Research Article

Differential Subordinations for Certain Meromorphically Multivalent Functions Defined by Dziok-Srivastava Operator

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1. Introduction

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),$$

which are analytic in the punctured open unit disk $\mathbb{U}_0 = \{z : 0 < |z| < 1\}$ with a pole at $z = 0$. Also let the Hadamard product (or convolution) of the following functions:

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j} z^{n-p} \quad (j = 1, 2)$$

be given by

$$(f_1 * f_2)(z) := z^{-p} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n-p} = (f_2 * f_1)(z).$$
Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g < f$ or (more precisely) $g(z) < f(z)$ ($z \in \mathbb{U}$), if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $g(z) = f(w(z))$ ($z \in \mathbb{U}$). In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$g(z) < f(z) \quad (z \in \mathbb{U}) \iff g(0) = f(0), \quad g(\mathbb{U}) \subset f(\mathbb{U}).$$

(1.4)

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

which are analytic in $\mathbb{U}$. A function $f(z) \in A$ is said to be in the class $S^*(\alpha)$ if

$$\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

(1.6)

for some $\alpha$ ($\alpha < 1$). When $0 \leq \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order $\alpha$ in $\mathbb{U}$. A function $f(z) \in A$ is said to be prestarlike of order $\alpha$ in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \quad (\alpha < 1),$$

(1.7)

where the symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in $\mathbb{U}$. We denote this class by $R(\alpha)$ (see [1]). Clearly a function $f(z) \in A$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in $\mathbb{U}$ and $R(1/2) = S^*(1/2)$.

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s),$$

(1.8)

we define the generalized hypergeometric function $\genfrac{[}{]}{0pt}{}{qF_s(a_1, \ldots, a_q; \beta_1, \ldots, \beta_s; z)}{2}$ by

$$\genfrac{[}{]}{0pt}{}{qF_s(a_1, \ldots, a_q; \beta_1, \ldots, \beta_s; z)}{2} := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

(1.9)

$$\quad (q \leq s + 1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma(z)$, by

$$\begin{align*}
(x)_n &:= \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0; x \in \mathbb{C} \setminus \{0\}) \\ x(x+1) \cdots (x+n-1) & (n \in \mathbb{N}; x \in \mathbb{C}). \end{cases}
\end{align*}$$

(1.10)
Corresponding to a function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^p q F_s(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s; z), \quad (1.11)$$

we now consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma(p) \rightarrow \Sigma(p), \quad (1.12)$$

defined by means of the Hadamard product (or convolution) as follows:

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) := h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast f(z). \quad (1.13)$$

For convenience, we write

$$H_{p,q,s}(\alpha_1) := H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s). \quad (1.14)$$

Thus, after some calculations, we have

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z). \quad (1.15)$$

The operator $H_{p,q,s}(\alpha_1)$ is popularly known as the generalized Dziok-Srivastava operator. Many interesting subclasses of multivalent functions, associated with the operator $H_{p,q,s}(\alpha_1)$ and its various special cases, were investigated recently by (e.g.) Dziok and Srivastava [2–4], Liu [5], Liu and Srivastava [6, 7], Patel et al. [8], Wang et al. [9], and others.

Let $P$ be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in $\mathbb{U}$.

**Definition 1.1.** A function $f(z) \in \Sigma(p)$ is said to be in the class $T_{p,q,s}(\alpha, \lambda; h)$ if it satisfies the subordination condition

$$\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'' < h(z), \quad (1.16)$$

where $\lambda$ is a complex number and $h(z) \in P$.

The main object of this paper is to present a systematic investigation of the class $T_{p,q,s}(\alpha_1, \lambda; h)$ defined above by means of the generalized Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$.

For our purpose, we shall need the following lemmas to derive our main results for the class $T_{p,q,s}(\alpha_1, \lambda; h)$.

**Lemma 1.2** (see [10]). Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} zg'(z) < h(z), \quad (1.17)$$
where $\Re \mu > 0$, then

$$g(z) < \tilde{h}(z) = \mu z^{-\mu} \int_{0}^{z} t^{\mu-1} h(t) dt < h(z)$$  \hspace{1cm} (1.18)

and $\tilde{h}(z)$ is the best dominant of (1.17).

**Lemma 1.3** (see [1]). Let $\alpha < 1$, $f(z) \in S^0(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in $\mathbb{U}$,

$$\frac{g \ast (fF)}{g \ast f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the closed convex hull of $F(\mathbb{U})$.

### 2. Properties of the Class $T_{p,q,s}(\alpha_1, \lambda; h)$

**Theorem 2.1.** Let $\lambda_1 < \lambda_2 \leq 0$. Then $T_{p,q,s}(\alpha_1, \lambda_1; h) \subset T_{p,q,s}(\alpha_1, \lambda_2; h)$.

**Proof.** Let $\lambda_1 < \lambda_2 \leq 0$ and suppose that

$$g(z) = -\frac{z^{p+1}(H_{p,q,s}(\alpha_1) f(z))'}{p}$$  \hspace{1cm} (2.1)

for $f(z) \in T_{p,q,s}(\alpha_1, \lambda_1; h)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating both sides of (2.1) with respect to $z$ and using (1.16), we have

$$\frac{(\lambda_1 - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_1}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'' = g(z) - \frac{\lambda_1}{p + 1} z g'(z) < h(z).$$  \hspace{1cm} (2.2)

Hence an application of Lemma 1.2 yields

$$g(z) < h(z).$$  \hspace{1cm} (2.3)

Noting that $0 < \lambda_2 / \lambda_1 < 1$ and that $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.1) to (2.3) that

$$\frac{(\lambda_2 - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_2}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))''

= \frac{\lambda_2}{\lambda_1} \left( \frac{(\lambda_1 - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_1}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'' \right)

+ \left( 1 - \frac{\lambda_2}{\lambda_1} \right) g(z) < h(z).$$  \hspace{1cm} (2.4)

Thus $f(z) \in T_{p,q,s}(\alpha_1, \lambda_2; h)$ and the proof of Theorem 2.1 is completed.  \hspace{1cm} $\square$
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**Theorem 2.2.** Let $0 < b_1 < b_2$. Then $T_{p,q,s}(b_2, \lambda; h) \subset T_{p,q,s}(b_1, \lambda; h)$.

**Proof.** Define a function $g(z)$ by

$$g(z) = z + \sum_{n=1}^{\infty} \frac{(b_1)_n}{(b_2)_n} z^{n+1} \quad (z \in \mathbb{U}; 0 < b_1 < b_2). \quad (2.5)$$

Then

$$z^{p+1} h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z) = g(z) \in A, \quad (2.6)$$

where

$$h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z) \quad (2.7)$$

is defined as in (1.11), and

$$\frac{z}{(1 - z)^{b_2}} * g(z) = \frac{z}{(1 - z)^{b_1}}. \quad (2.8)$$

By (2.8), we see that

$$\frac{z}{(1 - z)^{b_2}} * g(z) \in S^*(1 - \frac{b_1}{2}) \subseteq S^*(1 - \frac{b_2}{2}) \quad (0 < b_1 < b_2), \quad (2.9)$$

which implies that

$$g(z) \in R(1 - \frac{b_2}{2}). \quad (2.10)$$

Let $f(z) \in T_{p,q,s}(b_2, \lambda; h)$. It is easy to verify that

$$z^{p+1} (H_{p,q,s}(b_1) f(z))' = (z^p h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z)) * \left( z^{p+1} (H_{p,q,s}(b_2) f(z))' \right) \quad (2.11)$$

$$z^{p+2} (H_{p,q,s}(b_1) f(z))'' = (z^p h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z)) * \left( z^{p+2} (H_{p,q,s}(b_2) f(z))'' \right). \quad (2.12)$$

From (2.11), (2.12), and (2.6), we deduce that

$$\frac{(1 - 1/p)}{p} z^{p+1} (H_{p,q,s}(b_1) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_1) f(z))''$$

$$= \frac{g(z)}{z} \ast w(z) = \frac{g(z) * (zw(z))}{g(z) * z}, \quad (2.13)$$
where
\[ w(z) := \frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(b_2) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_2) f(z))'' < h(z). \] (2.14)

Since the function \( z \) belongs to the function class \( S^*(1 - b_2/2) \) and \( h(z) \) is convex univalent in \( \mathbb{U} \), it follows from (2.12), (2.13), (2.14), and Lemma 1.3 that
\[ \frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(b_1) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_1) f(z))'' < h(z). \] (2.15)

Thus \( f(z) \in T_{p,q,s}(b_1, \lambda; h) \) and the proof of Theorem 2.2 is completed. \( \Box \)

**Theorem 2.3.** Let \( f(z) \in T_{p,q,s}(\alpha_1, \lambda; h) \) and \( g(z) \in \Sigma(p) \) and
\[ \Re\{z^p g(z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \] (2.16)

Then
\[ (f * g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h). \] (2.17)

**Proof.** For \( f(z) \in T_{p,q,s}(\alpha_1, \lambda; h) \) and \( g(z) \in \Sigma(p) \), we have
\[
\begin{align*}
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1)(f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1)(f * g)(z))'' \\
= \frac{(\lambda - 1)}{p} (z^p g(z))' (z^{p+1} (H_{p,q,s}(\alpha_1) f(z))') + \frac{\lambda}{p(p+1)} (z^p g(z))'' (z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'') \\
= (z^p g(z))' \psi(z),
\end{align*}
\] (2.18)

where
\[ \psi(z) = \frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))''. \] (2.19)

In view of (2.16), the function \( z^p g(z) \) has the Herglotz representation
\[ z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}), \] (2.20)

where \( \mu(x) \) is a probability measure defined on the unit circle \( |x| = 1 \) and
\[ \int_{|x|=1} d\mu(x) = 1. \] (2.21)
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Since $h(z)$ is convex univalent in $U$, it follows from (2.18) to (2.20) that
\[
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) (f \ast g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1) (f \ast g)(z))'' \\
= \int_{|x|=1} \varphi(xz) d\mu(x) < h(z).
\]
(2.22)

This shows that $(f \ast g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$ and the theorem is proved. \(\square\)

**Theorem 2.4.** Let $f(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$, $g(z) \in \Sigma(p)$ and
\[
z^{p+1} g(z) \in R(\alpha) \quad (\alpha < 1).
\]
(2.23)
Then
\[
(f \ast g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h).
\]
(2.24)

**Proof.** For $f(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$ and $g(z) \in \Sigma(p)$, from (2.18) we have
\[
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(f \ast g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(f \ast g)(z))'' \\
= \frac{(z^{p+1} g(z)) \ast (z \varphi(z))}{(z^{p+1} g(z)) \ast z} \quad (z \in U),
\]
(2.25)
where $\varphi(z)$ is defined as in (2.19). Since $h(z)$ is convex univalent in $U$, $\varphi(z) < h(z)$,
\[
z^{p+1} g(z) \in R(\alpha), \quad z \in S^\ast(\alpha) \quad (\alpha < 1),
\]
(2.26)
it follows from (2.25) and Lemma 1.3 the desired result. \(\square\)

**Theorem 2.5.** Let $\lambda < 0$, $\beta > 0$ and $f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta)$. If $\beta \leq \beta_0$, where
\[
\beta_0 = \frac{1}{2} \left( 1 + \frac{1}{\lambda} \int_0^1 \frac{u^{-(p+1)/\lambda - 1}}{1 + u} \, du \right)^{-1},
\]
(2.27)
then $f(z) \in T_{p,q,s}(0; h)$. The bound $\beta_0$ is sharp when $h(z) = 1/(1 - z)$.

**Proof.** Let us define
\[
\varphi(z) = -\frac{z^{p+1} (H_{p,q,s}(\alpha_1) f(z))'}{p}
\]
(2.28)
for \( f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta) \) with \( \lambda < 0 \) and \( \beta > 0 \). Then we have

\[
g(z) - \frac{\lambda}{p+1} z g'(z) = \frac{(\lambda-1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1)f(z))^\prime + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1)f(z))^\prime
d < \beta h(z) + 1 - \beta.
\]

Hence an application of Lemma 1.2 yields

\[
g(z) < -\frac{\beta(p+1)}{\lambda} z^{(p+1)/\lambda} \int_0^\lambda t^{-(p+1)/\lambda-1} h(t) dt + 1 - \beta = (h \ast \psi)(z),
\]

where

\[
\psi(z) = \frac{\beta(p+1)}{\lambda} z^{(p+1)/\lambda} \int_0^\lambda \frac{t^{-(p+1)/\lambda-1}}{1-t} dt + 1 - \beta.
\]

If \( 0 < \beta \leq \beta_0 \), where \( \beta_0 > 1 \) is given by (2.27), then it follows from (2.31) that

\[
\text{Re} \ \psi(z) = -\frac{\beta(p+1)}{\lambda} \int_0^1 u^{-(p+1)/\lambda-1} \text{Re} \left( \frac{1}{1 - uz} \right) du + 1 - \beta
\]

\[
> -\frac{\beta(p+1)}{\lambda} \int_0^1 u^{-(p+1)/\lambda-1} \frac{u}{1+u} du + 1 - \beta
\]

\[
\geq \frac{1}{\lambda} \quad (z \in \mathbb{U}; \lambda < 0).
\]

Now, by using the Herglotz representation for \( \psi(z) \), from (2.28) and (2.30), we arrive at

\[
-\frac{z^{p+1} (H_{p,q,s}(\alpha_1)f(z))^\prime}{p} < (h \ast \psi)(z) < h(z)
\]

because \( h(z) \) is convex univalent in \( \mathbb{U} \). This shows that \( f(z) \in T_{p,q,s}(0; h) \).

For \( h(z) = 1/(1-z) \) and \( f(z) \in \Sigma(p) \) defined by

\[
-\frac{z^{p+1} (H_{p,q,s}(\alpha_1)f(z))^\prime}{p} = -\frac{\beta(p+1)}{\lambda} z^{(p+1)/\lambda} \int_0^\lambda t^{-(p+1)/\lambda-1} \frac{t}{1-t} dt + 1 - \beta,
\]

it is easy to verify that

\[
\frac{(\lambda-1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1)f(z))^\prime + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1)f(z))^\prime = \beta h(z) + 1 - \beta.
\]
Thus \( f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta) \). Also, for \( \beta > \beta_0 \), we have

\[
\text{Re}\left\{ -\frac{z^{p+1}}{p} (H_{p,q,s}(\alpha_1) f(z))' \right\} \rightarrow -\frac{\beta(p+1)}{\lambda} \int_0^1 \frac{u^{-(p+1)/\lambda)}-1}{1 + u} du + 1 - \beta < \frac{1}{2} \quad (z \rightarrow -1),
\]

which implies that \( f(z) \notin T_{p,q,s}(0; h) \). Hence the bound \( \beta_0 \) cannot be increased when \( h(z) = 1/(1 - z) \).

\[\square\]

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**References**
