Research Article

On Asymptotic Behaviour of Solutions to $n$-Dimensional Systems of Neutral Differential Equations

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This paper presents the properties and behaviour of solutions to a class of $n$-dimensional functional differential systems of neutral type. Sufficient conditions for solutions to be either oscillatory, or $\lim_{t \to \infty} y_i(t)/t = 0$, or $\lim_{t \to \infty} |y_i(t)/t| = \infty$, $i = 1, 2, \ldots, n$, are established. One example is given.

1. Introduction

The authors have investigated some properties of solutions to $n$-dimensional functional differential systems

$$[y_1(t) - a(t)y_1(g(t))]' = p_1(t)y_2(t),$$

$$y'_i(t) = p_i(t)y_{i+1}(t), \quad i = 2, 3, \ldots, n - 1, \quad (1.1)$$

$$y'_n(t) = \sigma p_n(t)f(y_1(h(t))), \quad t \geq t_0,$$

in [1]. We studied the properties of solutions presupposing that both functions $a(t)$ and $y_1(t)$ were bounded and there were presented theorems where sufficient conditions to every solution with the first component of the solution $y_1(t)$ to be either oscillatory, or $\lim_{t \to \infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$. 

The goal of this paper is to enquire about the behaviour of the solution to \( n \)-dimensional functional differential system of neutral type (1.1) under no restriction to \( a(t) \) and to the first component \( y_1(t) \) of solution \( y(t) \). Results are given in theorems where sufficient conditions are stated to every solution to have the next properties: a solution to be either oscillatory, or \( \lim_{t \to -\infty} y_i(t) = 0 \), or \( \lim_{t \to -\infty} |y_i(t)| = \infty, i = 1, 2, \ldots, n \).

The system (1.1) is investigated under the assumptions \( \sigma \in \{-1, 1\}, n \geq 3 \), and throughout this paper, the next conditions are considered:

(a) \( a : [t_0, \infty) \to (0, \infty) \) is a continuous function;

(b) \( g : [t_0, \infty) \to \mathbb{R} \) is a continuous and increasing function, \( \lim_{t \to -\infty} g(t) = \infty \);

(c) \( p_i : [t_0, \infty) \to [0, \infty), i = 1, 2, \ldots, n \), are continuous functions; \( p_n \) not identically equal to zero in any neighbourhood of infinity, \( \int_{t_0}^{\infty} p_j(t)\,dt = \infty, j = 1, 2, \ldots, n - 1 \);

(d) \( h : [t_0, \infty) \to \mathbb{R} \) is a continuous and increasing function, \( \lim_{t \to -\infty} h(t) = \infty \);

(e) \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function; moreover, for \( u \neq 0, uf(u) > 0 \) and \( |f(u)| \geq K|u| \) hold, where \( K \) is a positive constant.

For a function \( y_1(t) \),

\[
z_1(t) = y_1(t) - a(t)y_1(g(t))
\]

is defined, and for \( t_1 \geq t_0 \), we introduce

\[
\tilde{t}_1 = \min\{t_1, g(t_1), h(t_1)\}.
\]

A vector function \( y = (y_1, \ldots, y_n) \) is a solution to the system (1.1) if there is a \( t_1 \geq t_0 \) such that \( y \) is continuous on \( [\tilde{t}_1, \infty) \); functions \( z_1(t), y_i(t), i = 2, 3, \ldots, n \) are continuously differentiable on \( [t_1, \infty) \) and \( y \) satisfies (1.1) on \( [t_1, \infty) \).

\( W \) denotes the set of all solutions \( y = (y_1, \ldots, y_n) \) to the system (1.1) that exist on some interval \( [T_y, \infty) \subset [t_0, \infty) \) and satisfy the condition

\[
\sup \left\{ \sum_{i=1}^{n} |y_i(t)| : t \geq T \right\} > 0 \quad \text{for any} \ T \geq T_y.
\]

A solution \( y \in W \) is considered nonoscillatory if there exists a \( T_y \geq t_0 \) such that every component is different from zero for \( t \geq T_y \). Otherwise a solution \( y \in W \) is said to be oscillatory.

Properties of solutions to similar differential equations and systems like system (1.1) have been studied in [1–6] and in the references cited therein. Problems of existence of solutions to neutral differential systems were analysed, for example, in [7, 8].
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It will be useful to define two types of recursion formulae. Let \( i_k \in \{1, 2, \ldots, n\}, k = 1, 2, \ldots, n \), and \( t, u \in [t_0, \infty) \). One has

\[
I_0(u, t) \equiv 1,
\]

\[
I_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_t^u p_{i_1}(x) I_{k-1}(x; t; p_{i_2}, p_{i_3}, \ldots, p_{i_k}) \, dx,
\]

(1.5)

\[
J_0(u, t) \equiv 1,
\]

\[
J_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_t^u p_{i_1}(x) J_{k-1}(u, x; p_{i_2}, p_{i_3}, \ldots, p_{i_k}) \, dx.
\]

(1.6)

It is easy to prove that the following identities hold:

\[
I_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = J_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k})
\]

(1.7)

for \( k = 1, 2, \ldots, n \).

Functions \( g^{-1}(t) \), \( h^{-1}(t) \) denote the inverse functions to \( g(t), h(t) \).

2. Preliminaries

Lemma 2.1 (see [9, Lemma 1]). Let \( y \in W \) be a solution of (1.1) with \( y_1(t) \neq 0 \) on \([t_1, \infty), t_1 \geq t_0\). Then \( y \) is nonoscillatory and \( z_1(t), y_2(t), \ldots, y_n(t) \) are monotone on some ray \([T, \infty), T \geq t_1\).

Let \( y \in W \) be a non-oscillatory solution of (1.1). By (1.1) and (c), it follows that the function \( z_1(t) \) from (1.2) has to be eventually of constant sign, so that either

\[
y_1(t)z_1(t) > 0
\]

(2.1)

or

\[
y_1(t)z_1(t) < 0
\]

(2.2)

for sufficiently large \( t \).

We mention for the comfort of proofs a classification of non-oscillatory solutions of the system (1.1) which was introduced by the authors in [1].

Assume first that (2.1) holds.

By [9, Lemma 4], the statement in Lemma 2.2 follows.

Lemma 2.2. Let \( y = (y_1, y_2, \ldots, y_n) \in W \) be a non-oscillatory solution to (1.1) on \([t_1, \infty), t_1 \geq t_0\), and assume that (2.1) holds. Then there exists an integer \( l \in \{1, 2, \ldots, n\} \) such that \( \sigma \cdot (-1)^{n+l+1} = 1 \) or \( l = n \), and \( t_2 \geq t_1 \) such that for \( t \geq t_2 \)

\[
y_i(t)z_1(t) > 0, \quad i = 1, 2, \ldots, l,
\]

\[
(-1)^{i+l}y_i(t)z_1(t) > 0, \quad i = l + 1, \ldots, n.
\]

(2.3)
Denote by $N^+_1$ the set of non-oscillatory solutions to (1.1) satisfying (2.3). Now assume that (2.2) holds.

By the aid of Kiguradze’s lemma, it is easy to prove Lemma 2.3.

**Lemma 2.3.** Let $y = (y_1, y_2, \ldots, y_n) \in W$ be a non-oscillatory solution to (1.1) on $[t_1, \infty)$, and assume that (2.2) holds. Then there exists an integer $l \in \{1, 2, \ldots, n\}$ and $\sigma \cdot (-1)^{n_l} = 1$ or $l = n$, and $t_2 \geq t_1$ such that for $t \geq t_2$ either

$$y_1(t)z_1(t) < 0,$$

$$(-1)^{i_1}y_i(t)z_1(t) < 0, \quad i = 2, \ldots, n,$$

or

$$y_1(t)z_1(t) < 0,$$

$$y_i(t)z_1(t) > 0, \quad i = 2, 3, \ldots, l,$$

$$(-1)^{i_2}y_i(t)z_1(t) > 0, \quad i = l + 1, \ldots, n.$$

Denote by $N^-_1$ the set of nonoscillatory solutions to (1.1) satisfying (2.4), and by $N^+_1$ the set of non-oscillatory solutions to (1.1) satisfying (2.5). Denote by $N$ the set of all non-oscillatory solutions to (1.1). Obviously by Lemmas 2.2 and 2.3, we have the classification of non-oscillatory solutions to the system (1.1):

$n$ odd, $\sigma = 1$:

$$N = N^+_2 \cup N^+_4 \cup \cdots \cup N^+_{n-1} \cup N^+_n \cup N^-_1 \cup N^-_3 \cup \cdots \cup N^-_n,$$  \hspace{1cm} (2.6)

$n$ odd, $\sigma = -1$:

$$N = N^+_1 \cup N^+_3 \cup \cdots \cup N^+_n \cup N^-_2 \cup N^-_4 \cup \cdots \cup N^-_{n-1} \cup N^-_n,$$  \hspace{1cm} (2.7)

$n$ even, $\sigma = 1$:

$$N = N^+_1 \cup N^+_3 \cup \cdots \cup N^+_{n-1} \cup N^+_n \cup N^-_2 \cup N^-_4 \cup \cdots \cup N^-_{n-1} \cup N^-_n,$$  \hspace{1cm} (2.8)

$n$ even, $\sigma = -1$:

$$N = N^+_2 \cup N^+_4 \cup \cdots \cup N^+_n \cup N^-_1 \cup N^-_3 \cup \cdots \cup N^-_{n-1} \cup N^-_n.$$  \hspace{1cm} (2.9)

The next lemma can be proved similarly as Lemma 2 in [9].
Lemma 2.4. Let \( y = (y_1, y_2, \ldots, y_n) \in W \) be a non-oscillatory solution to (1.1) on \([t_1, \infty), t_1 \geq t_0,\) and let \( \lim_{t \to \infty} |z_1(t)| = L_1, \lim_{t \to \infty} |y_k(t)| = L_k, k = 2, \ldots, n. \) Then

\[
\begin{align*}
k \geq 2, & \quad L_k > 0 \implies L_i = \infty, \quad i = 1, \ldots, k - 1, \\
1 \leq k < n, & \quad L_k < \infty \implies L_i = 0, \quad i = k + 1, \ldots, n.
\end{align*}
\] (2.10)

Remark 2.5. If \( g(t) < t, \) and \( 0 < a(t) \leq \lambda^* < 1, \) \((\lambda^* \text{ is a constant}),\) then from [9], we have \( N_{\lambda^*} = \emptyset, \) \( k \in \{2, 3, \ldots, n\}.\)

Lemma 2.6 (see [10, Lemma 2.2]). In addition to conditions (a) and (b) suppose that

\[
1 \leq a(t), \quad t \geq t_0.
\] (2.11)

Let \( y_1(t) \) be a continuous non-oscillatory solution to the functional inequality

\[
y_1(t) \left[ y_1(t) - a(t)y_1(g(t)) \right] > 0
\] (2.12)

defined in a neighbourhood of infinity. Suppose that \( g(t) > t \) for \( t \geq t_0. \) Then \( y_1(t) \) is bounded. If, moreover,

\[
1 < \lambda^* \leq a(t), \quad t \geq t_0
\] (2.13)

for some positive constant \( \lambda^*, \) then \( \lim_{t \to \infty} y_1(t) = 0. \)

3. Main Results

Theorem 3.1. Suppose that

\[
0 < a(t) \leq \lambda^* < 1, \quad \text{for some constant } \lambda^*, \quad t \geq t_0,
\] (3.1)

\[
g(t) < h(t) < t \quad \text{for } t \geq t_0,
\] (3.2)

\[
\alpha : [t_0, \infty) \to \mathbb{R} \text{ is a continuous function, } \alpha(t) < t, \quad \lim_{t \to \infty} \alpha(t) = \infty,
\] (3.3)

\[
\int_{x_1}^{\infty} p_1(x_1) \int_{x_2}^{\infty} p_2(x_2) \int_{x_3}^{\infty} p_3(x_3) \cdots \int_{x_{n-1}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_n}^{\infty} p_n(x_n) dx_n \cdots dx_1 = \infty,
\] (3.4)

\[
\limsup_{l \to \infty} K_l^{-2} (t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*), p_{l-1}(*), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_l) + \int_{h^{-1}(t)}^{\infty} p_n(x_n) dx_n > 1
\] (3.5)

for \( l = 3, 5, \ldots, n - 2, \)

\[
\limsup_{l \to \infty} K_l^{-1} (t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_{h^{-1}(t)}^{\infty} p_n(x_n) dx_n > 1.
\] (3.6)
If \( n \) is odd and \( \sigma = -1 \), then every solution \( y \in W \) to (1.1) is oscillatory or \( \lim_{t \to \infty} y_i(t) = 0 \), \( i = 1, 2, \ldots, n \).

**Proof.** Let \( y \in W \) be a non-oscillatory solution to (1.1). The Expression (2.7) holds. Taking into account Remark 2.5, one may write

\[
N = N_1^+ \cup N_3^+ \cup \cdots \cup N_n^+.
\]  

(7)

Without loss of generality we may suppose that \( y_1(t) \) is positive for \( t \geq t_2 \).

(i) Let \( y \in N_1^+ \) on \( [t_2, \infty) \). In this case, we can write for \( t \geq t_2 \)

\[
y_1(t) > 0, z_1(t) > 0, y_2(t) < 0, y_3(t) > 0, \ldots, y_n(t) > 0,
\]  

(8)

and \( \lim_{t \to \infty} z_1(t) = L_1 \geq 0 \). We claim that \( L_1 = 0 \). Otherwise \( L_1 > 0 \). Then

\[
L_1 \leq z_1(h(t)) \leq y_1(h(t)) \quad \text{for} \quad t \geq t_3,
\]  

(9)

where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the last equation of (1.1) from \( x_{n-1} \) to \( x_{n-1}^* \), we get for \( x_{n-1} \geq t_3 \)

\[
y_n(x_{n-1}) - y_n(x_{n-1}^*) = \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)f(y_1(h(x_n)))dx_n.
\]  

(10)

From (10) with regard to (e), (8), and (9), we have for \( x_{n-1}^* \to \infty \)

\[
y_n(x_{n-1}) \geq KL_1 \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n, \quad x_{n-1} \geq t_3.
\]  

(11)

Multiplying (11) by \( p_{n-1}(x_{n-1}) \) and then using the \( (n - 1) \)th equation of the system (1.1), we get for \( x_{n-1} \geq t_3 \)

\[
y_{n-1}(x_{n-1}) \geq KL_1 p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n.
\]  

(12)

Integrating (12) from \( x_{n-2} \) to \( x_{n-2}^* \to \infty \), and then using (8), we get for \( x_{n-2} \geq t_3 \)

\[
-y_{n-1}(x_{n-2}) \geq KL_1 \int_{x_{n-2}}^{x_{n-2}^*} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n dx_{n-1}.
\]  

(13)

Multiplying (13) by \( p_{n-2}(x_{n-2}) \) and then using the \( (n - 2) \)th equation of the system (1.1), and the new inequality we integrate from \( x_{n-3} \) to \( x_{n-3}^* \to \infty \) we employ (8) and for \( x_{n-3} \geq t_3 \)

\[
y_{n-2}(x_{n-3}) \geq KL_1 \int_{x_{n-3}}^{x_{n-3}^*} p_{n-2}(x_{n-2}) \int_{x_{n-2}}^{x_{n-2}^*} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n dx_{n-1} dx_{n-2}.
\]  

(14)
Similarly for $x_1 \geq t_3$, we have

$$-z_1'(t) \geq KL_1 p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots p_{n-1}(x_{n-1})$$

$$\times \int_{x_{n-1}}^{\infty} p_n(x_n) dx_n dx_{n-1} \cdots dx_2. \tag{3.15}$$

Integrating (3.15) from $T$ to $T^* \to \infty$ and then using (3.8), we get for $T \geq t_3$

$$z_1(T) \geq KL_1 \int_{T}^{\infty} p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \cdots p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) dx_n dx_{n-1} \cdots dx_1, \tag{3.16}$$

which is a contradiction to (3.4). Hence $\lim_{t \to \infty} z_1(t) = 0$.

Then $z_1(t) \leq 1$, $t \geq t_4$, where $t_4 \geq t_3$ is sufficiently large and

$$y_1(t) \leq a(t) y_1(g(t)) + 1 \leq \lambda^* y_1(g(t)) + 1, \quad t \geq t_4. \tag{3.17}$$

We prove that $y_1(t)$ is bounded indirectly. Let $y_1(t)$ be unbounded. Then there exists a sequence $\{\tilde{t}_n\}_{n=1}^{\infty}$, $\tilde{t}_n \geq t_4$, where $n = 1, 2, \ldots$, $\tilde{t}_n \to \infty$ as $n \to \infty$,

$$\lim_{n \to \infty} y_1(\tilde{t}_n) = \infty, \quad y_1(\tilde{t}_n) = \max_{t_4 \leq s \leq \tilde{t}_n} y_1(s). \tag{3.18}$$

It follows from (3.1), (3.2), and (3.17),

$$y_1(\tilde{t}_n) \leq \lambda^* y_1(g(\tilde{t}_n)) + 1 \leq \lambda^* y_1(\tilde{t}_n) + 1, \tag{3.19}$$

$$y_1(\tilde{t}_n) \leq \frac{1}{1 - \lambda^*}, \quad n = 1, 2, \ldots.$$  

That is a contradiction to $\lim_{n \to \infty} y_1(\tilde{t}_n) = \infty$, and the function $y_1(t)$ is bounded.

We claim that $\lim_{t \to \infty} y_1(t) = 0$ and prove it indirectly. Let $\lim\sup_{t \to \infty} y_1(t) = s > 0$. Let $\{t_n\}_{n=1}^{\infty}$, $t_n \geq t_4$, $n = 1, 2, \ldots$, be such a kind of sequence, that $t_n \to \infty$ as $n \to \infty$, and $\lim\sup_{n \to \infty} y_1(t_n) = s$. Then $\lim\sup_{n \to \infty} y_1(g(t_n)) \leq s$. From (1.2) and (3.1),

$$z_1(t_n^*) \geq y_1(t_n^*) - \lambda^* y_1(g(t_n^*)), \quad n = 1, 2, \ldots, \tag{3.20}$$

$$y_1(g(t_n^*)) \geq \frac{y_1(t_n^*) - z_1(t_n^*)}{\lambda^*}, \quad n = 1, 2, \ldots$$

follow.

From the last inequality, we have

$$s \geq \frac{s}{\lambda^*}, \quad \lambda^* \geq 1. \tag{3.21}$$
That is a contradiction to condition (3.1) and $\lim_{t \to \infty} y_1(t) = 0 = \lim_{t \to \infty} y_i(t)$. Since $\lim_{t \to \infty} z_1(t) = L_1 = 0$ and from Lemma 2.4, imply $\lim_{t \to \infty} y_i(t) = 0$, $i = 2, 3, \ldots, n$. 

(II) Let $y \in N_1^+$, for some $l = 3, 5, \ldots, n - 2$, on $[t_2, \infty)$. In this case, one can consider for $t \geq t_2$

$$y_1(t) > 0, z_1(t) > 0, y_2(t) > 0, \ldots, y_l(t) > 0, y_{l+1}(t) < 0, \ldots, y_n(t) > 0. \quad (3.22)$$

Integrating the first equation of the system (1.1) from $\alpha(t)$ to $t$ and using (3.22) above, we get

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) y_2(x_1) \, dx, \quad t \geq t_3, \quad (3.23)$$

where $t_3 \geq t_2$ is sufficiently large. Integrating step by step 2nd, 3rd, $\ldots, (l - 1)$th equations of the system (1.1) and subsequently substituting into (3.23) for $t \geq t_3$, we obtain

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) \int_{x_l}^{x_2} p_2(x_2) \cdots \int_{x_3}^{x_l} p_{l-1}(x_{l-1}) y_l(x_{l-1}) \, dx_{l-1} \, dx_{l-2} \cdots \, dx_1. \quad (3.24)$$

Integrating $l$th, $(l+1)$th, $\ldots, (n-1)$th equation of the system (1.1) and using (3.22), we have

$$y_l(x_{l-1}) \geq -\int_{x_{l-1}}^{x_l} p_{l-1}(x_{l-1}) y_{l+1}(x_{l-1}) \, dx_{l-1},$$

$$-y_{l+1}(x_l) \geq \int_{x_l}^{x_{l+1}} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) \, dx_{l+1},$$

$$y_{l+2}(x_{l+1}) \geq -\int_{x_{l+1}}^{x_{l+2}} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) \, dx_{l+2}, \quad (3.25)$$

$$\vdots$$

$$-y_{n-1}(x_{n-2}) \geq \int_{x_{n-2}}^{x_{n-1}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \, dx_{n-1}.$$  

Combining expressions (3.24) and (3.25) and using (3.22), we get for $t \geq t_3$

$$z_1(t) \geq y_n(t) \int_{\alpha(t)}^{t} p_1(x_1) \int_{x_1}^{x_2} p_2(x_2) \cdots \int_{x_3}^{x_l} p_{l-1}(x_{l-1}) \int_{x_{l-1}}^{x_l} p_l(x_l)$$

$$\times \int_{x_1}^{x_{l+1}} p_{l+1}(x_{l+1}) \cdots \int_{x_{n-2}}^{x_{n-1}} p_{n-1}(x_{n-1}) \, dx_{n-1} \, dx_{n-2} \cdots \, dx_1. \quad (3.26)$$

The formula above may be rewritten by (1.5) and (1.6) for $t \geq t_3$ to

$$z_1(t) \geq y_n(t) I_{l-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*)) \times J_{n-l+1}((*), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}), \quad (3.27)$$
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Integrating the last equation of (1.1) from \( t \to t^* \to \infty \) and using (e), (1.2), and (3.22), we obtain for \( t \geq t_4 \) where \( t_4 \geq t_3 \) is sufficiently large,

\[
y_n(t) \geq K \int_t^{\infty} p_n(x_n)z_1(h(x_n))\,dx_n. \tag{3.28}
\]

From (3.2), (3.27), and (3.28) and the monotonicity of \( z_1(h) \), we have

\[
z_1(t) \geq K I_{n-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(\ast) \times \int_n^{\infty} ([\ast], \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))
\]

\[
\times \int_t^{\infty} p_n(x_n)z_1(h(x_n))\,dx_n
\]

\[
\geq z_1(t) K I_{n-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(\ast) \times \int_n^{\infty} ([\ast], \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))
\]

\[
\times \int_h^{-1}(t) p_n(x_n)\,dx_n, \tag{3.29}
\]

\[
1 \geq K I_{n-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(\ast) \times \int_n^{\infty} ([\ast], \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))
\]

\[
\times \int_h^{-1}(t) p_n(x_n)\,dx_n
\]

for \( t \geq t_4 \), which is a contradiction to (3.5), and it gives

\[
N_3^+ \cup N_3^+ \cup \cdots \cup N_{n-2}^+ = \emptyset. \tag{3.30}
\]

(III) Let \( y \in N_n^+ \) on \([t_2, \infty)\). In this case we consider for the components of solution \( y(t) \) and for function \( z_1 \)

\[
z_1(t) > 0, \quad y_i(t) > 0, \quad i = 1, 2, \ldots, n, \quad t \geq t_2. \tag{3.31}
\]

Analogically as in the previous part of the proof,

\[
z_1(t) \geq y_n(t) I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}), \quad t \geq t_3, \tag{3.32}
\]

holds and also (3.28), and for \( t \geq t_3 \)

\[
1 \geq K I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_h^{-1}(t) p_n(x_n)\,dx_n, \tag{3.33}
\]

which is a contradiction to (3.6) and \( N_n^+ = \emptyset \).
Theorem 3.2. Suppose that (3.1)–(3.4) are employed and (3.5) holds for \( l = 3, 5, \ldots, n - 1 \) and

\[
\int_s^\infty p_n(x_n) \int_{h(x_n)}^{x_n} p_1(x_1) \int_{h(x_2)}^{x_2} p_2(x_2) \cdots \int_{h(x_{n-1})}^{x_{n-1}} p_{n-1}(x_{n-1}) \, dx_{n-1} \cdots \, dx_2 \, dx_1 \, dx_n = \infty \quad (3.34)
\]

for \( s \) sufficiently large.

If \( n \) is even and \( \sigma = 1 \), then every solution \( y \in W \) to the system (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \), for \( i = 1, 2, \ldots, n \), or \( \lim_{t \to \infty} |y_i(t)| = \infty \), for \( i = 1, 2, \ldots, n \).

Proof. Let \( y \in W \) be a non-oscillatory solution to (1.1). Expression (2.8) holds. Taking into account Remark 2.5,

\[
N = N_1^+ \cup N_3^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+.
\]

Without loss of generality we may suppose that \( y_1(t) \) is positive for \( t \geq t_2 \).

(I) Let \( y \in N_1^+ \) on \([t_2, \infty)\). In this case, for \( t \geq t_2 \)

\[
y_1(t) > 0, \quad z_1(t) > 0, \quad y_2(t) < 0, \quad y_4(t) > 0, \quad y_2(t) < 0, \quad \ldots, \quad y_n(t) < 0.
\]

(3.36)

We may choose analogical approach as in Theorem 3.1 part (I). Equation (3.9) holds and we replace (3.11) by inequality

\[
-y_n(x_{n-1}) \geq KL_1 \int_{x_{n-1}}^\infty p_n(x_n) \, dx_n, \quad x_{n-1} \geq t_3.
\]

Moreover (3.15) holds and similarly as in the proof of Theorem 3.1 case (I). We prove that \( \lim_{t \to \infty} y_i(t) = 0 \), for \( i = 1, 2, \ldots, n \).

(II) Let \( y \in N_1^+ \) on \([t_2, \infty)\), for some \( l = 3, 5, \ldots, n - 1 \). In this case, for \( t \geq t_2 \),

\[
y_1(t) > 0, \quad z_1(t) > 0, \quad y_2(t) > 0, \quad \ldots, \quad y_{l+1}(t) > 0, y_{l+1}(t) < 0, \quad \ldots, \quad y_n(t) < 0.
\]

(3.38)

The analogical approach as in Theorem 3.1 part (II) follows out.

Instead of inequality (3.27), we get for \( t \geq t_3 \)

\[
z_1(t) \geq -y_n(t) L_{l-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*)) \times J_{n-l+1}(\alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1})
\]

(3.39)

and instead of (3.28) for \( t \geq t_4 \)

\[
-y_n(t) \geq K \int_t^\infty p_n(x_n) z_1(h(x_n)) \, dx_n,
\]

(3.40)

and in the end we gain the contradiction to (3.5).
(III) Let \( y \in N_1^* \) on \([t_2, \infty)\). In this case (3.31) holds. Integrating the last equation of the system (1.1) and on the basis of (3.31), (3.2), (e), and (1.2), we have

\[
y_n(t) \geq K \int_s^t p_n(x_n)z_1(h(x_n)) \, dx_n, \quad t \geq s \geq t_3,
\]

(3.41)

where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the first equation of the system (1.1) from \( h(s) \) to \( h(x_n) \) and employing (3.31), we obtain

\[
z_1(h(x_n)) \geq \int_{h(s)}^{h(x_n)} p_1(x_1) y_2(x_1) \, dx_1, \quad s \geq t_3.
\]

(3.42)

Combining (3.41) and (3.42), we have for \( t \geq s \geq t_3 \)

\[
y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(t)} p_1(x_1) y_2(x_1) \, dx_1 \, dx_n.
\]

(3.43)

Further consequently integrating the 2nd, 3rd, \ldots, \( (l-1) \)th equations of the system (1.1) and step by step substituting into (3.43), we get for \( t \geq s \geq t_3 \)

\[
y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} p_1(x_1) \int_{h(s)}^{x_1} p_2(x_2) \int_{h(s)}^{x_2} \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \, dx_{n-1} \, dx_{n-2} \cdots \, dx_2 \, dx_1 \, dx_n.
\]

(3.44)

On basis of (3.31), for \( x_{n-1} \geq t_3 \)

\[
y_n(x_{n-1}) \geq C, \quad 0 < C = \text{const.}, \quad \text{for } x_{n-1} \geq t_3,
\]

(3.45)

hold.

Combining (3.44) and (3.45) for \( t \geq s \geq t_3 \), we have

\[
y_n(t) \geq KC \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} p_1(x_1) \int_{h(s)}^{x_1} p_2(x_2) \int_{h(s)}^{x_2} \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) \, dx_{n-1} \, dx_{n-2} \cdots \, dx_2 \, dx_1 \, dx_n.
\]

(3.46)

From the inequality above and relation (3.34), we obtain \( \lim_{t \to \infty} y_n(t) = \infty \). Lemma 2.4 implies \( \lim_{t \to \infty} z_1(t) = \infty \) and \( \lim_{t \to \infty} y_i(t) = \infty, i = 2, 3, \ldots, n-1 \). Since \( z_1(t) < y_1(t) \) for \( t \geq t_2 \), so \( \lim_{t \to \infty} y_1(t) = \infty \) and the final conclusion is \( \lim_{t \to \infty} |y_i(t)| = \infty, i = 1, 2, \ldots, n. \)
Theorem 3.3. Suppose that (3.3) holds and

\[ 1 < \lambda^* \leq a(t) \quad \text{for some constant } \lambda^*, \quad t \geq t_0, \quad (3.47) \]

\[ t < g(t) < h(t) \quad \text{for } t \geq t_0, \quad (3.48) \]

\[ \int_1^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) \int_{x_2}^\infty p_3(x_3) \cdots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \]

\[ \times \int_{x_{n-1}}^\infty \frac{p_n(x_n)dx_ndx_{n-1} \cdots dx_1}{a(g^{-1}(h(x_n)))} = \infty, \quad (3.49) \]

\[ \limsup_{t \to \infty} K_{t-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(\ast) \times \int_{J_{t-1}}(\ast), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1})) \]

\[ \times \int_t^\infty \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))} > 1, \quad (3.50) \]

for \( l = 3, 5, \ldots, n - 2, \)

\[ \limsup_{t \to \infty} K_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_t^\infty \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))} > 1. \quad (3.51) \]

If \( n \) is odd and \( \sigma = 1 \) then every solution \( y \in W \) to (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \), \( i = 1, 2, \ldots, n. \)

Proof. Let \( y \in W \) be a non-oscillatory solution to (1.1). Expression (2.6) holds. Without loss of generality we may suppose that \( y_1(t) \) is positive for \( t \geq t_2. \)

(I) Let \( y \in N^+_1 \cup N^+_2 \cup \cdots \cup N^+_{n-1} \cup N^+_n \) on \([t_2, \infty)\). Lemma 2.6 implies \( \lim_{t \to \infty} y_1(t) = 0 \). In this case, for \( t \geq t_2, \)

\[ 0 < z_1(t) < y_1(t), \quad (3.52) \]

and so \( \lim_{t \to \infty} z_1(t) = 0 \) which is a contradiction to the fact that the \( z_1(t) \) is positive and a nondecreasing function on the interval \([t_2, \infty)\) and

\[ N^+_2 \cup N^+_3 \cup \cdots \cup N^+_{n-1} \cup N^+_n = \emptyset. \quad (3.53) \]

(II) Let \( y \in N^-_1 \) on \([t_2, \infty)\). In this case, we can write for \( t \geq t_2 \)

\[ y_1(t) > 0, z_1(t) < 0, \quad y_2(t) > 0, \quad y_3(t) < 0, \ldots, \quad y_n(t) < 0. \quad (3.54) \]

We indirectly prove \( \lim_{t \to \infty} z_1(t) = 0. \)

Since \( z_1(t) \) is nondecreasing \( \lim_{t \to \infty} z_1(t) = -L_1, \quad L_1 > 0, \quad L_1 = \text{const.} \), and

\[ z_1(t) \leq -L_1 \quad \text{for } t \geq t_2. \quad (3.55) \]
Because \( z_1(t) > -a(t)y_1(g(t)) \),

\[
z_1\left(g^{-1}(h(t))\right) > -a\left(g^{-1}(h(t))\right)y_1(h(t)),
\]

(3.56)

\[
-y_1(h(t)) < \frac{z_1\left(g^{-1}(h(t))\right)}{a\left(g^{-1}(h(t))\right)}, \quad t \geq t_2
\]

(3.57)

follows.

From (3.55) and (3.57), we get

\[
-L_1 \geq z_1\left(g^{-1}(h(x_n))\right) \geq -a\left(g^{-1}(h(x_n))\right)y_1(h(x_n)), \quad x_n > t_2.
\]

(3.58)

By (c), (e), the last equation of (1.1), and (3.58), we get for \( x_n > t_2 \)

\[
\frac{KL_1p_n(x_n)}{a\left(g^{-1}(h(x_n))\right)} \leq Kp_n(x_n)y_1(h(x_n)) \leq p_n(x_n)f\left(y_1(h(x_n))\right) = y'_n(x_n).
\]

(3.59)

Integrating (3.59) from \( x_{n-1} \) to \( x^*_n \to \infty \), we get

\[
KL_1 \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_n}{a\left(g^{-1}(h(x_n))\right)} \leq -y_n(x_{n-1}) \quad \text{for } x_{n-1} \geq t_2.
\]

(3.60)

Multiplying (3.60) by \( p_{n-1}(x_{n-1}) \) and then using the \((n-1)\)th equation of system (1.1), we get for \( x_{n-1} \geq t_2 \)

\[
KL_1p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_n}{a\left(g^{-1}(h(x_n))\right)} \leq -y_{n-1}(x_{n-1}).
\]

(3.61)

Integrating (3.61) from \( x_{n-2} \) to \( x^*_n \to \infty \), we get for \( x_{n-2} \geq t_2 \)

\[
KL_1 \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_ndx_{n-1}}{a\left(g^{-1}(h(x_n))\right)} \leq y_{n-1}(x_{n-2}).
\]

(3.62)

Similarly we continue by the same way until we derive for \( x_1 \geq t_2 \)

\[
KL_1p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_ndx_{n-1}\cdots dx_2}{a\left(g^{-1}(h(x_n))\right)} \leq z_1(x_1).
\]

(3.63)
Integrating (3.63) from $T$ to $T^* \to \infty$, we get for $T \geq t_2$

$$KL_1 \int_T^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) \int_{x_2}^\infty p_3(x_3) \cdots \int_{x_{n-1}}^\infty p_{n-1}(x_{n-1}) \times \int_{x_{n-1}}^\infty p_n(x_n)dx_n dx_{n-1} \cdots dx_1 \frac{a(g^{-1}(h(x_n)))}{a^n} \leq -z_1(T).$$  (3.64)

That contradicts (3.49), and consequently $\lim_{t \to \infty} z_1(t) = 0$ holds.

We prove that $y_1(t)$ is bounded and $\lim_{t \to \infty} y_1(t) = 0$. There is some positive constant $B > 0, z_1(t) \geq -B$ for $t \geq t_2$, and by (1.2) and (3.47), one has for $t \geq t_2$

$$y_1(t) = a(t)y_1(g(t)) + z_1(t) \geq a(t)y_1(g(t)) - B \geq \lambda^* y_1(g(t)) - B.$$  (3.65)

We prove indirectly that $y_1(t)$ is bounded. Let us suppose that $y_1(t)$ is unbounded. Then $y_1(g(t))$ is unbounded, and there is a sequence

$$\left\{ \tilde{t}_n \right\}_{n=1}^\infty, \quad \tilde{t}_n \geq t_2, \quad n = 1, 2, \ldots, \quad \tilde{t}_n \to \infty \quad \text{as} \quad n \to \infty,$$

$$\lim_{n \to \infty} y_1(\tilde{t}_n) = \infty, \quad y_1(g(\tilde{t}_n)) = \max_{t_2 \leq s \leq g(\tilde{t}_n)} y_1(s).$$  (3.66)

By (3.65)

$$\lambda^* y_1(g(\tilde{t}_n)) \leq y_1(\tilde{t}_n) + B \leq y_1(g(\tilde{t}_n)) + B,$$

$$y_1(g(\tilde{t}_n)) \leq \frac{B}{\lambda^* - 1}, \quad n = 1, 2, \ldots.$$  (3.67)

That is a contradiction to $\lim_{n \to \infty} y_1(g(\tilde{t}_n)) = \infty$, and the function $y_1(t)$ is bounded. We claim that $\lim_{t \to \infty} y_1(t) = 0$, and we will prove it indirectly.

Let $\lim \sup_{t \to \infty} y_1(g(t)) = s, \quad 0 < s, \quad s = \text{const}$. Then $\lim \sup_{t \to \infty} y_1(t) = s$.

Let $\left\{ t^*_n \right\}_{n=1}^\infty, \quad t^*_n \geq t_2, \quad n = 1, 2, \ldots$, be such a kind of sequence that $\lim_{n \to \infty} t^*_n = \infty$ and $\lim \sup_{n \to \infty} y_1(g(t^*_n)) = s$.

Then, $\lim \sup_{n \to \infty} y_1(t^*_n) \leq s$.

By (1.2) and (3.47),

$$z_1(t^*_n) \leq y_1(t^*_n) - \lambda^* y_1(g(t^*_n)), \quad n = 1, 2, \ldots,$$

$$y_1(g(t^*_n)) \leq \frac{y_1(t^*_n) - z_1(t^*_n)}{\lambda^*}, \quad n = 1, 2, \ldots.$$  (3.68)

follows.

By the last inequality, we have

$$s = \lim \sup_{t \to \infty} y_1(g(t^*_n)) \leq \frac{\lim \sup_{t \to \infty} y_1(t^*_n)}{\lambda^*} \leq \frac{s}{\lambda^*}.\quad (3.69)$$
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1 ≥ λ∗ holds. That is a contradiction to (3.47). It means lim sup_{t \to \infty} y_1(t) = 0 and also lim sup_{t \to \infty} y_1(t) = 0. Moreover, y_1(t) > 0 holds, so lim inf_{t \to \infty} y_1(t) = 0 and this leads to lim_{t \to \infty} y_1(t) = 0.

By Lemma 2.4 it follows that

$$\lim_{t \to \infty} y_i(t) = 0, \quad i = 2, 3, \ldots, n.$$  \hfill (3.70)

(III) Let \( y \in N_1^−, l = 3, 5, \ldots, n−2, \) on \([t_2, \infty).\) In this case for, \( t \geq t_2,\)

$$y_l(t) > 0, \quad z_1(t) < 0, \quad y_2(t) < 0, \ldots, \quad y_l(t) < 0, \quad y_{l+1}(t) > 0, \ldots, \quad y_n(t) < 0.$$  \hfill (3.71)

Integrating the first equation of (1.1) from \( a(t) \) to \( t \) and using (3.71), we get

$$z_1(t) \geq \int_{a(t)}^{t} p_1(x_1) y_2(x_1) dx_1, \quad t \geq t_3,$$  \hfill (3.72)

where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the 2nd, 3rd, \ldots, \((l − 1)\)th equations of the system (1.1), and substituting into (3.72), we get for \( t \geq t_3\)

$$z_1(t) \leq \int_{a(t)}^{t} p_1(x_1) \int_{a(t)}^{x_1} p_2(x_2) \cdots \int_{a(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) y_l(x_{l-1}) dx_{l-1} dx_{l-2} \cdots dx_1.$$  \hfill (3.73)

Integrating \((l)\)th, \((l + 1)\)th, \ldots, \((n − 1)\)th equations of the system (1.1) we gain the syste

$$y_l(x_{l−1}) \leq −\int_{x_{l−1}}^{x_l} p_l(x_l) y_{l+1}(x_l) dx_l,$$

$$−y_{l+1}(x_1) \leq \int_{x_1}^{x_{l−2}} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) dx_{l+1},$$

$$y_{l+2}(x_{l+1}) \leq −\int_{x_{l+1}}^{x_{l−2}} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) dx_{l+2},$$

$$\vdots$$

$$−y_{n−1}(x_{n−2}) \leq \int_{x_{n−2}}^{x_{n−1}} p_{n−1}(x_{n−1}) y_n(x_{n−1}) dx_{n−1}.$$  \hfill (3.74)
We combine the formulae (3.73) and (3.74), and with regard to (3.71), we get for \( t \geq t_3 \)

\[
\begin{align*}
z_1(t) &\leq y_n(t) \int_{a(t)}^{t} p_1(x_1) \int_{a(t)}^{x_1} p_2(x_2) \cdots \int_{a(t)}^{x_{i-1}} p_{i-1}(x_{i-1}) \int_{a(t)}^{x_{i-2}} p_i(x_i) \\
& \quad \times \int_{a(t)}^{x_{i-2}} p_{i+1}(x_{i+1}) \cdots \int_{a(t)}^{x_n} p_{n-1}(x_{n-1}) dx_{n-1} dx_{n-2} \cdots dx_1.
\end{align*}
\] (3.75)

Employing (1.5) and (1.6) the equation above may be rewritten to

\[
z_1(t) \leq y_n(t) I_{i-2}(t, a(t); p_1, p_2, \ldots, p_{i-2}(*)) \times J_{n-i+1}((*), a(t); p_{n-1}, \ldots, p_1))
\] (3.76)

for \( t \geq t_3 \).

Integrating the last equation of (1.1) from \( t \) to \( t^* \rightarrow \infty \) and using (e) and (3.71),

\[
y_n(t) \leq -K \int_{t}^{\infty} p_n(x_n) y_1(h(x_n)) dx_n, \quad t \geq t_3.
\] (3.77)

From (3.2), (3.57) in regard to (3.76), (3.77) and monotonicity of \( z_1(g^{-1}(h)) \), we get for \( t \geq t_3 \)

\[
\begin{align*}
z_1(t) &\leq K I_{i-2}(t, a(t); p_1, p_2, \ldots, p_{i-2}(*)) \times J_{n-i+1}((*), a(t); p_{n-1}, \ldots, p_1)) \\
& \quad \times \int_{t}^{\infty} \frac{p_n(x_n) z_1(g^{-1}(h(x_n))) dx_n}{a(g^{-1}(h(x_n)))} \\
& \leq z_1(t) K I_{i-2}(t, a(t); p_1, p_2, \ldots, p_{i-2}(*)) \times J_{n-i+1}((*), a(t); p_{n-1}, \ldots, p_1)) \\
& \quad \times \int_{t}^{\infty} \frac{p_n(x_n) dx_n}{a(g^{-1}(h(x_n)))},
\end{align*}
\] (3.78)

which means for \( t \geq t_3 \)

\[
1 \geq K I_{i-2}(t, a(t); p_1, p_2, \ldots, p_{i-2}(*)) \times J_{n-i+1}((*), a(t); p_{n-1}, \ldots, p_1)) \\
\quad \times \int_{t}^{\infty} \frac{p_n(x_n) dx_n}{a(g^{-1}(h(x_n)))}.
\] (3.79)

This is a contradiction to (3.50) and

\[
N_3^- \cup N_5^- \cup \cdots \cup N_{n-2}^- = \emptyset.
\] (3.80)

(IV) Let \( y \in N_{r}^- \), on \( [t_2, \infty) \).

In this case, we can write for \( t \geq t_2 \)

\[
y_1(t) > 0, \quad z_1(t) < 0, \quad y_i(t) < 0, \quad i = 2, 3, \ldots, n.
\] (3.81)
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We may lead the proof analogically as in the previous part of the proof and we will prove that (3.77), (3.57), and

\[ z_1(t) \leq y_n(t) I_{n-1}(t, a(t); p_1, p_2, \ldots, p_{n-1}) \]  

(3.82)

hold and also

\[ 1 \geq K I_{n-1}(t, a(t); p_1, p_2, \ldots, p_{n-1}) \int_t^\infty \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))}, \quad t \geq t_3 \]  

(3.83)

which is a contradiction to (3.51) and \( N_n = \emptyset \). \( \Box \)

**Theorem 3.4.** Suppose that (3.3), (3.47)--(3.49) hold and condition (3.50) is fulfilled for \( l = 3, 5, \ldots, n - 1 \), and

\[ \int_s^\infty \frac{p_n(x_n)}{a(g^{-1}(h(x_n)))} \int_{g^{-1}(h(s))}^{x_1} p_1(x_1) \int_{g^{-1}(h(s))}^{x_2} p_2(x_2) \]

\[ \cdots \int_{g^{-1}(h(s))}^{x_{n-2}} p_{n-1}(x_{n-1})dx_{n-1}dx_{n-2} \cdots dx_1dx_n = \infty \]

(3.84)

for \( s \geq t_0 \).

If \( n \) is even and \( \sigma = -1 \), then every solution \( y \in W \) to (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0, i = 1, 2, \ldots, n \), or \( \lim_{t \to \infty} |z_i(t)| = \infty \) and \( \lim_{t \to \infty} |y_i(t)| = \infty, i = 2, \ldots, n \).

**Proof.** Let \( y \in W \) be a non-oscillatory solution to (1.1), Expression (2.9) holds.

(I) Let \( y \in N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ \). Analogically as in the proof of Theorem 3.3 (I), we prove that

\[ N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ = \emptyset. \]  

(3.85)

(II) Let \( y \in N_i^+ \) on \([t_2, \infty)\). Similarly to the proof of Theorem 3.3 (II), we prove \( \lim_{t \to \infty} y_i(t) = 0, i = 1, 2, \ldots, n \).

(III) Let \( y \in N_i^- \), for some \( l = 3, 5, \ldots, n - 1 \), for \( t \in [t_2, \infty) \). Likewise as proof of Theorem 3.3 (III), for sets \( N_i^- \) we prove that \( N_3^- \cup N_5^- \cup \cdots \cup N_{n-1}^- = \emptyset \).

(IV) Let \( y \in N_n^- \) for \( t \in [t_2, \infty) \). Analogically to the proof of case (III) of Theorem 3.2, we claim \( \lim_{t \to \infty} |z_1(t)| = \infty, \lim_{t \to \infty} |y_i(t)| = \infty, i = 2, \ldots, n \). \( \Box \)
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Example 3.5. We consider system (1.1) as follows:

\[ \left( y_1(t) - \frac{1}{2} y_1 \left( \frac{t}{4} \right) \right)' = \frac{t}{2} y_2(t), \]
\[ y_2'(t) = \frac{1}{2} e^{\frac{t}{4}} y_3(t), \]
\[ y_3'(t) = \frac{1}{2} e^{\frac{t}{8}} y_4(t), \]
\[ y_4'(t) = \frac{1}{16} \left( e^{-\frac{3t}{8}} + \frac{5}{8} e^{-\frac{9t}{8}} \right) y_1 \left( \frac{t}{2} \right), \quad t \geq 1. \]

All assumptions of Theorem 3.2 are satisfied, and every solution \( y \in W \) to (3.86) is either oscillatory or

\[ \lim_{t \to \infty} y_i(t) = 0, \quad i = 1, 2, 3, 4, \quad \text{or} \quad \lim_{t \to \infty} |y_i(t)| = \infty, \quad i = 1, 2, 3, 4. \]

One of the solutions has particular components as follows:

\[ y_1(t) = e^{t}, \quad y_2(t) = e^{\frac{t}{2}} - \frac{1}{8} e^{-\frac{t}{4}}, \]
\[ y_3(t) = e^{\frac{t}{4}} + \frac{1}{16} e^{-\frac{t}{2}}, \quad y_4(t) = \frac{1}{2} \left( e^{\frac{t}{8}} - \frac{1}{8} e^{-\frac{5t}{8}} \right), \quad t \geq 1, \]

and in this case

\[ \lim_{t \to \infty} y_i(t) = \infty, \quad i = 1, 2, 3, 4. \]

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References


