frac{1}{c} x(t) \right], \tag{2.13}$$

that is,

$$Ax(t) = -c(Ex)(t - \delta(t)). \tag{2.14}$$

Let $f \in C_\omega$ be arbitrary. We are looking for $x$ such that

$$(Ax)(t) = f(t). \tag{2.15}$$

that is,

$$-c(Ex)(t - \delta(t)) = f(t). \tag{2.16}$$

Therefore,

$$(Ex)(t) = -\frac{f(t + \delta(t))}{c} =: f_1(t), \tag{2.17}$$

and hence

$$x(t) = \left( E^{-1} f_1 \right)(t) = f_1(t) + \sum_{j=1}^{\infty} \left( B_j^i f_1 \right)(t) = -\frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} B_j^i \frac{f(t + \delta(t))}{c}, \tag{2.18}$$
proving that $A^{-1}$ exists and satisfies

$$
\left[ A^{-1}f \right](t) = \frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} \frac{B_{j} f(t + \delta(t))}{c} = \frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f \left( s + \delta(t) + \sum_{i=1}^{j-1} \delta(D_{i}) \right),
$$

$$
\left| \left[ A^{-1}f \right](t) \right| = \left| \frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f \left( s + \delta(t) + \sum_{i=1}^{j-1} \delta(D_{i}) \right) \right| \leq \frac{\| f \|}{|c| - 1}.
$$

(2.19)

Statements (1) and (2) are proved. From the above proof, (3) can easily be deduced. $\square$

**Lemma 2.2.** If $c < 0$ and $|c| < \alpha$, one has for $y \in K$ that

$$
\frac{\alpha - |c|}{1 - c^{2}} \| y \| \leq \left( A^{-1}y \right)(t) \leq \frac{1}{1 - |c|} \| y \|.
$$

(2.20)

**Proof.** Since $c < 0$ and $|c| < \alpha < 1$, by Lemma 2.1, we have for $y \in K$ that

$$
\left( A^{-1}y \right)(t) = y(t) + \sum_{j=1}^{\infty} c^{j} y \left( s - \sum_{i=1}^{j-1} \delta(D_{i}) \right)
$$

$$
= y(t) + \sum_{j \geq 1 \text{ even}} c^{j} y \left( s - \sum_{i=1}^{j-1} \delta(D_{i}) \right) - \sum_{j \geq 1 \text{ odd}} c^{j} y \left( s - \sum_{i=1}^{j-1} \delta(D_{i}) \right)
$$

$$
\geq \alpha \| y \| + \alpha \sum_{j \geq 1 \text{ even}} c^{j} \| y \| - \| y \| \sum_{j \geq 1 \text{ odd}} |c|^{j}
$$

(2.21)

$$
= \frac{\alpha}{1 - c^{2}} \| y \| - \frac{|c|}{1 - c^{2}} \| y \|
$$

$$
= \frac{\alpha - |c|}{1 - c^{2}} \| y \|.
$$

$\square$

**Lemma 2.3.** If $c > 0$ and $c < 1$ then for $y \in K$ one has

$$
\frac{\alpha}{1 - c} \| y \| \leq \left( A^{-1}y \right)(t) \leq \frac{1}{1 - c} \| y \|.
$$

(2.22)
Proof. Since $c > 0$ and $c < 1$, $\alpha < 1$, by Lemma 2.1, we have for $y \in K$ that
\[
(A^{-1} y)(t) = y(t) + \sum_{j \geq 1} c^j y \left( s - \sum_{i=1}^{j-1} \delta_i(D_i) \right) \\
\geq \alpha \|y\| + \alpha \|y\| \sum_{j \geq 1} c^j \\
= \frac{\alpha}{1 - c} \|y\|.
\] (2.23)

3. Periodic Solutions for Neutral Differential Equation

In this section, we consider the second-order neutral differential equation
\[
(x(t) - cx(t - \delta(t)))'' = f(t, x(t)) + g(t, x(t - \tau(t))) + e(t),
\] (3.1)
where $\tau, e \in C_\omega$ and $\int_0^\omega e(t)\,dt = 0$; $f$ and $g$ are continuous functions defined on $\mathbb{R}^2$ and periodic in $t$ with $f(t, \cdot) = f(t + \omega, \cdot), g(t, \cdot) = g(t + \omega, \cdot)$, $f(t, 0) = 0$, $f(t, u) \geq 0$, or $f(t, u) \leq 0$ for all $(t, u) \in \mathbb{R}^2$.

We first recall Mawhin’s continuation theorem which our study is based upon. Let $X$ and $Y$ be real Banach spaces and $L : D(L) \subset X \to Y$ a Fredholm operator with index zero, where $D(L)$ denotes the domain of $L$. This means that $\text{Im} L$ is closed in $Y$ and $\text{dim Ker} L = \text{dim}(Y/\text{Im} L) < +\infty$. Consider supplementary subspaces $X_1, Y_1$, of $X, Y$ respectively, such that $X = \text{Ker} L \oplus X_1, Y = \text{Im} L \oplus Y_1$, and let $P_1 : X \to \text{Ker} L$ and $Q_1 : Y \to Y_1$ denote the natural projections. Clearly, $\text{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_{P_1} := L|_{D(L)\cap X_1}$ is invertible. Let $L_{P_1}^{-1}$ denote the inverse of $L_{P_1}$.

Let $\Omega$ be an open bounded subset of $X$ with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be $L$-compact in $\overline{\Omega}$ if $Q_1 N(\overline{\Omega})$ is bounded and the operator $L_{P_1}^{-1} (I - Q_1) N : \overline{\Omega} \to X$ is compact.

Lemma 3.1 (Gaines and Mawhin [8]). Suppose that $X$ and $Y$ are two Banach spaces and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Y$ is $L$-compact on $\overline{\Omega}$. Assume that the following conditions hold:

1) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;
2) $Nx \notin \text{Im} L$, for all $x \in \partial\Omega \cap \text{Ker} L$;
3) $\deg \{JQ_1 N, \Omega \cap \text{Ker} L, 0\} \neq 0$, where $J : \text{Im} Q_1 \to \text{Ker} L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use Mawhin’s continuation theorem to study the existence of $\omega$-periodic solutions for (3.1), we rewrite (3.1) in the following form:

\[
(Ax_1)'(t) = x_2(t),
\]
\[
x_2(t) = f(t, x_1(t)) + g(t, x_1(t - \tau(t))) + e(t).
\] (3.2)
Clearly, if \( x(t) = (x_1(t), x_2(t))^T \) is an \( \omega \)-periodic solution to (3.2), then \( x_1(t) \) must be an \( \omega \)-periodic solution to (3.1). Thus, the problem of finding an \( \omega \)-periodic solution for (3.1) reduces to finding one for (3.2).

Recall that \( C_\omega = \{ \phi \in C([0, \omega) : \phi(t + \omega) \equiv \phi(t) \} \) with norm \( \| \phi \| = \max_{t \in [0, \omega]} |\phi(t)| \).

Define \( X = Y = C_\omega \times C_\omega = \{ x = (x_1(\cdot), x_2(\cdot)) \in C([0, \omega) : x(t) = x(t + \omega), t \in \mathbb{R} \} \) with norm \( \| x \| = \max \{ \| x_1 \|, \| x_2 \| \} \). Clearly, \( X \) and \( Y \) are Banach spaces. Moreover, define

\[
L : D(L) = \left\{ x \in C^1([0, \omega) : x(t + \omega) = x(t) \right\} \subset X \rightarrow Y
\]

by

\[
(Lx)(t) = \begin{pmatrix}
(Ax_1)'(t) \\
x'_2(t)
\end{pmatrix}
\]

(3.4)

and \( N : X \rightarrow Y \) by

\[
(Nx)(t) = \begin{pmatrix}
x_2(t) \\
f(t, x'_1(t)) + g(t, x_1(t - \tau(t))) + e(t)
\end{pmatrix}.
\]

(3.5)

Then (3.2) can be converted to the abstract equation \( Lx = Nx \). From the definition of \( L \), one can easily see that

\[
\text{Ker} L \equiv \mathbb{R}^2, \quad \text{Im} L = \left\{ y \in Y : \int_0^\omega \begin{pmatrix}
y_1(s) \\
y_2(s)
\end{pmatrix} ds = \begin{pmatrix}
0 \\
0
\end{pmatrix} \right\}.
\]

(3.6)

So \( L \) is a Fredholm operator with index zero. Let \( P_1 : X \rightarrow \text{Ker} L \) and \( Q_1 : Y \rightarrow \text{Im} Q_1 \subset \mathbb{R}^2 \) be defined by

\[
P_1 x = \begin{pmatrix}
(Ax_1)(0) \\
x_2(0)
\end{pmatrix}, \quad Q_1 y = \frac{1}{\omega} \int_0^\omega \begin{pmatrix}
y_1(s) \\
y_2(s)
\end{pmatrix} ds,
\]

(3.7)

then \( \text{Im} P_1 = \text{Ker} L, \text{Ker} Q_1 = \text{Im} L \). Setting \( L_{P_1} = L|_{D(L) \cap \text{Ker} P_1} \) and \( L^{-1}_{P_1} : \text{Im} L \rightarrow D(L) \) denotes the inverse of \( L_{P_1} \), then

\[
[L^{-1}_{P_1} y](t) = \begin{pmatrix}
(A^{-1}Fy_1)(t) \\
(Fy_2)(t)
\end{pmatrix},
\]

(3.8)

\[
[Fy_1](t) = \int_0^t y_1(s) ds, \quad [Fy_2](t) = \int_0^t y_2(s) ds.
\]

From (3.5) and (3.8), it is clear that \( Q_1 N \) and \( L^{-1}_{P_1} (I - Q_1) N \) are continuous and \( Q_1 N(\overline{\Omega}) \) is bounded, and then \( L^{-1}_{P_1} (I - Q_1) N(\overline{\Omega}) \) is compact for any open bounded \( \Omega \subset X \) which means \( N \) is \( L \)-compact on \( \overline{\Omega} \).
Now we give our main results on periodic solutions for (3.1).

**Theorem 3.2.** Suppose there exist positive constants $K_1, D, M, b$ with $M > \|e\|$ such that:

1. (H1) $|f(t, u)| \leq K_1|u| + b$, for $(t, u) \in \mathbb{R} \times \mathbb{R}$;
2. (H2) $\text{sgn} \cdot g(t, x) > \|e\|$, for $|x| > D$;
3. (H3) $g(t, x) \geq -M$, for $|x| \leq -D$ and $t \in \mathbb{R}$.

Then (3.1) has at least one solution with period $\omega$ if $0 < \omega^{1/2}(1 + |c|)^{1/2} \sqrt{2K_1/|1 - |c|| - |c|\delta_1} < 1$, where $\delta_1 = \max_{t \in [0, \omega]} |\delta(t)|$.

**Proof.** By construction (3.2) has an $\omega$-periodic solution if and only if the following operator equation

$$Lx = Nx$$ (3.9)

has an $\omega$-periodic solution. From (3.8), we see that $N$ is $L$-compact on $\overline{\Omega}$, where $\Omega$ is any open, bounded subset of $C_x$. For $\lambda \in (0, 1]$ define

$$\Omega_1 = \{x \in C_x : Lx = \lambda Nx\}. \quad (3.10)$$

Then $x = (x_1, x_2)^T \in \Omega_1$ satisfies

$$\begin{align*}
(Ax_1)'(t) &= \lambda x_2(t), \\
\lambda f(t, x_1(t)) &= \lambda g(t, x_1(t - \tau(t))) + \lambda e(t).
\end{align*} \quad (3.11)$$

We first claim that there is a constant $\xi \in \mathbb{R}$ such that

$$|x_1(\xi)| \leq D. \quad (3.12)$$

In view of $\int_0^\omega (Ax_1)'(t)dt = 0$, we know that there exist two constants $t_1, t_2 \in [0, \omega]$ such that $(Ax_1)'(t_1) \geq 0, (Ax_1)'(t_2) \leq 0$. From the first equation of (3.11), we have $x_2(t) = (1/\lambda)(Ax_1)'(t)$, so

$$\begin{align*}
x_2(t_1) &= \frac{1}{\lambda} (Ax_1)'(t_1) \geq 0, \\
x_2(t_2) &= \frac{1}{\lambda} (Ax_1)'(t_2) \leq 0.
\end{align*} \quad (3.13)$$

Let $t_3, t_4 \in [0, \omega]$ be, respectively, a global maximum and minimum point of $x_2(t)$. Clearly, we have

$$\begin{align*}
x_2(t_3) &\geq 0, \quad x_2'(t_3) = 0, \\
x_2(t_4) &\leq 0, \quad x_2'(t_4) = 0.
\end{align*} \quad (3.14)$$
Since \( f(t, x'_1) \geq 0 \) or \( f(t, x'_1) \leq 0 \), w.l.o.g., suppose \( f(t, x'_1) \geq 0 \), for \((t, x'_1) \in [0, \omega] \times \mathbb{R}\). Then
\[
-g(t_3, x_1(t_3 - \tau(t_3))) - e(t_3) = f(t, x'_1(t_3)) \geq 0, \\
g(t_3, x_1(t_3 - \tau(t_3))) \leq -e(t_3) \leq \|e\|. 
\] (3.15)

From \((H_2)\) we see that
\[
x_1(t_3 - \tau(t_3)) < D. 
\] (3.16)

Similarly, we have
\[
g(t_4, x_1(t_4 - \tau(t_4))) \geq -e(t_4) \geq -\|e\|, 
\] (3.17)

and again by \((H_2)\),
\[
x_1(t_4 - \tau(t_4)) < -D. 
\] (3.18)

Case 1. If \( x_1(t_3 - \tau(t_3)) \in (-D, D) \), define \( \xi = t_3 - \tau(t_3) \), obviously \( |x_1(\xi)| \leq D \).

Case 2. If \( x_1(t_3 - \tau(t_3)) < -D \), from (3.18) and the fact that \( x \) is a continuous function in \( \mathbb{R} \), there exists a constant \( \xi \) between \( x_1(t_3 - \tau(t_3)) \) and \( x_1(t_4 - \tau(t_4)) \) such that \( |x_1(\xi)| = D \). This proves (3.12).

Choose an integer \( k \) and a constant \( t_5 \in [0, \omega] \) such that \( \xi = \omega k + t_5 \), then \( |x_1(\xi)| = |x_1(t_5)| \leq D \). Hence
\[
|x_1(t)| \leq D + \int_0^\omega |x'_1(s)| ds. 
\] (3.19)

Substituting \( x_2(t) = (1/\lambda)(Ax_1)'(t) \) into the second equation of (3.11) yields
\[
\left( \frac{1}{\lambda} (Ax_1)(t) \right)'' = \lambda f(t, x'_1(t)) + \lambda g(t, x_1(t - \tau(t))) + \lambda e(t), 
\] (3.20)

that is,
\[
((Ax_1)(t))'' = \lambda^2 f(t, x'_1(t)) + \lambda^2 g(t, x_1(t - \tau(t))) + \lambda^2 e(t). 
\] (3.21)

Integrating both sides of (3.21) over \([0, \omega]\), we have
\[
\int_0^\omega [f(t, x'_1(t)) + g(t, x_1(t - \tau(t)))] dt = 0. 
\] (3.22)
On the other hand, multiplying both sides of (3.21) by \((Ax_1)(t)\) and integrating over \([0, \omega]\), we get

\[
\int_0^{\omega} (Ax_1(t))^2 dt = \int_0^{\omega} \|Ax_1(t)\|^2 dt - \lambda^2 \int_0^{\omega} f(t, x'_1(t))(Ax_1(t))dt - \lambda^2 \int_0^{\omega} e(t)(Ax_1(t))dt.
\]

Using \((H_1)\), we have

\[
\int_0^{\omega} \|Ax_1(t)\|^2 dt \leq \int_0^{\omega} [f(t, x'_1(t))]|x_1(t) - cx_1(t - \delta(t))|dt + \int_0^{\omega} |g(t, x_1(t - \tau(t)))||x_1(t) - cx_1(t - \delta(t))|dt + \int_0^{\omega} |e(t)||x_1(t) - cx_1(t - \delta(t))|dt
\]

\[
\leq (1 + |c|)||x_1|| \left[ K_1 \int_0^{\omega} |x'_1(t)| dt + b \omega + \int_0^{\omega} |g(t, x_1(t - \tau(t)))| dt + \omega ||e|| \right].
\]

Besides, we can assert that there exists some positive constant \(N_1\) such that

\[
\int_0^{\omega} |g(t, x_1(t - \tau(t)))| dt \leq 2 \omega N_1 + \omega b + K_1 \int_0^{\omega} |x'_1(t)| dt.
\]

In fact, in view of condition \((H_1)\) and (3.22) we have

\[
\int_0^{\omega} \{g(t, x_1(t - \tau(t))) - K_1 |x'_1(t)| - b\} dt \leq \int_0^{\omega} \{g(t, x_1(t - \tau(t))) - |f(t, x'_1(t))|\} dt
\]

\[
\leq \int_0^{\omega} \{g(t, x_1(t - \tau(t))) + f(t, x'_1(t))\} dt = 0.
\]

Define

\[
E_1 = \{t \in [0, \omega] : x_1(t - \tau(t)) > D\};
\]

\[
E_2 = \{t \in [0, \omega] : |x_1(t - \tau(t))| \leq D\} \cup \{t \in [0, \omega] : x_1(t - \tau(t)) < -D\}.
\]
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With these sets we get

\[
\int_{E_1} \left| g(t, x_1(t - \tau(t))) \right| dt \leq \omega \max \left\{ M, \sup_{t \in [0, \omega], x_1(t - \tau(t)) \leq D} \left| g(t, x_1) \right| \right\}.
\]

\[
\int_{E_1} \left\{ \left| g(t, x_1(t - \tau(t))) \right| - K_1 \left| x_1'(t) \right| - b \right\} dt
\]

\[
= \int_{E_1} \left\{ g(t, x_1(t - \tau(t))) - K_1 \left| x_1'(t) \right| - b \right\} dt
\]

\[
\leq - \int_{E_2} \left\{ g(t, x_1(t - \tau(t))) - K_1 \left| x_1'(t) \right| - b \right\} dt
\]

\[
\leq \int_{E_2} \left\{ \left| g(t, x_1(t - \tau(t))) \right| + K_1 \left| x_1'(t) \right| + b \right\} dt,
\]

which yields

\[
\int_{E_1} \left| g(t, x_1(t - \tau(t))) \right| dt \leq \int_{E_2} \left| g(t, x_1(t - \tau(t))) \right| dt + \int_{E_1 \cup E_2} (K_1 \left| x_1'(t) \right| + b) dt
\]

\[
= \int_{E_2} \left| g(t, x_1(t - \tau(t))) \right| dt + \omega b + K_1 \int_0^\omega \left| x_1'(t) \right| dt.
\]

That is,

\[
\int_0^\omega \left| g(t, x_1(t - \tau(t))) \right| dt = \int_{E_1} \left| g(t, x_1(t - \tau(t))) \right| dt + \int_{E_2} \left| g(t, x_1(t - \tau(t))) \right| dt
\]

\[
\leq 2 \int_{E_2} \left| g(t, x_1(t - \tau(t))) \right| dt + \omega b + K_1 \int_0^\omega \left| x_1'(t) \right| dt
\]

\[
\leq 2 \omega \max \left\{ M, \sup_{t \in [0, \omega], x_1(t - \tau(t)) \leq D} \left| g(t, x_1) \right| \right\} + \omega b + K_1 \int_0^\omega \left| x_1'(t) \right| dt
\]

\[
= 2 \omega D_1 + \omega b + K_1 \int_0^\omega \left| x_1'(t) \right| dt,
\]

where \( N_1 = \max\{ M, \sup_{t \in [0, \omega], x_1(t - \tau(t)) \leq D} \left| g(t, x_1) \right| \} \), proving (3.25).
Substituting (3.25) into (3.24) and recalling (3.19), we get

\[
\int_0^{\omega} |(Ax_1)'(t)|^2 \, dt \leq (1 + |c|)|x_1|_0 \left( 2K_1 \int_0^{\omega} |x_1'(t)| \, dt + 2\omega b + 2\omega N_1 + \omega \max_{t \in [0,\omega]} |e(t)| \right)
\]

\[
= (1 + |c|) \left[ 2K_1 \left( D + \int_0^{\omega} |x_1'(t)| \, dt \right) \int_0^{\omega} |x_1'(t)| \, dt 
\right. 
\left. + \left( 2\omega b + 2\omega N_1 + \omega \max_{t \in [0,\omega]} |e(t)| \right) \left( D + \int_0^{\omega} |x_1'(t)| \, dt \right) \right]
\]

\[
= (1 + |c|) \left[ 2K_1 D \int_0^{\omega} |x_1'(t)| \, dt + 2K_1 \left( \int_0^{\omega} |x_1'(t)| \, dt \right)^2 
+ N_2 \int_0^{\omega} |x_1'(t)| \, dt + N_2 D \right]
\]

\[
= 2K_1 (1 + |c|) \left( \int_0^{\omega} |x_1'(t)| \, dt \right)^2 
+ (1 + |c|)(N_2 + 2K_1 D) \int_0^{\omega} |x_1'(t)| \, dt 
+ (1 + |c|)N_2 D,
\]

(3.31)

where \( N_2 = 2\omega b + 2\omega N_1 + \omega \|e\| \). Since \( (Ax)(t) = x(t) - cx(t - \delta(t)) \), we have

\[
(Ax_1)'(t) = (x_1(t) - cx_1(t - \delta(t)))'
\]

\[
= x_1'(t) - cx_1'(t - \delta(t)) + cx_1'(t - \delta(t))\delta'(t)
\]

\[
= (Ax_1)'(t) + cx_1'(t - \delta(t))\delta'(t),
\]

(3.32)

By applying Lemma 2.1, we have

\[
\int_0^{\omega} |x_1'(t)| \, dt = \int_0^{\omega} \left| \left( A^{-1}Ax_1' \right)(t) \right| \, dt
\]

\[
\leq \frac{\int_0^{\omega} \left| (Ax_1)'(t) \right| \, dt}{|1 - |c||}
\]

\[
= \frac{\int_0^{\omega} \left| (Ax_1)'(t) - cx_1'(t - \delta(t))\delta'(t) \right| \, dt}{|1 - |c||}
\]

\[
\leq \frac{\int_0^{\omega} \left| (Ax_1)'(t) \right| \, dt + |c|\delta_1 \int_0^{\omega} |x_1'(t)| \, dt}{|1 - |c||},
\]

(3.33)
where \( \delta_1 = \max_{t \in [0, \alpha]} |\delta'(t)| \). Since \( 0 < \omega^{1/2} (1 + |c|)^{1/2} \sqrt{2K_1} / (|1 - |c|| - |c|\delta_1) \), then \( |1 - |c|| - |c|\delta_1 > 0 \), so we get

\[
\int_0^\alpha |x_1'(t)| dt \leq \frac{\omega^{1/2} (\int_0^\alpha (Ax_1)'(t) dt)^{1/2}}{|1 - |c|| - |c|\delta_1|} \leq \frac{\omega^{1/2} (\int_0^\alpha ((Ax_1)'(t))^2 dt)^{1/2}}{|1 - |c|| - |c|\delta_1|}.
\] (3.34)

Applying the inequality \((a + b)^k \leq a^k + b^k \) for \( a, b > 0 \), \( 0 < k < 1 \), it follows from (3.31) and (3.34) that

\[
\int_0^\alpha |x_1'(t)| dt \\
\leq \frac{\omega^{1/2}}{|1 - |c|| - |c|\delta_1|} \left[ (1 + |c|)^{1/2} \sqrt{2K_1} \int_0^\alpha |x_1'(t)| dt + (1 + |c|)^{1/2} \left( \int_0^\alpha |x_1'(t)| dt \right)^{1/2} \right] \times (N_2 + 2K_1D)^{1/2} + (1 + |c|)^{1/2} N_2 D^{1/2}].
\] (3.35)

Since \( \omega^{1/2} (1 + |c|)^{1/2} \sqrt{2K_1} / (|1 - |c|| - |c|\delta_1) < 1 \), it is easy to see that there exists a constant \( M_1 > 0 \) (independent of \( \lambda \)) such that

\[
\int_0^\alpha |x_1'(t)| dt \leq M_1.
\] (3.36)

It follows from (3.19) that

\[
\|x_1\| \leq D + \int_0^\alpha |x_1'(t)| dt \leq D + M_1 := M_2.
\] (3.37)

By the first equation of (3.11) we have \( \int_0^\alpha x_2(t) dt = \int_0^\alpha (Ax_1)'(t) dt = 0 \), which implies that there is a constant \( t_1 \in [0, \alpha] \) such that \( x_2(t_1) = 0 \), hence \( \|x_2\| \leq \int_0^\alpha |x_2'(t)| dt \). By the second equation of (3.11) we obtain

\[
x_2'(t) = \lambda f(t, x_1'(t)) + \lambda g(x_1(t - \tau(t))) + \lambda e(t).
\] (3.38)

So, from \((H_1)\) and (3.25), we have

\[
|x_2| \leq \int_0^\alpha |f(t, x_1'(t))| dt + \int_0^\alpha |g(t, x_1(t - \tau(t))))| dt + \int_0^\alpha |e(t)| dt
\]

\[
\leq 2K_1 M_1 + 2\omega b + 2\omega N_1 + \omega \|e\| := M_3.
\] (3.39)
Let $M_4 = \sqrt{M_2^2 + M_3^2} + 1$, $\Omega = \{x = (x_1, x_2)^T : \|x_1\| < M_4, \|x_2\| < M_4\}$, then for all $x \in \partial \Omega \cap \text{Ker } L$

\[
Q_1Nx = \frac{1}{\omega} \int_0^\omega \left( f(t, x_1(t)) + g(t, x_1(t - \tau(t))) + e(t) \right) dt.
\]

If $Q_1Nx = 0$, then $x_2(t) = 0$, $x_1 = M_4$ or $-M_4$. But if $x_1(t) = M_4$, we know

\[
0 = \int_0^\omega g(M_4)dt,
\]

that is, $g(M_4) = 0$. From assumption $(H_2)$, we know $M_4 \leq D$, which yields a contradiction, one can argue similarly if $x_1 = -M_4$. We also have $Q_1Nx \neq 0$, that is, for all $x \in \partial \Omega \cap \text{Ker } L, x \notin \text{Im } L$, so conditions (1) and (2) of Lemma 3.1 are both satisfied. Define the isomorphism $J : \text{Im } Q_1 \rightarrow \text{Ker } L$ as follows:

\[
J(x_1, x_2)^T = (x_2, x_1)^T.
\]

Let $H(\mu, x) = \mu x + (1 - \mu)Q_1N x, (\mu, x) \in [0, 1] \times \Omega$, then, for all $(\mu, x) \in (0, 1) \times (\partial \Omega \cap \text{Ker } L)$,

\[
H(\mu, x) = \left( \mu x_1(t) + \frac{1 - \mu}{\omega} \int_0^\omega \left[ f(t, x_1(t)) + g(t, x_1(t - \tau(t))) + e(t) \right] dt \right)
\]

\[
(\mu + (1 - \mu))x_2(t)
\]

We have $\int_0^\omega e(t)dt = 0$. So, we can get

\[
H(\mu, x) = \left( \mu x_1(t) + \frac{1 - \mu}{\omega} \int_0^\omega \left[ f(t, x_1(t)) + g(t, x_1(t - \tau(t))) \right] dt \right)
\]

\[
(\mu + (1 - \mu))x_2(t)
\]

$\forall (\mu, x) \in (0, 1) \times (\partial \Omega \cap \text{Ker } L)$.

From $(H_2)$, it is obvious that $x^TH(\mu, x) > 0$, for all $(\mu, x) \in (0, 1) \times (\partial \Omega \cap \text{Ker } L)$. Hence

\[
\text{deg}[Q_1N, \Omega \cap \text{Ker } L, 0] = \text{deg}[H(0, x), \Omega \cap \text{Ker } L, 0]
\]

\[
= \text{deg}[H(1, x), \Omega \cap \text{Ker } L, 0]
\]

\[
= \text{deg}[I, \Omega \cap \text{Ker } L, 0] \neq 0.
\]

So condition (3) of Lemma 3.1 is satisfied. By applying Lemma 3.1, we conclude that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^T$ on $\Omega \cap D(L)$, that is, (3.1) has an $\omega$-periodic solution $x_1(t)$. \qed
By using a similar argument, we can obtain the following theorem.

**Theorem 3.3.** Suppose there exist positive constants $K_1, D, M, b$ with $M > \|e\|$ such that:

- $(H_1)$ $|f(t, u)| \leq K_1|u| + b$, for $(t, u) \in \mathbb{R} \times \mathbb{R}$;
- $(H_2)$ $\text{sgn} x \cdot g(t, x) > \|e\|$, for $|x| > D$;
- $(H_3)$ $g(t, x) \leq M$, for $x \geq D$ and $t \in \mathbb{R}$,

then (3.1) has at least one solution with period $\omega$ if $0 < \omega(1 + |c|)^{1/2}/(1 + c - |c|) < 1$.

**Remark 3.4.** If $\int_0^\omega e(t)dt \neq 0$ and $f(t,0) \neq 0$, the problem of existence of $\omega$-periodic solutions to (3.1) can be converted to the existence of $\omega$-periodic solutions to:

$$(x(t) - cx(t - \delta(t)))'' = f_1(t, x'(t)) + g_1(t, x(t - \tau(t))) + e_1(t), \quad (3.46)$$

where $f_1(t, x) = f(t, x) - f(t, 0)$, $g_1(t, x) = g(t, x) + (1/\omega) \int_0^\omega e(t)dt + f(t, 0)$, and $e_1(t) = e(t) - (1/\omega) \int_0^\omega e(t)dt$. Clearly, $\int_0^\omega e_1(t)dt = 0$ and $f_1(t,0) = 0$, and (3.46) can be discussed by using Theorem 3.2 (or Theorem 3.3).

4. **Positive Periodic Solutions for Neutral Equations**

Consider the following second-order neutral functional differential equation:

$$(x(t) - cx(t - \delta(t)))'' = -a(t)x(t) + \lambda b(t)f(x(t - \tau(t))), \quad (4.1)$$

where $\lambda$ is a positive parameter; $f \in C(\mathbb{R}, [0, \infty))$, and $f(x) > 0$ for $x > 0$; $a \in C(\mathbb{R}, (0, \infty))$ with $\max\{a(t) : t \in [0, \omega]\} < (\pi/\omega)^2$, $b \in C(\mathbb{R}, (0, \infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, $a(t), b(t)$, and $\tau(t)$ are $\omega$-periodic functions.

Define the Banach space $X$ as in Section 2, and let $C_\omega^+ = \{x \in C(\mathbb{R}, (0, \infty)) : x(t + \omega) = x(t)\}$. Denote

$$M = \max\{a(t) : t \in [0, \omega]\}, \quad m = \min\{a(t) : t \in [0, \omega]\}, \quad \beta = \sqrt{M},$$

$$L = \frac{1}{2\beta\sin(\beta/2)}, \quad l = \frac{\cos(\beta/2)}{2\beta\sin(\beta/2)}, \quad k = l(M + m) + LM,$$

$$k_1 = \frac{k - \sqrt{k^2 - 4LMm}}{2LM}, \quad a = \frac{l[m - (M + m)|c|]}{LM(1 - |c|)}.$$

It is easy to see that $M, m, \beta, L, l, k, k_1 > 0$.

Now we consider (4.1). First let

$$\overline{f}_0 = \lim_{x \to 0^+} \frac{f(x)}{x}, \quad \overline{f}_\infty = \lim_{x \to \infty} \frac{f(x)}{x}, \quad \underline{f}_0 = \lim_{x \to 0^-} \frac{f(x)}{x}, \quad \underline{f}_\infty = \lim_{x \to \infty} \frac{f(x)}{x}, \quad (4.3)$$
and denote
\[ i_0 = \text{number of } 0's \text{ in } \left( f_{0}, f_{\infty} \right), \quad i_0 = \text{number of } 0's \text{ in } \left( f_{0}, f_{\infty} \right); \]
\[ i_{\infty} = \text{number of } \infty's \text{ in } \left( f_{0}, f_{\infty} \right), \quad i_{\infty} = \text{number of } \infty's \text{ in } \left( f_{0}, f_{\infty} \right). \]
\[ (4.4) \]

It is clear that \( i_0, i_{0}, i_{\infty}, i_{\infty} \in \{0, 1, 2\}. \) We will show that (4.1) has \( i_0 \) or \( i_{\infty} \) positive \( w \)-periodic solutions for sufficiently large or small \( \lambda \), respectively.

In the following we discuss (4.1) in two cases, namely, the case where \( c < 0 \) and \( c > -\min\{k_1, m/(M + m)\} \) (note that \( c > -m/(M + m) \) implies \( \alpha > 0 \); \( c > -k_1 \) implies \( |c| < \alpha \) and the case where \( c > 0 \) and \( c < \min\{m/(M + m), (LM - lm)/(L - 1)(M - lm)\} \) (note that \( c < m/(M + m) \) implies \( \alpha > 0 \); \( c < (LM - lm)/(L - 1)(M - lm) \) implies \( \alpha < 1 \)). Obviously, we have \( |c| < 1 \) which makes Lemma 2.1 applicable for both cases and also Lemmas 2.2 or 2.3, respectively.

Let \( K = \{x \in X : x(t) \geq a||x||\} \) denote the cone in \( X \) as defined in Section 2, where \( a \) is just as defined above. We also use \( K_r = \{x \in K : ||x|| < r\} \) and \( \partial K_r = \{x \in K : ||x|| = r\} \). Let \( y(t) = (Ax)(t) \), then from Lemma 2.1 we have \( x(t) = (A^{-1}y)(t) \). Hence (4.1) can be transformed into
\[ y''(t) + a(t)\left(A^{-1}y\right)(t) = \lambda b(t)f\left(\left(A^{-1}y\right)(t - \tau(t))\right), \]
\[ (4.5) \]
which can be further rewritten as
\[ y''(t) + a(t)y(t) - a(t)H(y(t)) = \lambda b(t)f\left(\left(A^{-1}y\right)(t - \tau(t))\right), \]
\[ (4.6) \]
where \( H(y(t)) = y(t) - (A^{-1}y)(t) = -c(A^{-1}y)(t - \delta(t)) \).

Now we discuss the two cases separately.

### 4.1. Case I

Assume \( c < 0 \) and \( c > -\min\{k_1, m/(M + m)\} \).

**Lemma 4.1 (see [7]).** The equation
\[ y''(t) + My(t) = h(t), \quad h \in C^+_{\omega}, \]
\[ (4.7) \]
has a unique \( \omega \)-periodic solution
\[ y(t) = \int_t^{t+\omega} G(t, s)h(s)ds, \]
\[ (4.8) \]
where
\[ G(t, s) = \frac{\cos \beta((\omega/2) + t - s)}{2\beta \sin(\beta \omega/2)}, \quad s \in [t, t + \omega]. \]
\[ (4.9) \]
Lemma 4.2 (see [7]). One has $\int_{t}^{t+\omega} G(t,s)ds = 1/M$. Furthermore, if $\max\{a(t) : t \in [0,\omega]\} < (\pi/\omega)^2$, then $0 < l \leq G(t,s) \leq L$ for all $t \in [0,\omega]$ and $s \in [t,t+\omega]$.

Now we consider

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = h(t), \quad h \in C^+_\omega,$$

and define operators $T, \hat{H} : X \to X$ by

$$(Th)(t) = \int_{t}^{t+\omega} G(t,s)h(s)ds, \quad (\hat{H}y)(t) = M - a(t)y(t) + a(t)H(y(t)).$$

Clearly $T, \hat{H}$ are completely continuous $(Th)(t) > 0$ for $h(t) > 0$ and $\|\hat{H}\| \leq (M-m+M(|c|/1-|c|))$).

By Lemma 4.1, the solution of (4.10) can be written in the form

$$y(t) = (Th)(t) + (T\hat{H}y)(t).$$

In view of $c < 0$ and $c > -\min\{k_1, m/(M+m)\}$, we have

$$\left\|T\hat{H}\right\| \leq \|T\|\|\hat{H}\| \leq \frac{M-m+M|c|}{M(1-|c|)} < 1,$$

and hence

$$y(t) = \left(I - T\hat{H}\right)^{-1}(Th)(t).$$

Define an operator $P : X \to X$ by

$$(Ph)(t) = \left(I - T\hat{H}\right)^{-1}(Th)(t).$$

Obviously, for any $h \in C^+_\omega$, if $\max\{a(t) : t \in [0,\omega]\} < (\pi/\omega)^2$, $y(t) = (Ph)(t)$ is the unique positive $\omega$-periodic solution of (4.10).

Lemma 4.3. $P$ is completely continuous and

$$(Th)(t) \leq (Ph)(t) \leq \frac{M(1-|c|)}{m-(M+m)|c|}\|Th\|, \quad \forall h \in C^+_\omega.$$
Proof. By the Neumann expansion of $P$, we have

$$P = \left( I - \hat{H} \right)^{-1} T$$

$$= \left( I + \hat{H} + \left( \hat{H} \right)^2 + \cdots + \left( \hat{H} \right)^n + \cdots \right) T$$

(4.17)

$$= T + T\hat{H}T + \left( \hat{H} \right)^2 T + \cdots + \left( \hat{H} \right)^n T + \cdots .$$

Since $T$ and $\hat{H}$ are completely continuous, so is $P$. Moreover, by (4.17), and recalling that $\|\hat{H}\| \leq (M - m + m|c|)/M(1 - |c|) < 1$, we get

$$\langle Th \rangle(t) \leq \langle Ph \rangle(t) \leq \frac{M(1 - |c|)}{m - (M + m)|c|} \| Th \|. \quad (4.18)$$

Define an operator $Q : X \to X$ by

$$Qy(t) = P\left( \lambda b(t)f \left( \left( A^{-1}y \right) \left( t - \tau(t) \right) \right) \right). \quad (4.19)$$

Lemma 4.4. One has $Q(K) \subset K$.

Proof. From the definition of $Q$, it is easy to verify that $Qy(t + \omega) = Qy(t)$. For $y \in K$, we have from Lemma 4.3 that

$$Qy(t) = P\left( \lambda b(t)f \left( \left( A^{-1}y \right) \left( t - \tau(t) \right) \right) \right)$$

$$\geq \lambda \int_t^{t+\omega} G(t, s)b(s)f \left[ \left( A^{-1}y \right) \left( s - \tau(s) \right) \right] ds$$

$$= \lambda \int_t^{t+\omega} G(t, s)b(s)f \left[ \left( A^{-1}y \right) \left( s - \tau(s) \right) \right] ds \geq \lambda M \int_0^{\omega} b(s)f \left[ \left( A^{-1}y \right) \left( s - \tau(s) \right) \right] ds. \quad (4.20)$$

On the other hand,

$$Qy(t) = P\left( \lambda b(t)f \left( \left( A^{-1}y \right) \left( t - \tau(t) \right) \right) \right) \leq \frac{M(1 - |c|)}{m - (M + m)|c|} \| T\left( \lambda b(t)f \left( \left( A^{-1}y \right) \left( t - \tau(t) \right) \right) \right) \|$$
existence of positive periodic solutions for $Q$ equivalent to the existence of fixed points for the operator $Q$.

**Proof.**

By Lemmas 2.2, 4.2, and 4.3, we have for $Q$ that

$$
\frac{M(1 - |c|)}{m - (M + m)|c|} \max_{t \in [0, \omega]} \int_t^{t+\omega} G(t, s) b(s) f \left( \left( A^{-1} y \right)(s - \tau(s)) \right) ds
$$

$$
\leq \frac{M(1 - |c|)}{m - (M + m)|c|} L \int_0^{\omega} b(s) f \left( \left( A^{-1} y \right)(s - \tau(s)) \right) ds.
$$

Therefore,

$$
Qy(t) \geq \frac{L [m - (M + m)|c|]}{LM(1 - |c|)} \|Qy\| = \alpha \|Qy\|,
$$

(4.22)

that is, $Q(K) \subset K$. \hfill \Box

From the continuity of $P$, it is easy to verify that $Q$ is completely continuous in $X$. Comparing (4.6) to (4.10), it is obvious that the existence of periodic solutions for (4.6) is equivalent to the existence of fixed points for the operator $Q$ in $X$. Recalling Lemma 4.4, the existence of positive periodic solutions for (4.6) is equivalent to the existence of fixed points of $Q$ in $K$. Furthermore, if $Q$ has a fixed point $y$ in $K$, it means that $(A^{-1} y)(t)$ is a positive $\omega$-periodic solutions of (4.1).

**Lemma 4.5.** If there exists $\eta > 0$ such that

$$
f \left( \left( A^{-1} y \right)(t - \tau(t)) \right) \geq \left( A^{-1} y \right)(t - \tau(t)) \eta, \quad \text{for } t \in [0, \omega], \ y \in K,
$$

(4.23)

then

$$
\|Qy\| \geq \lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^{\omega} b(s) ds \|y\|, \quad y \in K.
$$

(4.24)

**Proof.** By Lemmas 2.2, 4.2, and 4.3, we have for $y \in K$ that

$$
Qy(t) = P \left( \lambda b(t) f \left( \left( A^{-1} y \right)(t - \tau(t)) \right) \right)
$$

$$
\geq T \left( \lambda b(t) f \left( \left( A^{-1} y \right)(t - \tau(t)) \right) \right)
$$

$$
= \lambda \int_t^{t+\omega} G(t, s) b(s) f \left( \left( A^{-1} y \right)(s - \tau(s)) \right) ds
$$

$$
\geq \lambda \eta \int_0^{\omega} b(s) \left( A^{-1} y \right)(s - \tau(s)) ds
$$

$$
\geq \lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^{\omega} b(s) ds \|y\|.
$$

(4.25)
Hence
\[ \| Qy \| \geq \lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s) \| y \|, \quad y \in K. \quad (4.26) \]

**Lemma 4.6.** If there exists \( \varepsilon > 0 \) such that
\[ f\left( (A^{-1}y)(t - \tau(t)) \right) \leq (A^{-1}y)(t - \tau(t)) \varepsilon, \quad \text{for } t \in [0, \omega], \ y \in K, \quad (4.27) \]
then
\[ \| Qy \| \leq \lambda \varepsilon \frac{LM \int_0^\omega b(s) ds}{m - (M + m)|c|} \| y \|, \quad y \in K. \quad (4.28) \]

**Proof.** By Lemmas 2.2, 4.2, and 4.3, we have
\[
\| Qy(t) \| \leq \lambda \frac{M(1 - |c|)}{m - (M + m)|c|} \int_0^\omega b(s) f\left( (A^{-1}y)(s - \tau(s)) \right) ds \\
\leq \lambda \frac{M(1 - |c|)}{m - (M + m)|c|} \varepsilon \int_0^\omega b(s) (A^{-1}y)(s - \tau(s)) ds \\
\leq \lambda \varepsilon \frac{LM \int_0^\omega b(s) ds}{m - (M + m)|c|} \| y \|. \quad (4.29)
\]

Define
\[
F(r) = \max \left\{ f(t) : 0 \leq t \leq \frac{r}{1 - |c|} \right\}, \\
f_1(r) = \min \left\{ f(t) : \frac{\alpha - |c|}{1 - c^2} r \leq t \leq \frac{r}{1 - |c|} \right\}. \quad (4.30)
\]

**Lemma 4.7.** If \( y \in \partial K_r \), then
\[ \| Qy \| \geq \lambda f_1(r) \int_0^\omega b(s) ds. \quad (4.31) \]

**Proof.** By Lemma 2.2, we obtain \( ((\alpha - |c|)/(1 - c^2)) r \leq (A^{-1}y)(t - \tau(t)) \leq r / (1 - |c|) \) for \( y \in \partial K_r \), which yields \( f((A^{-1}y)(t - \tau(t))) \geq f_1(r) \). The lemma now follows analog to the proof of Lemma 4.5. \( \square \)
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Lemma 4.8. If $y \in \partial K_r$, then

$$\|Qy\| \leq \frac{\lambda LM(1-|c|)F(r)}{m-(M+m)|c|} \int_0^\omega b(s)ds.$$  \hspace{1cm} (4.32)

Proof. By Lemma 2.2, we can have $0 \leq (A^{-1}y)(t-\tau(t)) \leq r/(1-|c|)$ for $y \in \partial K_r$, which yields $f((A^{-1}y)(t-\tau(t))) \leq F(r)$. Similar to the proof of Lemma 4.6, we get the conclusion.

We quote the fixed point theorem which our results will be based on.

Lemma 4.9 (see [9]). Let $X$ be a Banach space and $K$ a cone in $X$. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K_r} \rightarrow K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Now we give our main results on positive periodic solutions for (4.1).

Theorem 4.10. (a) If $\tilde{i}_0 = 1$ or 2, then (4.1) has $\tilde{i}_0$ positive $\omega$-periodic solutions for $\lambda > 1/f_1(1)\int_0^\omega b(s)ds > 0$;

(b) If $\tilde{i}_\infty = 1$ or 2, then (4.1) has $i_\infty$ positive $\omega$-periodic solutions for $0 < \lambda < (m-(M+m)|c|)/LM(1-|c|)F(1)\int_0^\omega b(s)ds$;

(c) If $\tilde{i}_\infty = 0$ or $i_0 = 0$, then (4.1) has no positive $\omega$-periodic solutions for sufficiently small or sufficiently large $\lambda > 0$, respectively.

Proof. (a) Choose $r_1 = 1$. Take $\lambda_0 = 1/f_1(1)\int_0^\omega b(s)ds > 0$, then for all $\lambda > \lambda_0$, we have from Lemma 4.7 that

$$\|Qy\| > \|y\|, \quad \text{for } y \in \partial K_{r_1}. \hspace{1cm} (4.33)$$

Case 1. If $\tilde{i}_0 = 0$, we can choose $0 < \tilde{r}_2 < r_1$, so that $f(u) \leq \varepsilon u$ for $0 \leq u \leq \tilde{r}_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \frac{LM \int_0^\omega b(s)ds}{m-(M+m)|c|} < 1. \hspace{1cm} (4.34)$$

Letting $r_2 = (1-|c|)\tilde{r}_2$, we have $f((A^{-1}y)(t-\tau(t))) \leq \varepsilon (A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $0 \leq (A^{-1}y)(t-\tau(t)) \leq \|y\|/(1-|c|) \leq \tilde{r}_2$ for $y \in \partial K_{r_2}$. In view of Lemma 4.6 and (4.34), we have for $y \in \partial K_{r_2}$ that

$$\|Qy\| \leq \lambda \varepsilon \frac{LM \int_0^\omega b(s)ds}{m-(M+m)|c|} \|y\| \leq \|y\|. \hspace{1cm} (4.35)$$

It follows from Lemma 4.9 and (4.33) that

$$i(Q, K_{r_1}, K) = 1, \quad i(Q, K_{r_2}, K) = 0. \hspace{1cm} (4.36)$$
thus \(i(Q, K_{r_1} \setminus \overline{K}_{r_2}, K) = -1\) and \(Q\) has a fixed point \(y\) in \(K_{r_1} \setminus \overline{K}_{r_2}\), which means \((A^{-1}y)(t)\) is a positive \(\omega\)-positive solution of (4.1) for \(\lambda > \lambda_0\).

**Case 2.** If \(\overline{f}_\infty = 0\), there exists a constant \(\overline{H} > 0\) such that \(f(u) \leq \varepsilon u\) for \(u \geq \overline{H}\), where the constant \(\varepsilon > 0\) satisfies

\[
\lambda \varepsilon \frac{L M \int_0^\omega b(s)ds}{m - (M + m)|c|} < 1. \tag{4.37}
\]

Letting \(r_3 = \max\{2r_1, \overline{H}(1 - c^2)/(|\alpha - |c||)\}, \) we have \(f((A^{-1}y)(t - \tau(t))) \leq \varepsilon(A^{-1}y)(t - \tau(t))\) for \(y \in K_{r_2}\). By Lemma 2.2, we have \((A^{-1}y)(t - \tau(t))) \geq ((|\alpha - |c||)/(1 - c^2))\|y\| \geq \overline{H}\) for \(y \in \partial K_{r_2}\).

Thus by Lemma 4.6 and (4.37), we have for \(y \in \partial K_{r_2}\) that

\[
\|Qy\| \leq \lambda \varepsilon \frac{L M \int_0^\omega b(s)ds}{m - (M + m)|c|} \|y\| < \|y\|. \tag{4.38}
\]

Recalling from Lemma 4.9 and (4.33) that

\[
i(Q, K_{r_1}, K) = 1, \quad i(Q, K_{r_1}, K) = 0, \tag{4.39}
\]

then \(i(Q, K_{r_1} \setminus \overline{K}_{r_2}, K) = 1\) and \(Q\) has a fixed point \(y\) in \(K_{r_1} \setminus \overline{K}_{r_2}\), which means \((A^{-1}y)(t)\) is a positive \(\omega\)-positive solution of (4.1) for \(\lambda > \lambda_0\).

**Case 3.** If \(\overline{f}_0 = \overline{f}_\infty = 0\), from the above arguments, there exist \(0 < r_2 < r_1 < r_3\) such that \(Q\) has a fixed point \(y_1(t)\) in \(K_{r_1} \setminus \overline{K}_{r_2}\) and a fixed point \(y_2(t)\) in \(K_{r_2} \setminus \overline{K}_{r_1}\). Consequently, \((A^{-1}y_1)(t)\) and \((A^{-1}y_2)(t)\) are two positive \(\omega\)-periodic solutions of (4.1) for \(\lambda > \lambda_0\).

(b) Let \(r_1 = 1\). Take \(\lambda_0 = (m - (M + m)|c|)/LM(1 - |c||F(r_1) \int_0^\omega b(s)ds) > 0\); then by Lemma 4.8 we know if \(\lambda < \lambda_0\) then

\[
\|Qy\| < \|y\|, \quad y \in \partial K_{r_1}. \tag{4.40}
\]

**Case 1.** If \(f_\infty = \infty\), we can choose \(0 < \overline{r}_2 < r_1\) so that \(f(u) \geq \eta u\) for \(0 \leq u \leq \overline{r}_2\), where the constant \(\eta > 0\) satisfies

\[
\lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds > 1. \tag{4.41}
\]

Letting \(r_2 = (1 - |c|)\overline{r}_2\), we have \(f((A^{-1}y)(t - \tau(t))) \geq \eta(A^{-1}y)(t - \tau(t))\) for \(y \in K_{r_2}\). By Lemma 2.2, we have \(0 \leq (A^{-1}y)(t - \tau(t))) \leq \|y\|/(1 - |c|) \leq \overline{r}_2\) for \(y \in \partial K_{r_2}\). Thus by Lemma 4.5 and (4.41),

\[
\|Qy\| \geq \lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds \|y\| > \|y\|. \tag{4.42}
\]
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It follows from Lemma 4.9 and (4.40) that

\[ i(Q, K_r, K) = 0, \quad i(Q, K_r, K) = 1, \quad (4.43) \]

which implies \( i(Q, K_r \setminus \overline{K}_r, K) = 1 \) and \( Q \) has a fixed point \( y \) in \( K_r \setminus \overline{K}_r \). Therefore, \((A^{-1} y)(t)\) is a positive \( \omega \)-periodic solution of (4.1) for \( 0 < \lambda < \lambda_0 \).

**Case 2.** If \( f_{-\infty} = \infty \), there exists a constant \( \tilde{H} > 0 \) such that \( f(u) \geq \eta u \) for \( u \geq \tilde{H} \), where the constant \( \eta > 0 \) satisfies

\[ \lambda \ln \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds > 1. \quad (4.44) \]

Letting \( r_\ell = \max \{2r_1, \tilde{H}(1 - c^2)/(\alpha - |c|)\} \), we have \( f((A^{-1} y)(t - \tau(t))) \geq \eta(A^{-1} y)(t - \tau(t)) \) for \( y \in K_r \). By Lemma 2.2, we have \((A^{-1} y)(t - \tau(t)) \geq ((\alpha - |c|)/(1 - c^2))\|y\| \geq \tilde{H} \) for \( y \in \partial K_r \). Thus by Lemma 4.5 and (4.44), we have for \( y \in \partial K_r \) that

\[ \|Qy\| \geq \lambda \eta \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds\|y\| > \|y\|. \quad (4.45) \]

It follows from Lemma 4.9 and (4.40) that

\[ i(Q, K_r, K) = 0, \quad i(Q, K_r, K) = 1. \quad (4.46) \]

that is, \( i(Q, K_r \setminus \overline{K}_r, K) = -1 \) and \( Q \) has a fixed point \( y \) in \( K_r \setminus \overline{K}_r \). That means \((A^{-1} y)(t)\) is a positive \( \omega \)-periodic solution of (4.1) for \( 0 < \lambda < \lambda_0 \).

**Case 3.** If \( f_{-\infty} = f_{\infty} = \infty \), from the above arguments, \( Q \) has a fixed point \( y_1 \) in \( K_r \setminus \overline{K}_r \) and a fixed point \( y_2 \) in \( K_r \setminus \overline{K}_r \). Consequently, \((A^{-1} y_1)(t)\) and \((A^{-1} y_2)(t)\) are two positive \( \omega \)-periodic solutions of (4.1) for \( 0 < \lambda < \lambda_0 \).

(c) By Lemma 2.2, if \( y \in K \), then \((A^{-1} y)(t - \tau(t)) \geq ((\alpha - |c|)/(1 - c^2))\|y\| > 0 \) for \( t \in [0, \omega] \).

**Case 1.** If \( f_0 = 0 \), we have \( f_{-\infty} > 0 \) and \( f_{\infty} > 0 \). Let \( b_1 = \min\{f(u)/u; u > 0\} > 0 \), then we obtain

\[ f(u) \geq b_1 u, \quad u \in [0, +\infty). \quad (4.47) \]

Assume \( y(t) \) is a positive \( \omega \)-periodic solution of (4.1) for \( \lambda > \lambda_0 \), where \( \lambda_0 = (1 - c^2)/lb_1 (\alpha - |c|) \int_0^\omega b(s)ds > 0 \). Since \( Qy(t) = y(t) \) for \( t \in [0, \omega] \), then by Lemma 4.5, if \( \lambda > \lambda_0 \) we have

\[ \|y\| = \|Qy\| \geq \lambda lb_1 \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds\|y\| > \|y\|, \quad (4.48) \]

which is a contradiction.
Case 2. If \( \bar{t}_\infty = 0 \), we have \( \bar{f}_0 < \infty \) and \( \bar{f}_\infty < \infty \). Let \( b_2 = \max\{|f(u)/u : u > 0\} > 0 \), then we obtain

\[
f(u) \leq b_2 u, \quad u \in [0, \infty).
\]

(4.49)

Assume \( y(t) \) is a positive \( \omega \)-periodic solution of (4.1) for \( 0 < \lambda < \lambda_0 \), where \( \lambda_0 = (m - (M + m)|c|)/b_2 LM \int_0^\omega b(s)ds \). Since \( Qy(t) = y(t) \) for \( t \in [0, \omega] \), it follows from Lemma 4.6 that

\[
\|y\| = \|Qy\| \leq \lambda b_2 \frac{LM \int_0^\omega b(s)ds}{m - (M + m)|c|} \|y\| < \|y\|,
\]

which is a contradiction. \( \square \)

**Theorem 4.11.** (a) If there exists a constant \( b_1 > 0 \) such that \( f(u) \geq b_1 u \) for \( u \in [0, +\infty) \), then (4.1) has no positive \( \omega \)-periodic solution for \( \lambda > (1 - c^2)/lb_1(\alpha - |c|) \int_0^\omega b(s)ds \).

(b) If there exists a constant \( b_2 > 0 \) such that \( f(u) \leq b_2 u \) for \( u \in [0, +\infty) \), then (4.1) has no positive \( \omega \)-periodic solution for \( 0 < \lambda < (m - (M + m)|c|)/b_2 LM \int_0^\omega b(s)ds \).

**Proof.** From the proof of (c) in Theorem 4.10, we obtain this theorem immediately. \( \square \)

**Theorem 4.12.** Assume \( \bar{t}_0 = \bar{t}_0 = \bar{t}_\infty = 0 \) and that one of the following conditions holds:

1. \( \bar{f}_0 \leq f_\infty \);
2. \( \bar{f}_0 > f_\infty \);
3. \( f_\infty \leq f_0 \leq f_\infty \);
4. \( f_\infty \leq f_0 \leq \bar{f}_0 \).

If

\[
\frac{1 - c^2}{l(\alpha - |c|) \int_0^\omega b(s)ds \max\{f_0, \bar{f}_0, f_\infty, \bar{f}_\infty\}} < \lambda < \frac{m - (M + m)|c|}{LM \int_0^\omega b(s)ds \min\{f_0, \bar{f}_0, f_\infty, \bar{f}_\infty\}},
\]

then (4.1) has one positive \( \omega \)-periodic solution.

**Proof.** We have the following cases.

**Case 1.** If \( \bar{f}_0 \leq f_\infty \), then

\[
\frac{1 - c^2}{\bar{f}_\infty l(\alpha - |c|) \int_0^\omega b(s)ds} < \lambda < \frac{m - (M + m)|c|}{\bar{f}_\infty LM \int_0^\omega b(s)ds}.
\]

(4.51)
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It is easy to see that there exists an $0 < \varepsilon < f_\infty$ such that

$$\frac{1 - c^2}{(\tilde{f}_\infty - \varepsilon)l(\alpha - |c|) \int_0^\omega b(s)ds} < \lambda < \frac{m - (M + m)|c|}{(f_0 + \varepsilon)LM \int_0^\omega b(s)ds}. \quad (4.53)$$

For the above $\varepsilon$, we choose $\tilde{r}_1 > 0$ such that $f(u) \leq (f_0 + \varepsilon)u$ for $0 \leq u \leq \tilde{r}_1$. Letting $r_1 = (1 - |c|)\tilde{r}_1$, we have $f((A^{-1}y)(t - \tau(t))) \leq (f_0 + \varepsilon)(A^{-1}y)(t - \tau(t))$ for $y \in K_r$. By Lemma 2.2, we have $0 \leq (A^{-1}y)(t - \tau(t)) \leq \|\cdot\|/(1 - |c|) \leq \tilde{r}_1$ for $K \in \partial K_r$. Thus by Lemma 4.6 we have for $y \in \partial K_r$ that

$$\|Qy\| \leq \lambda(f_0 + \varepsilon) \frac{LM \int_0^\omega b(s)ds}{m - (M + m)|c|} \|y\| < \|y\|. \quad (4.54)$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \geq (\tilde{f}_\infty - \varepsilon)u$ for $u \geq \tilde{H}$. Letting $r_2 = \max\{2r_1, \tilde{H}(1 - c^2)/(|\alpha| - |c|)\}$, we have $f((A^{-1}y)(t - \tau(t))) \geq (\tilde{f}_\infty - \varepsilon)(A^{-1}y)(t - \tau(t))$ for $y \in K_r$. By Lemma 2.2, we have $(A^{-1}y)(t - \tau(t)) \geq ((|\alpha| - |c|)/(1 - c^2))\|y\| \geq \tilde{H}$ for $y \in \partial K_r$. Thus by Lemma 4.5, for $y \in \partial K_r$

$$\|Qy\| \geq \lambda(f_\infty - \varepsilon) \frac{|\alpha| - |c|}{1 - c^2} \int_0^\omega b(s)ds \|y\| > \|y\|. \quad (4.55)$$

It follows from Lemma 4.9 that

$$i(Q, K_r, K) = 1, \quad i(Q, K_r, K) = 0, \quad (4.56)$$

thus $i(Q, K_r \setminus \bar{K}_r, K) = -1$ and $Q$ has a fixed point $y$ in $K_r \setminus \bar{K}_r$. So $(A^{-1}y)(t)$ is a positive $\omega$-periodic solution of (4.1).

Case 2. If $f_0 > \tilde{f}_\infty$, in this case, we have

$$\frac{1 - c^2}{\int_0^\infty l(\alpha - |c|) \int_0^\omega b(s)ds} < \lambda < \frac{m - (M + m)|c|}{f_0 LM \int_0^\omega b(s)ds}. \quad (4.57)$$

It is easy to see that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{1 - c^2}{(\tilde{f}_0 - \varepsilon)l(\alpha - |c|) \int_0^\omega b(s)ds} < \lambda < \frac{m - (M + m)|c|}{(f_0 + \varepsilon)LM \int_0^\omega b(s)ds}. \quad (4.58)$$

For the above $\varepsilon$, we choose $\tilde{r}_1 > 0$ such that $f(u) \geq (\tilde{f}_0 - \varepsilon)u$ for $0 \leq u \leq \tilde{r}_1$. Letting $r_1 = (1 - |c|)\tilde{r}_1$, we have $f((A^{-1}y)(t - \tau(t))) \geq (\tilde{f}_0 - \varepsilon)(A^{-1}y)(t - \tau(t))$ for $y \in K_r$. By Lemma 2.2,
we have \(0 \leq (A^{-1}y)(t - \tau(t)) \leq \|y\|/(1 - |c|) \leq \tau_1\) for \(y \in \partial K_{r_1}\). Thus we have by Lemma 4.5 that for \(y \in \partial K_{r_1}\)

\[
\|Qy\| \geq \lambda \left( f_0 - \varepsilon \right) \left| \alpha - |c| \right| (1 - c^2) \int_0^\alpha b(s)ds > \|y\|.
\]

(4.59)

On the other hand, there exists a constant \(\tilde{H} > 0\) such that \(f(u) \leq (f_\infty + \varepsilon)u\) for \(u \geq \tilde{H}\). Letting \(r_2 = \max\{2r_1, \tilde{H}(1 - c^2)/(\alpha - |c|)\}\), we have \(f((A^{-1}y)(t - \tau(t))) \leq (f_\infty + \varepsilon)(A^{-1}y)(t - \tau(t))\) for \(y \in K_{r_2}\). By Lemma 2.2 we have \((A^{-1}y)(t - \tau(t)) \geq ((\alpha - |c|)/(1 - c^2))\|y\| \geq \tilde{H}\) for \(y \in \partial K_{r_2}\). Thus by Lemma 4.6, for \(y \in \partial K_{r_2}\)

\[
\|Qy\| \leq \lambda \left( f_\infty + \varepsilon \right) \frac{LM \int_0^\alpha b(s)ds}{m - (M + m)|c|} \|y\|.
\]

(4.60)

It follows from Lemma 4.9 that

\[
i(Q, K_{r_1}, K) = 0, \quad i(Q, K_{r_2}, K) = 1.
\]

(4.61)

Thus \(i(Q, K_{r_1} \setminus \overline{K}_{r_2}, K) = -1\) and \(Q\) has a fixed point \(y \in K_{r_1} \setminus \overline{K}_{r_2}\), proving that \((A^{-1}y)(t)\) is a positive \(\omega\)-periodic solution of (4.1).

Case 3. One has \(f_0 \leq f_\infty \leq f_0 \leq f_\infty\). The proof is the same as in Case 1.

Case 4. One has \(f_\infty \leq f_0 \leq f_\infty \leq f_0\). The proof is the same as in Case 2. \(\square\)

4.2. Case II

Assume \(c > 0\) and \(c < \min\{m/(M + m), (LM - lm)/(L - l)M - lm\}\).

Define

\[
f_2(r) = \min\left\{ f(t) : \frac{\alpha}{1 - c} r \leq t \leq \frac{r}{1 - c} \right\}.
\]

(4.62)

Similarly as in Section 4.1, we get the following results.

Theorem 4.13. (a) If \(\tilde{t}_0 = 1\) or 2, then (4.1) has \(i_0\) positive \(\omega\)-periodic solutions for \(\lambda > 1/f_2(1) \int_0^\alpha b(s)ds \geq 0\).

(b) If \(\tilde{t}_\infty = 1\) or 2, then (4.1) has \(i_\infty\) positive \(\omega\)-periodic solutions for \(0 < \lambda < (m - (M + m)c)/LM(1 - c)F(1) \int_0^\alpha b(s)ds\).

(c) If \(\tilde{t}_\infty = 0\) or \(\tilde{t}_0 = 0\), then (4.1) has no positive \(\omega\)-periodic solution for sufficiently small or large \(\lambda > 0\), respectively.

Theorem 4.14. (a) If there exists a constant \(b_1 > 0\) such that \(f(u) \geq b_1u\) for \(u \in [0, +\infty)\), then (4.1) has no positive \(\omega\)-periodic solution for \(\lambda > (1 - c)/\lambda b_1 \int_0^\alpha b(s)ds\).
(b) If there exists a constant $b_2 > 0$ such that $f(u) \leq b_2u$ for $u \in [0, +\infty)$, then (4.1) has no positive $\omega$-periodic solution for $0 < \lambda < (m - (M + m)c)/b_2\int_0^\omega b(s)ds$.

**Theorem 4.15.** Assume $\mathcal{I}_0 = \mathcal{I}_\infty = \mathcal{I}_1 = 0$ hold and that one of the following conditions holds:

1. $\overline{f}_0 \leq f_\infty$;
2. $f_0 > \overline{f}_\infty$;
3. $f_0 \leq f_\infty \leq \overline{f}_0 \leq \overline{f}_\infty$;
4. $f_\infty \leq f_\infty \leq \overline{f}_\infty \leq \overline{f}_0$.

If

$$\frac{1 - c}{\lambda a \int_0^\omega b(s)ds \max\{f_\infty, \overline{f}_\infty, \overline{f}_\infty\}} < \lambda < \frac{m - (M + m)c}{LM \int_0^\omega b(s)ds \min\{f_\infty, \overline{f}_\infty, \overline{f}_\infty\}},$$

then (4.1) has one positive $\omega$-periodic solution.

**Remark 4.16.** In a similar way, one can consider the second-order neutral functional differential equation $(x(t) - cx(t - \delta(t)))'' - a(t)x(t) = -\lambda b(t)f(x(t - \tau(t)))$.

### 5. Examples

**Example 5.1.** Consider the following equation:

$$\left( x(t) - 15\left( t - \frac{1}{60}\sin 4t \right) \right)'' = x'(t) \sin 4t + \arctan \left( \frac{x(t - \sin 4t)}{1 + \cos^3(4t)} \right) + \cos 4t. \tag{5.1}$$

Comparing (5.1) to (3.1), we have $\omega = \pi/2$, $f(t, x) = x(t) \sin 4t$, $g(t, x) = \arctan(x/(1 + \cos^3(4t)))$, $c = 15$, $\delta(t) = (1/60)\sin 4t$, $\tau(t) = \sin 4t$, $e(t) = \cos 4t$ and $\delta_1 = \max_{t \in [0, \omega]}(1/15)\cos 4t| = 1/15$, and we can easily choose $D > \pi/2$ and $M = \pi/2$ such that $(H_2)$ and $(H_3)$ holds. Regarding assumption $(H_1)$ note that

$$|f(t, x'(t))| \leq |x'(t)|, \tag{5.2}$$

that is, $(H_1)$ holds with $K_1 = 1, b = 0$, and

$$\frac{\omega^{1/2}(1 + |c|)^{1/2}\sqrt{2K_1}}{|1 - |c|| - |c|\delta_1|} = \sqrt{\pi/2}(1 + 15)^{1/2}\sqrt{2} \approx 4\sqrt{\pi}/13 < 1. \tag{5.3}$$

Hence by Theorem 3.2, (5.1) has at least one $\pi/2$-periodic solution.
Example 5.2. Consider the following neutral functional differential equation:

\[
\left( u(t) + \frac{7}{30} u(t - \sin t) \right)'' + \frac{1}{16} u(t) = \lambda (1 - \sin t) u^2(t - \tau(t)) a^{\mu(t - \tau(t))},
\]

(5.4)

where \( \lambda \) and \( 0 < a < 1 \) are two positive parameters, \( \tau(t + 2\pi) = \tau(t) \).

Comparing (5.4) to (4.1), we see that \( \delta(t) = \sin t, c = -7/30, a(t) = 1/16, b(t) = 1 - \sin t, \omega = 2\pi, f(u) = u^2 a^u \). Clearly, \( M = 1/16 < (\pi/2\pi)^2 = 1/4, f_0 = 0, f_\infty = 0, I_0 = 2. \)

By Theorem 4.10, we easily get the following conclusion: (5.4) has two positive \( \omega \)-periodic solutions for \( \lambda > 1/4\pi r_1 \), where \( r_1 = \min \{ f(0.27), f(30/23) \} \).

In fact, by simple computations, we have

\[
M = m = \frac{1}{16}, \quad \beta = \frac{1}{4}, \quad L = \frac{1}{2\beta \sin(\beta 2\pi/2)} = 2\sqrt{2}, \quad l = \frac{\cos(\beta 2\pi/2)}{(2\beta \sin(\beta 2\pi/2))} = 2, \quad k = \frac{2 + \sqrt{2}}{8}, \quad k_1 = \frac{\sqrt{2} + 1 - \sqrt{3}}{2}, \quad \alpha = \frac{8}{23} \sqrt{2},
\]

\[
|c| = \frac{7}{30} < \min \left\{ k_1, \frac{m}{M + m} \right\} = \frac{\sqrt{2} + 1 - \sqrt{3}}{2}, \quad |c| = \frac{7}{30} < \frac{8}{23} \sqrt{2} = \alpha,
\]

\[
f_1(1) = \min \left\{ f(t) : 0.27 \approx \frac{(8/23)\sqrt{2} - (7/30)}{1 - (7/30)^2} \leq t \leq \frac{30}{23} \right\} = \min \left\{ f(0.27), f\left( \frac{30}{23} \right) \right\} = r_1,
\]

\[
\frac{1}{f_1(1)} \int_0^{r_1} b(s) ds = \frac{1}{4\pi r_1}.
\]

(5.5)

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References

