Research Article

A Note on the Generalized $q$-Bernoulli Measures with Weight $\alpha$

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We discuss a new concept of the $q$-extension of Bernoulli measure. From those measures, we derive some interesting properties on the generalized $q$-Bernoulli numbers with weight $\alpha$ attached to $\chi$.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$ (see [1–14]).

When we talk of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, and we use the notation of $q$-number as

$$[x]_q = \frac{1 - q^x}{1 - q},$$

(see [1–14]). Thus, we note that $\lim_{q \to 1} [x]_q = x$.

In [2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \quad (q\xi + 1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

(1.2)

with the usual convention of replacing $\xi_k$ by $\xi_k$. 

These numbers are $q$-extension of ordinary Bernoulli numbers $B_k$. But they do not remain finite when $q = 1$. So he modified (1.2) as follows:

$$
\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}
$$

with the usual convention of replacing $\beta^k$ by $\beta_{k,q}$.

The numbers $\beta_{k,q}$ are called the $k$-th Carlitz $q$-Bernoulli numbers.

In [1], Carlitz also considered the extended Carlitz’s $q$-Bernoulli numbers as follows:

$$
\beta_{h,0} = h, \quad q(q\beta_{h} + 1)^k - \beta_{h,k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}
$$

with the usual convention of replacing $(\beta_{h})^k$ by $\beta_{h,k,q}$.

Recently, Kim considered $q$-Bernoulli numbers, which are different extended Carlitz’s $q$-Bernoulli numbers, as follows: for $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$
\tilde{\beta}_{\alpha,0} = \frac{h}{[n]_q}, \quad q(q^{\alpha}\tilde{\beta}_{\alpha} + 1)^n - \tilde{\beta}_{\alpha,n,q} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}
$$

with the usual convention of replacing $(\tilde{\beta}_{\alpha})^k$ by $\tilde{\beta}_{\alpha,k,q}$ (see [3]).

The numbers $\tilde{\beta}_{\alpha,k,q}$ are called the $k$-th $q$-Bernoulli numbers with weight $\alpha$.

For fixed $d \in \mathbb{Z}_+$ with $(p, d) = 1$, we set

$$
X = X_d = \lim_{N \to \infty} \left( \mathbb{Z} \over dp^N\mathbb{Z} \right), \quad X_1 = \mathbb{Z}_p, \\
X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
$$

with the usual convention of replacing $(\tilde{\beta}_{\alpha})^k$ by $\tilde{\beta}_{\alpha,k,q}$.

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{p^N-1} f(x)q^x,
$$

(1.7)
(see [3, 4, 15, 16]). By (1.5) and (1.7), the Witt’s formula for the $q$-Bernoulli numbers with weight $\alpha$ is given by

$$
\int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \tilde{\beta}_{n,q}^{(\alpha)} \quad \text{where } n \in \mathbb{Z}_+.
$$

(1.8)

The $q$-Bernoulli polynomials with weight $\alpha$ are also defined by

$$
\tilde{\beta}_{n,q}^{(\alpha)}(x) = \sum_{i=0}^n \binom{n}{i} [x]_q^{n-i} q^{i\alpha} \tilde{\beta}_{i,q}^{(\alpha)}.
$$

(1.9)

From (1.7), (1.8), and (1.9), we can derive the Witt’s formula for $\tilde{\beta}_{n,q}^{(\alpha)}(x)$ as follows:

$$
\int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \tilde{\beta}_{n,q}^{(\alpha)}(x) \quad \text{where } n \in \mathbb{Z}_+.
$$

(1.10)

For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, the distribution relation for the $q$-Bernoulli polynomials with weight $\alpha$ are known that

$$
\tilde{\beta}_{n,q}^{(\alpha)}(x) = \left[\frac{d}{q} \right]_q^{n-1} \sum_{a=0}^{d-1} q^{a\alpha} \tilde{\beta}_{n,q}^{(\alpha)} \left(\frac{x + a}{d}\right),
$$

(1.11)

(see [3]). Recently, several authors have studied the $p$-adic $q$-Euler and Bernoulli measures on $\mathbb{Z}_p$ (see [8, 9, 11, 13, 14]). The purpose of this paper is to construct $p$-adic $q$-Bernoulli distribution with weight $\alpha = p$-adic $q$-Bernoulli unbounded measure with weight $\alpha$ on $\mathbb{Z}_p$ and to study their integral representations. Finally, we construct the generalized $q$-Bernoulli numbers with weight $\alpha$ and investigate their properties related to $p$-adic $q$-L-functions.

2. $p$-Adic $q$-Bernoulli Distribution with Weight $\alpha$

Let $X$ be any compact-open subset of $\mathbb{Q}_p$, such as $\mathbb{Z}_p$ or $\mathbb{Z}_p^*$. A $p$-adic distribution $\mu$ on $X$ is defined to be an additive map from the collection of compact open set in $X$ to $\mathbb{Q}_p$:

$$
\mu\left(\bigcup_{k=1}^n U_k\right) = \sum_{k=1}^n \mu(U_k) \quad \text{(additivity)},
$$

(2.1)

where $\{U_1, U_2, \ldots, U_n\}$ is any collection of disjoint compact openset sets in $X$.

The set $\mathbb{Z}_p$ has a topological basis of compact open sets of the form $a + p^n\mathbb{Z}_p$.

Consequently, if $U$ is any compact open subset of $\mathbb{Z}_p$, it can be written as a finite disjoint union of sets

$$
U = \bigcup_{j=1}^k \{a_j + p^n\mathbb{Z}_p\},
$$

(2.2)

where $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ with $0 \leq a_i < p^n$ for $i = 1, 2, \ldots, k$. 

Indeed, the $p$-adic ball $a + p^n\mathbb{Z}_p$ can be represented as the union of smaller balls

$$a + p^n\mathbb{Z}_p = \bigcup_{b=0}^{p-1} \left( a + bp^n + p^{n+1}\mathbb{Z}_p \right). \quad (2.3)$$

**Lemma 2.1.** Every map $\mu$ from the collection of compact-open sets in $X$ to $\mathbb{Q}_p$ for which

$$\mu\left( a + p^N\mathbb{Z}_p \right) = \bigcup_{b=0}^{p-1} \left( a + bp^N + dp^{N+1}\mathbb{Z}_p \right) \quad (2.4)$$

holds whenever $a + p^N\mathbb{Z}_p \subset X$, extends to a $p$-adic distribution (= $p$-adic unbounded measure) on $X$.

Now we define a map $\mu^{(a)}_{k,q}$ on the balls in $\mathbb{Z}_p$ as follows:

$$\mu^{(a)}_{k,q}(a + p^n\mathbb{Z}_p) = \left[ \left[ \frac{p^n}{p^n} \right] \right]_q - q^a f^{(a)}_{k,q^{p^n}} \left( \frac{[a]}{p^n} \right), \quad (2.5)$$

where $[a]_n$ is the unique number in the set $\{0, 1, 2, \ldots, p^n - 1\}$ such that $[a]_n \equiv a \pmod{p^n}$.

If $a \in \{0, 1, 2, \ldots, p^n - 1\}$, then

$$\sum_{b=0}^{p-1} \mu^{(a)}_{k,q}(a + bp^n + p^{n+1}\mathbb{Z}_p) = \sum_{b=0}^{p-1} \left[ \left[ \frac{p^n+1}{p^n+1} \right] \right]_q - q^a f^{(a)}_{k,q^{p^n+1}} \left( \frac{a + bp^n}{p^{n+1}} \right)$$

$$= q^a \left[ \left[ \frac{p^n}{p^n} \right] \right]_q \sum_{b=0}^{p-1} \left[ \left[ \frac{p^n}{p^n} \right] \right]_q - q^a f^{(a)}_{k,q^{p^n}} \left( \frac{[a/p^n] + b}{p} \right). \quad (2.6)$$

From (2.6), we note that $\mu^{(a)}_{k,q}$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if

$$\left[ \frac{p^n}{p^n} \right]_q \sum_{b=0}^{p-1} q^b f^{(a)}_{k,q^{p^n}} \left( \frac{[a/p^n] + b}{p} \right) = f^{(a)}_{k,q^{p^n}} \left( \frac{[a]}{p^n} \right). \quad (2.7)$$

**Theorem 2.2.** Let $\alpha \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Then we see that $\mu^{(a)}_{k,q}(a + p^n\mathbb{Z}_p)$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if

$$\left[ \frac{p^n}{p^n} \right]_q \sum_{b=0}^{p-1} q^b f^{(a)}_{k,q^{p^n}} \left( \frac{[a/p^n] + b}{p} \right) = f^{(a)}_{k,q^{p^n}} \left( \frac{[a]}{p^n} \right). \quad (2.8)$$

One sets

$$f^{(a)}_{k,q^n}(x) = \tilde{\beta}^{(a)}_{k,q^n}(x). \quad (2.9)$$
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From (2.5) and (2.9), one gets

\[
\mu^{(a)}_{k,q}(a + p^n z_p) = \frac{\lfloor p^n \rfloor^k}{\lfloor p^n \rfloor_q} q^a \tilde{\beta}^{(a)}_{k,q} \left( \frac{a}{p^n} \right). \tag{2.10}
\]

By (1.11), (2.10), and Theorem 2.2, we obtain the following theorem.

**Theorem 2.3.** Let \( \mu^{(a)}_{k,q} \) be given by

\[
\mu^{(a)}_{k,q}(a + dp^N z_p) = \frac{\lfloor dp^N \rfloor^k}{\lfloor dp^N \rfloor_q} q^a \tilde{\beta}^{(a)}_{k,q} \left( \frac{a}{dp^N} \right). \tag{2.11}
\]

Then \( \mu^{(a)}_{k,q} \) extends to a \( \mathbb{Q}(q) \)-valued distribution on the compact open sets \( U \subset X \).

From (2.11), one notes that

\[
\int_X d\mu^{(a)}_{k,q}(x) = \lim_{N \to \infty} \sum_{a=0}^{dp^N-1} \mu^{(a)}_{k,q}(x + dp^N z_p)
\]

\[
= \lim_{N \to \infty} \frac{\lfloor dp^N \rfloor^k}{\lfloor dp^N \rfloor_q} \sum_{a=0}^{dp^N-1} q^a \tilde{\beta}^{(a)}_{k,q} \left( \frac{a}{dp^N} \right). \tag{2.12}
\]

By (1.11) and (2.12), one gets

\[
\int_X d\mu^{(a)}_{k,q}(x) = \tilde{\beta}^{(a)}_{k,q}. \tag{2.13}
\]

Therefore, we obtain the following theorem.

**Theorem 2.4.** For \( \alpha \in \mathbb{N} \) and \( k \in \mathbb{Z}_+ \), one has

\[
\int_X d\mu^{(a)}_{k,q}(x) = \tilde{\beta}^{(a)}_{k,q}. \tag{2.14}
\]

Let \( \chi \) be Dirichlet character with conductor \( d \in \mathbb{N} \). Then one defines the generalized \( q \)-Bernoulli numbers attached to \( \chi \) as follows:

\[
\tilde{\beta}^{(a)}_{n,\chi,q} = \int_X \chi(x) [x]^n_d d\mu_q(x)
\]

\[
= \frac{[d]_q^n d^{-1}}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \tilde{\beta}^{(a)}_{n,q} \left( \frac{a}{d} \right). \tag{2.15}
\]
From (2.11) and (2.15), one can derive the following equation:

\[
\int_{X} \chi(x) \, d\mu_{k,q}^{(a)}(x) = \lim_{N \to \infty} \sum_{x=0}^{dpN} \chi(x) \mu_{k,q}^{(a)} \left( x + dp^N \nu \right)
\]

\[
= \lim_{N \to \infty} \left[ \frac{dpN}{[dpN]} \right]_{q}^{k} \chi(x) q^{\beta} \tilde{\mu}_{k,q}^{(a)} \left( \frac{x}{dpN} \right)
\]

\[
= \left[ \frac{dpN}{[dpN]} \right]_{q}^{k} \chi(x) \sum_{x=0}^{pN-1} q^{\alpha} \tilde{\mu}_{k,q}^{(a)} \left( \frac{a}{d} \right)
\]

\[
\int_{pX} \chi(x) \, d\mu_{k,q}^{(a)}(x) = \lim_{N \to \infty} \left[ \frac{dpN}{[dpN]} \right]_{q}^{k} \sum_{x=0}^{dpN} \chi(px) q^{\alpha} \tilde{\mu}_{k,q}^{(a)} \left( \frac{px}{dpN} \right)
\]

\[
= \frac{[p]}{[p]} q^{\alpha} \sum_{a=0}^{dpN-1} \chi(p) q^{\alpha} \tilde{\mu}_{k,q}^{(a)} \left( \frac{a}{d} \right)
\]

For \( \beta(\neq 1) \in X^{*} \), one has

\[
\int_{pX} \chi(x) \, d\mu_{k,q}^{(a)}(x) = \chi(p) \frac{[p]}{[p]} q^{\alpha} \tilde{\mu}_{k,q}^{(a)} \left( \frac{a}{d} \right)
\]

(2.16)

Therefore, we obtain the following theorem.

**Theorem 2.5.** For \( \beta(\neq 1) \in X^{*} \), one has

\[
\int_{X} \chi(x) \, d\mu_{k,q}^{(a)}(x) = \tilde{\mu}_{k,q}^{(a)}
\]

\[
\int_{pX} \chi(x) \, d\mu_{k,q}^{(a)}(x) = \chi(p) \frac{[p]}{[p]} q^{\alpha} \tilde{\mu}_{k,q}^{(a)}
\]

(2.17)
Define

\[ \mu_{k,\beta,q}^{(a)}(U) = \mu_{s,q}^{(a)}(U) - \beta^{-1} \left[ \left( \frac{1}{[\beta^{-1}]^q} \right)^k \right] \mu_{k,\beta,q}^{(a)}(\beta U). \]  

(2.19)

By a simple calculation, one gets

\[ \int_{pX} \chi(x) d\mu_{k,\beta,q}^{(a)}(\beta x) = \int_{pX} \chi(x) d\mu_{s,q}^{(a)}(x) - \beta^{-1} \left[ \left( \frac{1}{[\beta^{-1}]^q} \right)^k \right] \mu_{k,\beta,q}^{(a)}(\beta x) \]

\[ = \tilde{\rho}_{k,\beta,q}^{(a)} - \chi(p) \left[ [1/\beta]^q \right]^k \tilde{\rho}_{k,\beta,q}^{(a)} \]

(2.20)

By (2.19) and (2.20), one gets

\[ \int_{pX} \chi(x) d\mu_{k,\beta,q}^{(a)}(\beta x) = \int_{pX} \chi(x) d\mu_{s,q}^{(a)}(x) - \beta^{-1} \left[ \left( \frac{1}{[\beta^{-1}]^q} \right)^k \right] \mu_{k,\beta,q}^{(a)}(\beta x) \]

\[ = \tilde{\rho}_{k,\beta,q}^{(a)} - \chi(p) \left[ [1/\beta]^q \right]^k \tilde{\rho}_{k,\beta,q}^{(a)} - 1 \left[ [1/\beta]^q \right]^k \chi(1/\beta) \tilde{\rho}_{k,\beta,q}^{(a)} \]

\[ + \chi(p) \left[ [1/\beta]^q \right]^k \tilde{\rho}_{k,\beta,q}^{(a)}. \]

(2.21)

Now one defines the operator \( \chi^y = \chi^{y,k,a,q} \) on \( f(q) \) by

\[ \chi^y f(q) = \chi^{y,k,a,q} f(q) = \left[ \left[ y \right]^q \right]^k \chi(y) f(q^y). \]

(2.22)
Thus, by (2.22), one gets

$$\chi^{x,k,\alpha} \circ \chi^{y,k,\alpha} f(q) = \chi^{x,k,\alpha}[y]_q^k \chi(y) f(q)$$

$$= \frac{[y]_q^k}{[y]_q^k} \chi(y) \chi(x) \frac{[y]_q^k}{[y]_q^k} \chi(y) f(q)$$

$$= \frac{[xy]_q^k}{[xy]_q^k} \chi(xy) f(q)$$

$$= \chi^{xy,k,\alpha} f(q)$$

$$= \chi^{xy} f(q).$$

(2.23)

Let us define $\chi^x \chi^y = \chi^{x,k,\alpha} \circ \chi^{y,k,\alpha}$. Then one has

$$\chi^x \chi^y = \chi^{xy}.$$  

(2.24)

From the definition of $\chi^x$, one can easily derive the following equation;

$$\left(1 - \chi^x\right) \left(1 - \frac{1}{\beta} \chi^{1/\beta}\right) = 1 - \frac{1}{\beta} \chi^{1/\beta} - \chi^p + \frac{1}{\beta} \chi^{p/\beta}. $$

(2.25)

Let $f(q) = \tilde{\beta}^{(a)}_{k,\chi,q}$. Then one gets

$$\left(1 - \chi^x\right) \left(1 - \frac{1}{\beta} \chi^{1/\beta}\right) \tilde{\beta}^{(a)}_{k,\chi,q} = \frac{1}{\beta} \chi^{1/\beta} \tilde{\beta}^{(a)}_{k,\chi,q} - \frac{1}{\beta} \chi^{1/\beta} \tilde{\beta}^{(a)}_{k,\chi,q} - \frac{1}{\beta} \chi^{p/\beta} \tilde{\beta}^{(a)}_{k,\chi,q}$$

$$+ \frac{1}{\beta} \chi^{p/\beta} \tilde{\beta}^{(a)}_{k,\chi,q}.$$ 

(2.26)

By (2.21) and (2.26), one obtains the following equation:

$$\int_{X^*} \chi(x) \mu^{(a)}_{k,\beta,q} (\beta x) = \left(1 - \chi^x\right) \left(1 - \frac{1}{\beta} \chi^{1/\beta}\right) \tilde{\beta}^{(a)}_{k,\chi,q}$$

(2.27)

where $\beta(\neq 1) \in X^*$.

References


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