Research Article

Multiple Positive Solutions for Semilinear Elliptic Equations with Sign-Changing Weight Functions in $\mathbb{R}^N$

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Existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$-\Delta u + u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

$u > 0$ in $\mathbb{R}^N$,

$u \in H^1(\mathbb{R}^N)$.

where $\lambda > 0$, $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$), $a$, $b$ satisfy suitable conditions, and $b$ maybe changes sign in $\mathbb{R}^N$. The study is based on the extraction of the Palais-Smale sequences in the Nehari manifold.

1. Introduction

In this paper, we deal with the multiplicity of positive solutions for the following semilinear elliptic equation:

$$-\Delta u + u = a(x)|u|^{p-1} + \lambda b(x)|u|^{q-1} \quad \text{in} \quad \mathbb{R}^N,$$

$u > 0$ in $\mathbb{R}^N$,

$u \in H^1(\mathbb{R}^N)$.

where $\lambda > 0$, $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$) and $a$, $b$ are measurable functions and satisfy the following conditions:

$$(a1) \ 0 < a \in L^\infty(\mathbb{R}^N)$, where $\lim_{|x| \to \infty} a(x) = 1$, and there exist $C_0 > 0$ and $\delta_0 > 0$ such that

$$a(x) \geq 1 - C_0 e^{-\delta_0|x|} \quad \forall x \in \mathbb{R}^N. \quad (1.1)$$
(b1) \( b \in L^q(R^N) \) (\( q^+ = p/(p - q) \)), \( b^+ = \max \{b, 0\} \neq 0 \) \( b^- = \max \{-b, 0\} \) is bounded and \( b^- \) has a compact support \( K \) in \( R^N \).

(b2) There exist \( C_1 > 0, 0 < \delta_1 < \min \{\delta_0, q\} \) and \( R_0 > 0 \) such that

\[
b^+(x) - b(x) \geq C_1 e^{-\delta_1|x|} \quad \forall |x| \geq R_0,
\]

Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [1] considered the following equation:

\[
-\Delta u = u^{p-1} + \lambda u^{\alpha-1} \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\]

where \( \lambda > 0, 1 < q < 2 < p < 2^* \). They proved that there exists \( \lambda_0 > 0 \) such that \( (E_{\lambda}) \) admits at least two positive solutions for all \( \lambda \in (0, \lambda_0) \), has one positive solution for \( \lambda = \lambda_0 \) and no positive solution for \( \lambda > \lambda_0 \). Actually, Adimurthi et al. [2], Damascelli et al. [3], Korman [4], Ouyang and Shi [5], and Tang [6] proved that there exists \( \lambda_0 > 0 \) such that \( (E_{\lambda}) \) in the unit ball \( B^N(0; 1) \) has exactly two positive solutions for \( \lambda \in (0, \lambda_0) \), has exactly one positive solution for \( \lambda = \lambda_0 \) and no positive solution exists for \( \lambda > \lambda_0 \). For more general results of \( (E_{\lambda}) \) (involving sign-changing weights) in bounded domains, see, the work of Ambrosetti et al. in [7], of Garcia Azorero et al. in [8], of Brown and Wu in [9], of Brown and Zhang in [10], of Cao and Zhong in [11], of de Figueiredo et al. in [12], and their references.

However, little has been done for this type of problem in \( R^N \). We are only aware of the works [13–17] which studied the existence of solutions for some related concave-convex elliptic problems (not involving sign-changing weights). Furthermore, we do not know of any results for concave-convex elliptic problems involving sign-changing weight functions except [18, 19]. Wu in [18] have studied the multiplicity of positive solutions for the following equation involving sign-changing weights:

\[
-\Delta u + u = f_\lambda (x) u^{\alpha-1} + g_\mu (x) u^{\mu-1} \quad \text{in } R^N, \\
u > 0 \quad \text{in } R^N, \\
u \in H^1(R^N),
\]

where \( 1 < q < 2 < p < 2^* \) the parameters \( \lambda, \mu \geq 0 \). He also assumed that \( f_\lambda (x) = \lambda f_\cdot (x) + f_\cdot (x) \) is sign changing and \( g_\mu (x) = a(x) + \mu b(x) \), where \( a \) and \( b \) satisfy suitable conditions and proved that \( (E_{f_\lambda, g_\mu}) \) has at least four positive solutions.

In a recent work [19], Hsu and Lin have studied \( (E_{a, b}) \) in \( R^N \) with a sign-changing weight function. They proved there exists \( \lambda_0 > 0 \) such that \( (E_{a, b}) \) has at least two positive solutions for all \( \lambda \in (0, \lambda_0) \) provided that \( a, b \) satisfy suitable conditions and \( b \) maybe changes sign in \( R^N \).

Continuing our previous work [19], we consider \( (E_{a, b}) \) in \( R^N \) involving a sign-changing weight function with suitable assumptions which are different from the assumptions in [19].
In order to describe our main result, we need to define

\[ \Lambda_0 = \left( \frac{2-q}{(p-q\|a\|_{L^p})} \right)^{(2-q)/(p-2)} \left( \frac{p-2}{(p-q\|b^*\|_{L^p}^*)} \right)^{p(2-q)/(2(p-2)+q/2)} > 0, \]  

(1.3)

where \( \|a\|_{L^p} = \sup_{x \in \mathbb{R}^N} a(x), \|b^*\|_{L^{p^*}} = \left( \int_{\mathbb{R}^N} |b^*(x)|^q \, dx \right)^{1/q} \) and \( S_p \) is the best Sobolev constant for the imbedding of \( H^1(\mathbb{R}^N) \) into \( L^p(\mathbb{R}^N) \).

**Theorem 1.1.** Assume that (a1), (b1)-(b2) hold. If \( \lambda \in (0, (q/2)\Lambda_0) \), \((E_{a,\lambda b})\) admits at least two positive solutions in \( H^1(\mathbb{R}^N) \).

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we establish the existence of a local minimum. In Section 4, we prove the existence of a second solution of \((E_{a,\lambda b})\).

At the end of this section, we explain some notations employed. In the following discussions, we will consider \( H = H^1(\mathbb{R}^N) \) with the norm \( \|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx)^{1/2} \).

We denote by \( S_p \) the best constant which is given by

\[ S_p = \inf_{u \in H^1(\mathbb{R}^N)} \frac{\|u\|^2}{\left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{2/p}}. \]  

(1.4)

The dual space of \( H \) will be denoted by \( H^* \). \( \langle \cdot, \cdot \rangle \) denote the dual pair between \( H^* \) and \( H \). We denote the norm in \( L^{s}(\mathbb{R}^N) \) by \( \| \cdot \|_{L^{s}} \) for \( 1 \leq s \leq \infty \). \( B^N(x; r) \) is a ball in \( \mathbb{R}^N \) centered at \( x \) with radius \( r \). \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \). \( C, C_i \) will denote various positive constants, the exact values of which are not important.

### 2. Preliminary Results

Associated with (1.3), the energy functional \( J_\lambda : H \to \mathbb{R}^N \) defined by

\[ J_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} a(x)|u|^p \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} b(x)|u|^q \, dx, \]  

(2.1)

for all \( u \in H \) is considered. It is well-known that \( J_\lambda \in C^1(H, \mathbb{R}) \) and the solutions of \((E_{a,\lambda b})\) are the critical points of \( J_\lambda \).

Since \( J_\lambda \) is not bounded from below on \( H \), we will work on the Nehari manifold. For \( \lambda > 0 \) we define

\[ \mathcal{M}_\lambda = \{ u \in H \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0 \}. \]  

(2.2)

Note that \( \mathcal{M}_\lambda \) contains all nonzero solutions of \((E_{a,\lambda b})\) and \( u \in \mathcal{M}_\lambda \) if and only if

\[ \langle J'_\lambda(u), u \rangle = \|u\|^2 - \int_{\mathbb{R}^N} a(x)|u|^p \, dx - \lambda \int_{\mathbb{R}^N} b(x)|u|^q \, dx = 0. \]  

(2.3)

**Lemma 2.1.** \( J_\lambda \) is coercive and bounded from below on \( \mathcal{M}_\lambda \).
**Proof.** If \( u \in \mathcal{A}_1 \), then by (b1), (2.3), and the Hölder and Sobolev inequalities, one has

\[
J_1(u) = \frac{p - 2}{2p} \| u \|^2 - \frac{p - q}{pq} \int_{\mathbb{R}^N} b(x)|u|^q dx
\]

\[
\geq \frac{p - 2}{2p} \| u \|^2 - \lambda \left( \frac{p - q}{pq} \right) \int_{\mathbb{R}^N} b^\ast \| u \|^q.
\]

Since \( q < 2 < p \), it follows that \( J_1 \) is coercive and bounded from below on \( \mathcal{A}_1 \).

The Nehari manifold is closely linked to the behavior of the function of the form \( \varphi_u : t \rightarrow J_1(tu) \) for \( t > 0 \). Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [20] and are also discussed by Brown and Zhang in [10]. If \( u \in H \), we have

\[
\varphi_u(t) = \frac{t^2}{2} \| u \|^2 - \frac{tp}{p} \int_{\mathbb{R}^N} a(x)|u|^p dx - \frac{t^q}{q} \int_{\mathbb{R}^N} b(x)|u|^q dx,
\]

\[
\varphi_u'(t) = t \| u \|^2 - t^{p-1} \int_{\mathbb{R}^N} a(x)|u|^p dx - t^{q-1} \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx,
\]

\[
\varphi_u''(t) = \| u \|^2 - (p-1) t^{p-2} \int_{\mathbb{R}^N} a(x)|u|^p dx - (q-1) t^{q-2} \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx.
\]

It is easy to see that

\[
t \varphi_u'(t) = \| tu \|^2 - \int_{\mathbb{R}^N} a(x)|tu|^p dx - \lambda \int_{\mathbb{R}^N} b(x)|tu|^q dx,
\]

and so, for \( u \in H \setminus \{0\} \) and \( t > 0 \), \( \varphi_u'(t) = 0 \) if and only if \( tu \in \mathcal{A}_1 \) that is, the critical points of \( \varphi_u \) correspond to the points on the Nehari manifold. In particular, \( \varphi_u'(1) = 0 \) if and only if \( u \in \mathcal{A}_1 \). Thus, it is natural to split \( \mathcal{A}_1 \) into three parts corresponding to local minima, local maxima, and points of inflection. Accordingly, we define

\[
\mathcal{A}_1^+ = \{ u \in \mathcal{A}_1 : \varphi_u''(1) > 0 \},
\]

\[
\mathcal{A}_1^0 = \{ u \in \mathcal{A}_1 : \varphi_u''(1) = 0 \},
\]

\[
\mathcal{A}_1^- = \{ u \in \mathcal{A}_1 : \varphi_u''(1) < 0 \},
\]

and note that if \( u \in \mathcal{A}_1 \), that is, \( \varphi_u'(1) = 0 \), then

\[
\varphi_u''(1) = (2-q) \| u \|^2 - (p-q) \int_{\mathbb{R}^N} a(x)|u|^p dx,
\]

\[
= (2-p) \| u \|^2 - (q-p) \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx.
\]

We now derive some basic properties of \( \mathcal{A}_1^+, \mathcal{A}_1^0 \) and \( \mathcal{A}_1^- \).
Lemma 2.2. Suppose that $u_0$ is a local minimizer for $J_\lambda$ on $\mathcal{M}_\lambda$ and $u_0 \notin \mathcal{M}_\lambda^0$, then $J_\lambda'(u_0) = 0$ in $H^*$. 

Proof. See the work of Brown and Zhang in [10, Theorem 2.3].

Lemma 2.3. If $\lambda \in (0, \Lambda_0)$, then $\mathcal{M}_\lambda^0 = \emptyset$.

Proof. We argue by contradiction. Suppose that there exists $\lambda \in (0, \Lambda_0)$ such that $\mathcal{M}_\lambda^0 \neq \emptyset$. Then for $u \in \mathcal{M}_\lambda^0$ by (2.9) and the Sobolev inequality, we have

$$2 - \frac{q}{p - q} \|u\|^2 = \int_{\mathbb{R}^N} a(x)|u|^p dx \leq \|a\|_{L^q} S_p^{p/2}\|u\|^p,$$

(2.11)

and so

$$\|u\| \geq \left( \frac{2 - \frac{q}{p - q}}{(p - q)\|a\|_{L^q}} \right)^{1/(p-2)} S_p^{p/2(p-2)}. \tag{2.12}$$

Similarly, using (2.10), Hölder and Sobolev inequalities, we have

$$\|u\|^2 = \lambda \frac{p - q}{p - 2} \int_{\mathbb{R}^N} b(x)|u|^q dx \leq \lambda \frac{p - q}{p - 2} \|b^+\|_{L^q} S_p^{q/2}\|u\|^q,$$

(2.13)

which implies

$$\|u\| \leq \left( \lambda \frac{p - q}{p - 2} \|b^+\|_{L^q} \right)^{1/(2-q)} S_p^{q/2(2-q)}. \tag{2.14}$$

Hence, we must have

$$\lambda \geq \left( \frac{2 - \frac{q}{p - q}}{(p - q)\|a\|_{L^q}} \right)^{(2-q)/(p-2)} \left( \frac{p - 2}{(p - q)\|b^+\|_{L^q}} \right) S_p^{p(2-q)/2(p-2) + q/2} = \Lambda_0 \tag{2.15}$$

which is a contradiction. \hfill \Box

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $q_u : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

$$q_u(t) = t^{2-q}\|u\|^2 - t^{p-q} \int_{\mathbb{R}^N} a(x)|u|^p dx \quad \text{for } t > 0. \tag{2.16}$$

Clearly, $tu \in \mathcal{M}_\lambda$ if and only if $q_u(t) = \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx$. Moreover,

$$q'_u(t) = (2 - q)t^{1-q}\|u\|^2 - (p - q)t^{p-q-1} \int_{\mathbb{R}^N} a(x)|u|^p dx \quad \text{for } t > 0, \tag{2.17}$$
and so it is easy to see that if \( tu \in \mathcal{M}_\lambda \), then \( t^{q-1} q_u'(t) = q_u''(t) \). Hence, \( tu \in \mathcal{M}_\lambda^+ \) (or \( tu \in \mathcal{M}_\lambda^- \)) if and only if \( q_u'(t) > 0 \) (or \( q_u'(t) < 0 \)).

Let \( u \in H \setminus \{0\} \). Then, by (2.17), \( q_u \) has a unique critical point at \( t = t_{\text{max}}(u) \), where

\[
t_{\text{max}}(u) = \left( \frac{(2 - q) \|u\|^2}{(p - q) \int_{\mathbb{R}^N} a(x) |u|^p \, dx} \right)^{1/(p-2)} > 0,
\]

and clearly \( q_u \) is strictly increasing on \((0, t_{\text{max}}(u))\) and strictly decreasing on \((t_{\text{max}}(u), \infty)\) with \( \lim_{t \to \infty} q_u(t) = -\infty \). Moreover, if \( \lambda \in (0, \Lambda_0) \), then

\[
q_u(t_{\text{max}}(u)) = \left[ \left( \frac{2 - q}{p - q} \right)^{(2-q)/(p-2)} - \left( \frac{2 - q}{p - q} \right)^{(p-q)/(p-2)} \right] \frac{\|u\|^p}{\int_{\mathbb{R}^N} a(x) |u|^p \, dx} \left( \frac{\|u\|^2}{\int_{\mathbb{R}^N} a(x) |u|^p \, dx} \right)^{(2-q)/(p-2)} \int_{\mathbb{R}^N} b^+(x) |u|^q \, dx
\]

\[
\geq \lambda \|b^+\|_{L^q} \|u\|^q
\]

\[
\geq \lambda \int_{\mathbb{R}^N} b^+(x) |u|^q \, dx
\]

Therefore, we have the following lemma.

**Lemma 2.4.** Let \( \lambda \in (0, \Lambda_0) \) and \( u \in H \setminus \{0\} \).

(i) If \( \lambda \int_{\mathbb{R}^N} b(x) |u|^q \, dx \leq 0 \), then there exists a unique \( t^- = t^-(u) > t_{\text{max}}(u) \) such that \( t^- u \in \mathcal{M}_\lambda^- \), \( q_u \) is increasing on \((0, t^-)\) and decreasing on \((t^-, \infty)\). Moreover,

\[
J_1(t^- u) = \sup_{t \geq 0} J_1(t u).
\]

(ii) If \( \lambda \int_{\mathbb{R}^N} b(x) |u|^q \, dx > 0 \), then there exist unique \( 0 < t^+ = t^+(u) < t_{\text{max}}(u) < t^- = t^-(u) \) such that \( t^+ u \in \mathcal{M}_\lambda^+ \), \( t^- u \in \mathcal{M}_\lambda^- \), \( q_u \) is decreasing on \((0, t^+)\), increasing on \((t^+, t^-)\) and decreasing on \((t^-, \infty)\)

\[
J_1(t^+ u) = \inf_{0 \leq s \leq t_{\text{max}}(u)} J_1(s u), \quad J_1(t^- u) = \sup_{t \leq t^-} J_1(t u).
\]

(iii) \( \mathcal{M}_\lambda^- = \{ u \in H \setminus \{0\} : t^-(u) = 1/\|u\| t^-(u/\|u\|) = 1 \} \).

(iv) There exists a continuous bijection between \( \mathcal{U} = \{ u \in H \setminus \{0\} : \|u\| = 1 \} \) and \( \mathcal{M}_\lambda^- \). In particular, \( t^- \) is a continuous function for \( u \in H \setminus \{0\} \).

**Proof.** See the work of Hsu and Lin in [19, Lemma 2.5].
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We remark that it follows Lemma 2.4, \( \mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^- \) for all \( \lambda \in (0, \Lambda_0) \). Furthermore, by Lemma 2.4 it follows that \( \mathcal{N}_\lambda^+ \) and \( \mathcal{N}_\lambda^- \) are non-empty and by Lemma 2.1 we may define

\[
\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \quad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \tag{2.22}
\]

**Theorem 2.5.** (i) If \( \lambda \in (0, \Lambda_0) \), then we have \( \alpha_\lambda \leq \alpha_\lambda^+ < 0 \).
(ii) If \( \lambda \in (0, (q/2)\Lambda_0) \), then \( \alpha_\lambda^+ > d_0 \) for some \( d_0 > 0 \).

In particular, for each \( \lambda \in (0, (q/2)\Lambda_0) \), we have \( \alpha_\lambda^+ = \alpha_\lambda < 0 < \alpha_\lambda^- \).

**Proof.** See the work of Hsu and Lin in [19, Theorem 3.1]. \( \square \)

**Remark 2.6.** (i) If \( \lambda \in (0, \Lambda_0) \), then by (2.9), Hölder and Sobolev inequalities, for each \( u \in \mathcal{N}_\lambda^+ \) we have

\[
\|u\|^2 \leq \frac{p-q}{p-2} \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx \leq \frac{p-q}{p-2} \lambda \|b\|_{L^p} S^{-q/2}_p \|u\|^q \leq \frac{p-q}{p-2} \Lambda_0 \|b\|_{L^p} S^{-q/2}_p \|u\|^q,
\]

and so

\[
\|u\| \leq \left( \frac{p-q}{p-2} \Lambda_0 \|b\|_{L^p} S^{-q/2}_p \right)^{1/(2-q)} \quad \forall u \in \mathcal{N}_\lambda^+. \tag{2.24}
\]

(ii) If \( \lambda \in (0, (q/2)\Lambda_0) \), then by Lemma 2.4(i), (ii) and Theorem 2.5(ii), for each \( u \in \mathcal{N}_\lambda^- \) we have

\[
J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \geq \alpha_\lambda^- > 0. \tag{2.25}
\]

3. Existence of a Positive Solution

First, we define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in \( H \) for \( J_\lambda \) as follows.

**Definition 3.1.** (i) For \( c \in \mathbb{R} \), a sequence \( \{u_n\} \) is a (PS)\(_c\)-sequence in \( H \) for \( J_\lambda \) if \( J_\lambda(u_n) = c + o_n(1) \) and \( J_\lambda'(u_n) = o_n(1) \) strongly in \( H^* \) as \( n \to \infty \).
(ii) \( c \in \mathbb{R} \) is a (PS)-value in \( H \) for \( J_\lambda \) if there exists a (PS)\(_c\)-sequence in \( H \) for \( J_\lambda \).
(iii) \( J_\lambda \) satisfies the (PS)\(_c\)-condition in \( H \) if any (PS)\(_c\)-sequence \( \{u_n\} \) in \( H \) for \( J_\lambda \) contains a convergent subsequence.

Now we will ensure that there are (PS)\(_{\alpha_\lambda^-}\)-sequence and (PS)\(_{\alpha_\lambda^+}\)-sequence for \( \mathcal{N}_\lambda \) and \( \mathcal{N}_\lambda^- \), respectively, for the functional \( J_\lambda \).
Proposition 3.2. If \( \lambda \in (0, (q/2)\Lambda_0) \), then

(i) there exists a (PS)\(_{\alpha_1}\) sequence \( \{u_n\} \subset \mathcal{N}_\lambda \) in \( H \) for \( f_\lambda \).

(ii) there exists a (PS)\(_{\alpha_1}\) sequence \( \{u_n\} \subset \mathcal{N}_\lambda^* \) in \( H \) for \( f_\lambda \).

Proof. See Wu [21, Proposition 9]. \( \square \)

Now, we establish the existence of a local minimum for \( f_\lambda \) on \( \mathcal{N}_\lambda^* \).

Theorem 3.3. Assume (a1) and (b1) hold. If \( \lambda \in (0, (q/2)\Lambda_0) \), then there exists \( u_\lambda \in \mathcal{N}_\lambda^* \) such that

(i) \( f_\lambda(u_\lambda) = \alpha_\lambda = \alpha_1^* < 0 \),

(ii) \( u_\lambda \) is a positive solution of \((E_{a,\lambda b})\),

(iii) \( \|u_\lambda\| \to 0 \) as \( \lambda \to 0^+ \).

Proof. From Proposition 3.2(i) it follows that there exists \( \{u_n\} \subset \mathcal{N}_\lambda \) satisfying

\[
J_\lambda(u_n) = \alpha_\lambda + o_n(1) = \alpha_1^* + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } H^*.
\]  

(3.1)

By Lemma 2.1 we infer that \( \{u_n\} \) is bounded on \( H \). Passing to a subsequence (Still denoted by \( \{u_n\} \)), there exists \( u_\lambda \in H \) such that as \( n \to \infty \)

\[
\begin{align*}
u_n & \rightharpoonup u_\lambda \quad \text{weakly in } H, \\
u_n & \to u_\lambda \quad \text{almost everywhere in } \mathbb{R}^N, \\
u_n & \to u_\lambda \quad \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N) \quad \forall 1 \leq s < 2^*.
\end{align*}
\]  

(3.2)

By (b1), Egorov theorem and Hölder inequality, we have

\[
\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx + o_n(1) \quad \text{as } n \to \infty.
\]  

(3.3)

By (3.1) and (3.2), it is easy to see that \( u_\lambda \) is a solution of \((E_{a,\lambda b})\). From \( u_n \in \mathcal{N}_\lambda \) and (2.4), we deduce that

\[
\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \frac{q(p-2)}{2(p-q)}\|u_n\|^2 - \frac{pq}{p-q} J_\lambda(u_n).
\]  

(3.4)

Let \( n \to \infty \) in (3.4). By (3.1), (3.3) and \( \alpha_\lambda < 0 \), we get

\[
\lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx \geq - \frac{pq}{p-q} \alpha_\lambda > 0.
\]  

(3.5)

Thus, \( u_\lambda \in \mathcal{N}_\lambda \) is a nonzero solution of \((E_{a,\lambda b})\).
Next, we prove that \( u_n \to u_1 \) strongly in \( H \) and \( J_1(u_1) = \alpha_1 \). From the fact \( u_n, u_1 \in \mathcal{A}_\lambda \) and applying Fatou’s lemma, we get

\[
\alpha_1 \leq J_1(u_1) = \frac{p-2}{2p} \|u_1\|^2 - \frac{p-q}{pq} \lambda \int_{\mathbb{R}^N} b(x)|u_1|^q dx
\]

\[
\leq \liminf_{n \to \infty} \left( \frac{p-2}{2p} \|u_n\|^2 - \frac{p-q}{pq} \lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx \right)
\]

\[
\leq \liminf_{n \to \infty} J_1(u_n) = \alpha_1.
\]

This implies that \( J_1(u_1) = \alpha_1 \) and \( \lim_{n \to \infty} \|u_n\|^2 = \|u_1\|^2 \). Standard argument shows that \( u_n \to u_1 \) strongly in \( H \). By Theorem 2.5, for all \( \lambda \in (0, (q/2)\Lambda_0) \) we have that \( u_1 \in \mathcal{A}_\lambda \) and \( J_1(u_1) = \alpha_1^+ < \alpha_1 \) which implies \( u_1 \in \mathcal{A}_\lambda^+ \). Since \( J_1(u_1) = J_1(|u_1|) \) and \( |u_1| \in \mathcal{A}_\lambda^+ \), by Lemma 2.2 we may assume that \( u_1 \) is a nonzero nonnegative solution of \( (E_{a,\lambda,b}) \). By Harnack inequality [22] we deduce that \( u_1 > 0 \) in \( \mathbb{R}^N \). Finally, by (2.10), Hölder and Sobolev inequalities,

\[
\|u_1\|^{2-q} < \lambda \frac{p-q}{p-2} \|b^+\|_{L^p} S^{-q/2}_p,
\]

and thus we conclude the proof. \( \square \)

## 4. Second Positive Solution

In this section, we will establish the existence of the second positive solution of \( (E_{a,\lambda,b}) \) by proving that \( J_1 \) satisfies the (PS)\( _{\alpha_1^+} \)-condition.

**Lemma 4.1.** Assume that (a1) and (b1) hold. If \( \{u_n\} \subset H \) is a \( (PS)_c \)-sequence for \( J_1 \), then \( \{u_n\} \) is bounded in \( H \).

**Proof.** See the work of Hsu and Lin in [19, Lemma 4.1]. \( \square \)

Let us introduce the problem at infinity associated with \( (E_{a,\lambda,b}) \):

\[-\Delta u + u = u^{p-1} \quad \text{in} \quad \mathbb{R}^N, \quad u \in H, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N. \quad (E^\infty)\]

We state some known results for problem \( (E^\infty) \). First of all, we recall that by Lions [23] has studied the following minimization problem closely related to problem \( (E^\infty) \):

\[
S^\infty = \inf \{ J^\infty(u) : u \in H, \ u \neq 0, \ (J^\infty)'(u) = 0 \} > 0,
\]

where \( J^\infty(u) = (1/2)\|u\|^2 - (1/p) \int_{\mathbb{R}^N} |u|^p dx \). Note that a minimum exists and is attained by a ground state \( w_0 > 0 \) in \( \mathbb{R}^N \) such that

\[
S^\infty = J^\infty(w_0) = \sup_{t \geq 0} J^\infty(tw_0) = \left( \frac{1}{2} - \frac{1}{p} \right) S_p^{1/(p-2)},
\]

where \( S_p = \int_{\mathbb{R}^N} |x|^{p-2} x \cdot x dx \). Theorem 4.2.1. Assume that (a1) and (b1) hold. If \( \{u_n\} \subset H \) is a \( (PS)_c \)-sequence for \( J_1 \), then \( \{u_n\} \) is bounded in \( H \).

**Proof.** See the work of Hsu and Lin in [19, Lemma 4.1]. \( \square \)

Let us introduce the problem at infinity associated with \( (E_{a,\lambda,b}) \):

\[-\Delta u + u = u^{p-1} \quad \text{in} \quad \mathbb{R}^N, \quad u \in H, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N. \quad (E^\infty)\]

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S^\infty = J^\infty(w_0) = \sup_{t \geq 0} J^\infty(tw_0) = \left( \frac{1}{2} - \frac{1}{p} \right) S_p^{1/(p-2)},
\]

where \( S_p = \int_{\mathbb{R}^N} |x|^{p-2} x \cdot x dx \).
where $S_p = \inf_{u \in H \setminus \{0\}} \|u\|^2/(\int_{\mathbb{R}^N} |u|^pdx)^{2/p}$. Gidas et al. [24] showed that for every $\varepsilon > 0$, there exist positive constants $C_\varepsilon, C_2$ such that for all $x \in \mathbb{R}^N$,

$$C_\varepsilon \exp(- (1 + \varepsilon)|x|) \leq w_0(x) \leq C_2 \exp(-|x|).$$  \hfill (4.3)

We define

$$w_n(x) = w_0(x - ne), \quad \text{where } e = (0, 0, \ldots, 0, 1) \text{ is a unit vector in } \mathbb{R}^N.$$  \hfill (4.4)

Clearly, $w_n(x) \in H$.

**Lemma 4.2.** Let $\Omega$ be a domain in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ satisfies

$$\int_{\Omega} |f(x)e^{\sigma|x|}| \, dx < \infty \quad \text{for some } \sigma > 0,$$

then

$$\left(\int_{\Omega} f(x)e^{-\sigma|x-\bar{x}|} \, dx\right)e^{\sigma|x|} = \int_{\Omega} f(x)e^{\sigma(x,\bar{x})/|\bar{x}|} \, dx + o(1) \quad \text{as } |\bar{x}| \to \infty.$$  \hfill (4.6)

**Proof.** We know $\sigma|x| \leq \sigma|x| + \sigma|x - \bar{x}|$. Then,

$$\left| f(x)e^{-\sigma|x-\bar{x}|}e^{\sigma|x|} \right| \leq \left| f(x)e^{\sigma|x|} \right|.$$  \hfill (4.7)

Since $-\sigma|x - \bar{x}| + \sigma|x| = \sigma(x,\bar{x})/|\bar{x}| + o(1)$ as $|\bar{x}| \to \infty$, then the lemma follows from the Lebesgue dominated convergence theorem. \hfill $\Box$

**Lemma 4.3.** Under the assumptions (a1), (b1)-(b2) and $\lambda \in (0, \Lambda_0)$. Then there exists a number $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\sup_{t \geq 0} J_1(tw_n) < S^\infty.$$  \hfill (4.8)

In particular, $\alpha < S^\infty$ for all $\lambda \in (0, \Lambda_0)$.

**Proof.** (i) First, since $\|w_n\| = \|w_0\|$ for all $n \in \mathbb{N}$ and $J_1$ is continuous in $H$ and $J_1(0) = 0$, we infer that there exists $t_1 > 0$ such that

$$J_1(tw_n) < S^\infty \quad \forall n \in \mathbb{N}, \ t \in [0, t_1].$$  \hfill (4.9)
(ii) Since \( \lim_{|x| \to \infty} a(x) = 1 \), there exists \( n_1 \in \mathbb{N} \) such that if \( n \geq n_1 \), we get \( a(x) \geq 1/2 \) for \( x \in B^N(ne; 1) \). Then, for \( n \geq n_1 \)

\[
J_1(tw_n) = \frac{t^2}{2} \|w_n\|^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} a(x)|w_n|^p dx - \frac{t^q}{q} \int_{\mathbb{R}^N} \lambda b(x)|w_n|^q dx
\]

\[
\leq \frac{t^2}{2} \|w_n\|^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} a(x + ne)|w_0|^p dx + \frac{t^q}{q} \lambda \|b^-\|_{L^\infty} \int_{\mathbb{R}^N} |w_n|^q dx
\]

\[
\leq \frac{t^2}{2} \|w_n\|^2 - \frac{t^p}{2p} \int_{\mathbb{R}^N} |w_0|^p dx + \frac{t^q}{q} \lambda \|b^-\|_{L^\infty} \int_{\mathbb{R}^N} |w_0|^q dx
\]

\[
\to -\infty \quad \text{as} \quad t \to \infty.
\]

Thus, there exists \( t_2 > 0 \) such that for any \( t > t_2 \) and \( n > n_1 \) we get

\[
J_1(tw_n) < 0.
\]

(iii) By (i) and (ii), we need to show that there exists \( n_0 \) such that for \( n \geq n_0 \)

\[
\sup_{t_1 \leq t \leq t_2} J_1(tw_n) < S^\infty.
\]

We know that \( \sup_{t \geq 0} J^\infty(tw_0) = S^\infty \). Then, \( t_1 \leq t \leq t_2 \), we have

\[
J_1(tw_n) = \frac{1}{2} \|tw_n\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} a(x)(tw_n)^p dx - \frac{1}{q} \int_{\mathbb{R}^N} \lambda b(x)(tw_n)^q dx
\]

\[
\leq \frac{t^2}{2} \|w_n\|^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} |w_n|^p dx + \frac{t^q}{q} \int_{\mathbb{R}^N} \lambda \|b^-\|_{L^\infty} \int_{\mathbb{R}^N} |w_n|^q dx
\]

\[
\leq S^\infty + \frac{t^p}{p} \int_{\mathbb{R}^N} (1 - a)^+(x)|w_n|^p dx - \frac{t^q}{q} \int_{\mathbb{R}^N} \lambda b^+(x)|w_n|^q dx + \frac{t^2}{2} \int_{\mathbb{R}^N} \lambda b^-(x)|w_n|^q dx.
\]

Suppose \( a \) satisfies (a1), we get \( (1 - a)^+(x) \leq C_0 e^{-\delta_0 |x|} \) for all \( x \in \mathbb{R}^N \) and some positive constant \( \delta_0 \). By (4.3) and Lemma 4.3, there exists \( n_2 > n_1 \) such that for any \( n \geq n_2 \)

\[
\int_{\mathbb{R}^N} (1 - a)^+(x)|w_n|^p dx \leq C_3 e^{-\min(\delta_0,p) n}.
\]

By (b1) and (4.3), we get

\[
\int_{\mathbb{R}^N} \lambda b^-(x)|w_n|^q dx \leq \lambda \|b^-\|_{L^\infty} C_2 \int_{\mathbb{R}^N} e^{-q|x-ne|} dx
\]

\[
\leq \lambda C_3 e^{-qn}.
\]
By (b2), (4.3) and Lemma 4.3, we have
\[
\int_{\mathbb{R}^N} \lambda b^*(x) w_n^q dx \geq \lambda C_1 C_2 \int_{|x| \geq R_0} e^{-\delta_1|x|} e^{-\gamma(1+c)|x-\eta|} dx
\]
\[
\geq \lambda C e^{-\delta_1 n}. \tag{4.16}
\]

Since \(0 < \delta_1 < \min\{\delta_0, q\} \leq \min\{\delta_0, p\}\) and \(\lambda \in (0, \Lambda_0)\) and using (4.13)–(4.16), we have there exists \(n_0 > n_2\) such that for all \(n \geq n_0\), then
\[
\sup_{t \in \mathbb{R}} J_1(tw_n) < S^\infty, \quad \lambda \int_{\mathbb{R}^N} b(x)|w_n|^q dx > 0. \tag{4.17}
\]

This implies that if \(\lambda \in (0, \Lambda_0)\), then for all \(n \geq n_0\) we get
\[
\sup_{t \geq 0} J_1(tw_n) < S^\infty. \tag{4.18}
\]

From \(a(x) > 0\) for all \(x \in \mathbb{R}^N\) and (4.17), we have
\[
\int_{\mathbb{R}^N} a(x)|w_n|^p dx > 0, \quad \int_{\mathbb{R}^N} b(x)|w_n|^q dx > 0. \tag{4.19}
\]

Combining this with Lemma 2.4(ii), from the definition of \(a_\lambda^-\) and \(\sup_{t \geq 0} J_1(tw_n) < S^\infty\), for all \(\lambda \in (0, \Lambda_0)\), we obtain that there exists \(t_0 > 0\) such that \(t_0 w_n \in \mathcal{M}_{a_\lambda^-}\) and
\[
a_\lambda^- \leq J_1(t_0 w_n) \leq \sup_{t \geq 0} J_1(tw_n) < S^\infty. \tag{4.20}
\]

**Lemma 4.4.** Assume that (a1) and (b1) hold. If \(\{u_n\} \subset H\) is a (PS)\(_c\)-sequence for \(J_1\) with \(c \in (0, S^\infty)\), then there exists a subsequence of \(\{u_n\}\) converging weakly to a nonzero solution of \((E_{a,b})\) in \(\mathbb{R}^N\).

**Proof.** Let \(\{u_n\} \subset H\) be a (PS)\(_c\)-sequence for \(J_1\) with \(c \in (0, S^\infty)\). We know from Lemma 4.1 that \(\{u_n\}\) is bounded in \(H\), and then there exist a subsequence of \(\{u_n\}\) (still denoted by \(\{u_n\}\)) and \(u_0 \in H\) such that
\[
u_n \rightharpoonup u_0 \quad \text{weakly in } H,
\]
\[
u_n \to u_0 \quad \text{almost everywhere in } \mathbb{R}^N, \tag{4.21}
\]
\[
\nu_n \to u_0 \quad \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N) \quad \forall 1 \leq s < 2^*.
\]

It is easy to see that \(J'_1(u_0) = 0\) and by (b1), Egorov theorem and Hölder inequality, we have
\[
\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_0|^q dx + o_n(1). \tag{4.22}
\]
Next we verify that \( u_0 \not\equiv 0 \). Arguing by contradiction, we assume \( u_0 \equiv 0 \). By \((a1)\), for any \( \varepsilon > 0 \), there exists \( R_0 > 0 \) such that \( |a(x) - 1| < \varepsilon \) for all \( x \in [B^N(0; R_0)]^C \). Since \( u_n \to 0 \) strongly in \( L^s_{\text{loc}}(\mathbb{R}^N) \) for \( 1 \leq s < 2^* \), \( \{u_n\} \) is a bounded sequence in \( H \), therefore \( \int_{\mathbb{R}^N} |a(x) - 1| |u_n|^p \leq C \int_{B^N(0; R_0)} |u_n|^p + \varepsilon C \). Setting \( n \to \infty \), then \( \varepsilon \to 0 \), we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dx. \tag{4.23}
\]

We set

\[
l = \lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dx. \tag{4.24}
\]

Since \( f_{\lambda}(u_n) = o_n(1) \) and \( \{u_n\} \) is bounded, then by \((4.22)\), we can deduce that

\[
0 = \lim_{n \to \infty} \langle f_{\lambda}(u_n), u_n \rangle = \lim_{n \to \infty} \left( \|u_n\|^2 - \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx \right) \tag{4.25}
\]

that is,

\[
\lim_{n \to \infty} \|u_n\|^2 = l. \tag{4.26}
\]

If \( l = 0 \), then we get \( c = \lim_{n \to \infty} f_{\lambda}(u_n) = 0 \), which contradicts to \( c > 0 \). Thus we conclude that \( l > 0 \). Furthermore, by the definition of \( S_p \), we obtain

\[
\|u_n\|^2 \geq S_p \left( \int_{\mathbb{R}^N} |u_n|^p \, dx \right)^{2/p}. \tag{4.27}
\]

Then, as \( n \to \infty \), we have

\[
l = \lim_{n \to \infty} \|u_n\|^2 \geq S_p l^{2/p}, \tag{4.28}
\]

which implies that

\[
l \geq S_p^{p/(p-2)}. \tag{4.29}
\]
Hence, from (4.2) and (4.22)–(4.29), we get

\[
c = \lim_{n \to \infty} J_1(u_n)
= \frac{1}{2} \lim_{n \to \infty} ||u_n||^2 - \frac{1}{p} \lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)||u_n||^p dx - \frac{1}{q} \lim_{n \to \infty} \int_{\mathbb{R}^N} b(x)||u_n||^q dx
= \left( \frac{1}{2} - \frac{1}{p} \right) l
\geq \frac{p-2}{2p} S_p^{(p-2)} = S^\infty.
\]

(4.30)

This is a contradiction to \(c < S^\infty\). Therefore, \(u_0\) is a nonzero solution of \((E_{a,ib})\).

Now, we establish the existence of a local minimum of \(J_1\) on \(\mathcal{N}_\lambda\).

**Theorem 4.5.** Assume that (a1) and (b1)-(b2) hold. If \(\lambda \in (0,(q/2)\Lambda_0)\), then there exists \(U_\lambda \in \mathcal{N}_\lambda\) such that

(i) \(J_1(U_\lambda) = \alpha_\lambda^\ast\),

(ii) \(U_\lambda\) is a positive solution of \((E_{a,ib})\).

**Proof.** If \(\lambda \in (0,(q/2)\Lambda_0)\), then by Theorem 2.5(ii), Proposition 3.2(ii) and Lemma 4.3(ii), there exists a \((PS)_{\alpha_\lambda}\) sequence \(\{u_n\} \subset \mathcal{N}_\lambda\) in \(H\) for \(J_1\) with \(\alpha_\lambda^\ast \in (0,S^\infty)\). From Lemma 4.4, there exist a subsequence still denoted by \(\{u_n\}\) and a nonzero solution \(U_\lambda \in H\) of \((E_{a,ib})\) such that \(u_n \rightharpoonup U_\lambda\) weakly in \(H\).

First, we prove that \(U_\lambda \in \mathcal{N}_\lambda\). On the contrary, if \(U_\lambda \notin \mathcal{N}_\lambda\), then by \(\mathcal{N}_\lambda\) is closed in \(H\), we have \(||U_\lambda||^2 < \liminf_{n \to \infty} ||u_n||^2\). From (2.9) and \(a(x) > 0\) for all \(x \in \mathbb{R}^N\), we get

\[
\int_{\mathbb{R}^N} b(x)||U_\lambda||^q dx > 0, \quad \int_{\mathbb{R}^N} a(x)||U_\lambda||^p dx > 0.
\]

(4.31)

By Lemma 2.4(ii), there exists a unique \(t_\lambda^\ast\) such that \(t_\lambda^\ast U_\lambda \in \mathcal{N}_\lambda\). If \(u \in \mathcal{N}_\lambda\), then it is easy to see that

\[
J_1(u) = \frac{p-2}{2p} ||u||^2 - \frac{p-q}{pq} \int_{\mathbb{R}^N} b(x)||u||^q dx.
\]

(4.32)

From (3.1), \(u_n \in \mathcal{N}_\lambda\) and (4.32), we can deduce that

\[
\alpha_\lambda^\ast \leq J_1(t_\lambda^\ast U_\lambda) < \lim_{n \to \infty} J_1(t_\lambda^\ast u_n) \leq \lim_{n \to \infty} J_1(u_n) = \alpha_\lambda^\ast
\]

(4.33)

which is a contradiction. Thus, \(U_\lambda \in \mathcal{N}_\lambda\).

Next, by the same argument as that in Theorem 3.3, we get that \(u_n \rightharpoonup U_\lambda\) strongly in \(H\) and \(J_1(U_\lambda) = \alpha_\lambda^\ast > 0\) for all \(\lambda \in (0,(q/2)\Lambda_0)\). Since \(J_1(U_\lambda) = J_1(||U_\lambda||)\) and \(||U_\lambda|| \in \mathcal{N}_\lambda\) by Lemma 2.2 we may assume that \(U_\lambda\) is a nonzero nonnegative solution of \((E_{a,ib})\). Finally, by the Harnack inequality [22] we deduce that \(U_\lambda > 0\) in \(\mathbb{R}^N\).
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Now, we complete the proof of Theorem 1.1. By Theorems 3.3, 4.5, we obtain \((E_{a,b})\) has two positive solutions \(u_1\) and \(U_1\) such that \(u_1 \in \mathcal{M}_+^1, U_1 \in \mathcal{M}_+^1\). Since \(\mathcal{M}_+^1 \cap \mathcal{M}_+^1 = \emptyset\), this implies that \(u_1\) and \(U_1\) are distinct. It completes the proof of Theorem 1.1.

References


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