Research Article

Supercyclicity and Hypercyclicity of an Isometry Plus a Nilpotent

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Suppose that X is a separable normed space and the operators A and Q are bounded on X. In this paper, it is shown that if AQ = QA, A is an isometry, and Q is a nilpotent then the operator A + Q is neither supercyclic nor weakly hypercyclic. Moreover, if the underlying space is a Hilbert space and A is a co-isometric operator, then we give sufficient conditions under which the operator A + Q satisfies the supercyclicity criterion.

1. Introduction

Let x be a vector in a separable normed space X and T an operator on X. The orbit of x under T is defined by

\[ \text{orb}(T, x) = \{ T^n x : n = 0, 1, 2, \ldots \}. \] (1.1)

We recall that a vector x in X is cyclic for an operator T on X if the closed linear span of orb(T, x) is X; it is supercyclic, if the set of all scalar multiples of the elements of orb(T, x) is dense in X; also it is said to be (weakly) hypercyclic if orb(T, x) is (weakly) dense in X. An operator T is called cyclic, supercyclic, or (weakly) hypercyclic operator, respectively, if it has a cyclic, supercyclic, or (weakly) hypercyclic vector. Recently, the cyclicity of operators has attracted much attention from operator theorists. For a good source on this topic, see [1]. Hilden and Wallen in [2] proved that isometries on Hilbert spaces with dimension more than one are not supercyclic. Ansari and Bourdon in [3] and Miller in [4] independently proved
this fact on Banach spaces. Moreover, recently it is shown in [5] that m-isometric operators on Hilbert spaces, which forms a larger class than isometries, are neither supercyclic nor weakly hypercyclic. In this paper, it is shown that an isometry plus a nilpotent on normed spaces are neither supercyclic nor weakly hypercyclic if they commute. We also discuss this fact when the underlying space is a Hilbert space and the isometry is replaced by a co-isometry. We begin with some elementary properties of such operators. In what follows, as usual, for an operator $T$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma(T)$ are denoted, respectively, the approximate point spectrum, point spectrum, and spectrum of $T$. Also, $D$ denotes the open unit disc. Recall that an operator $Q$ on a normed space $\mathcal{X}$ is a nilpotent operator of order $N \geq 1$ if $Q^N = 0$ and $Q^{N-1} \neq 0$. From now on, we assume that $Q$ is a nilpotent operator of order $N \geq 1$ unless stated otherwise.

**Proposition 1.1.** Suppose that $\mathcal{X}$ is a normed space, and $A \in B(\mathcal{X})$ is an isometry such that $AQ = QA$. If $T = A + Q$, then

(i) $\sigma(T) = \sigma(A)$,

(ii) $\sigma_p(T) = \sigma_p(A)$,

(iii) $\sigma_{ap}(T) = \sigma_{ap}(A)$.

**Proof.** (i) Suppose that $\lambda \notin \sigma(A)$. Then it is easily seen that

$$
(T - \lambda)^{-1} = \sum_{k=1}^{N} (-1)^{k-1} (A - \lambda)^{-k} Q^{k-1}
$$

which implies that $\lambda \notin \sigma(T)$. Consequently, $\sigma(T) \subseteq \sigma(A)$. Since $A = T - Q$, a similar argument shows that $\sigma(A) \subseteq \sigma(T)$.

(ii) If $\lambda \in \sigma_p(A)$, there exits $x \neq 0$ such that $Ax = \lambda x$. Therefore,

$$
TQ^{N-1}x = AQ^{N-1}x = \lambda Q^{N-1}x.
$$

Now, if $Q^{N-1}x \neq 0$, then $\lambda \in \sigma_p(T)$; otherwise,

$$
TQ^{N-2}x = AQ^{N-2}x = \lambda Q^{N-2}x.
$$

Also, if $Q^{N-2}x \neq 0$ then $\lambda \in \sigma_p(T)$; otherwise, consider $Q^{N-3}x$ and continue this process to conclude that $Tx = Ax = \lambda x$ which implies that $\lambda \in \sigma_p(T)$. Hence, $\sigma_p(A) \subseteq \sigma_p(T)$. Moreover, since $A = T - Q$, using a similar method, we get $\sigma_p(T) \subseteq \sigma_p(A)$.

(iii) Let $\lambda \in \sigma_{ap}(T)$; then there exists a sequence $(x_n)_n$ in $\mathcal{X}$ such that $\|x_n\| = 1$ and

$$
Tx_n - \lambda x_n \rightarrow 0 \quad \text{as} \ n \rightarrow +\infty.
$$

Therefore,

$$
AQ^{N-1}x_n - \lambda Q^{N-1}x_n \rightarrow 0 \quad \text{as} \ n \rightarrow +\infty.
$$
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Suppose that there is \( c_1 > 0 \) so that

\[
\|QN^{-1}x_n\| > c_1 \tag{1.7}
\]

for all \( n \geq 1 \); then \( Ay_n - \lambda y_n \to 0 \) as \( n \to +\infty \) where

\[
y_n = \frac{QN^{-1}x_n}{\|QN^{-1}x_n\|} \tag{1.8}
\]

which, in turn, implies that \( \lambda \in \sigma_{ap}(A) \).

Now, if (1.7) does not hold, then we can assume, without loss of generality, that \( (x_n)_n \) satisfies

\[
QN^{-1}x_n \to 0 \text{ as } n \to +\infty. \tag{1.9}
\]

So by (1.5),

\[
AQN^{-2}x_n - \lambda QN^{-2}x_n \to 0 \text{ as } n \to +\infty. \tag{1.10}
\]

Now, if there is a constant \( c_2 > 0 \) such that

\[
\|QN^{-2}x_n\| > c_2 \tag{1.11}
\]

for all \( n \), then \( Az_n - \lambda z_n \to 0 \) where

\[
z_n = \frac{QN^{-2}x_n}{\|QN^{-2}x_n\|} \tag{1.12}
\]

which implies that \( \lambda \in \sigma_{ap}(A) \). Otherwise, we can assume, without loss of generality, that \( QN^{-2}x_n \to 0 \) as \( n \to \infty \) and by (1.5)

\[
AQN^{-3}x_n - \lambda QN^{-3}x_n \to 0 \tag{1.13}
\]

as \( n \to \infty \). The procedure continues to conclude that \( \lambda \in \sigma_{ap}(A) \). Since \( A = T - Q \), by a similar method \( \sigma_{ap}(A) \subseteq \sigma_{ap}(T) \).

In the remaining results of this section, the operators \( A \) and \( T \) are as in Proposition 1.1.

**Corollary 1.2.** Suppose that \( \mathcal{K} \) is a normed space. Then \( T - \lambda I \) is bounded below where \( |\lambda| \neq 1 \).

**Proof.** Since \( A \) is an isometry, \( \sigma_{ap}(T) = \sigma_{ap}(A) \subseteq \partial \mathbb{D} \). In fact, let \( \lambda \in \sigma_{ap}(A) \); then \( |\lambda| \leq \|A\| = 1 \); moreover, there exists a sequence \( (x_n)_n \) in \( \mathcal{K} \) with \( \|x_n\| = 1 \) and so \( (A - \lambda I)(x_n) \to 0 \) if \( n \to \infty \). Therefore,

\[
0 \leq 1 - |\lambda| \leq \|(A - \lambda I)(x_n)\| \to 0 \text{ as } n \to \infty, \tag{1.14}
\]
and so $|\lambda| = 1$. Now, if $|\lambda| \neq 1$, then $\lambda \not\in \sigma_{ap}(T)$ and so $T - \lambda$ is bounded below. \hfill \square

**Corollary 1.3.** Suppose that $\mathcal{X}$ is an infinite dimensional Banach space. Then the operator $T$ on $\mathcal{X}$ is not a compact operator.

**Proof.** If $T$ is a compact operator, then $0 \in \sigma(T) = \sigma(A)$. Thus $\overline{\mathcal{D}} \subseteq \sigma(T)$ which contradicts the fact that the spectrum of a compact operator is at most countable. \hfill \square

**Proposition 1.4.** If the operators $T$ and $A$ are defined on a normed space $\mathcal{X}$, then $\ker(T - \lambda) \subseteq \ker(A - \lambda)$ for every scalar $\lambda$.

**Proof.** Fix $\lambda \in \mathbb{C}$ and suppose that $Tx = \lambda x$ for some nonzero vector $x$. By Proposition 1.1, $\lambda \in \sigma_p(A)$ which implies that $|\lambda| = 1$. Therefore, if $n > N - 1$, we have

$$
\|x\|^2 = \|T^n x\|^2 = \left\| A^{n-(N-1)} \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{-k} x \right\|^2 \quad (1.15)
$$

Consequently,

$$
\|x\| \geq \binom{n}{N-1} \left[ \|Q^{N-1} x\| - \sum_{k=0}^{N-2} \frac{(N-1)! (n-N+1)!}{k! (n-k)!} \|Q^k A^{-1+k} x\| \right]. \quad (1.16)
$$

Since

$$
\lim_{n \to \infty} \frac{(N-1)! (n-N+1)!}{k! (n-k)!} = 0 \quad (1.17)
$$

for every $0 \leq k \leq N - 2$, we conclude that $\|Q^{N-1} x\| = 0$. Continue the above process to get $Qx = 0$, and so $Ax = \lambda x$. \hfill \square

**Corollary 1.5.** If $\mathcal{X}$ is a Hilbert space, then the eigenvectors of $T$ corresponding to distinct eigenvalues are orthogonal.

**Proof.** Let $x$ and $y$ be eigenvectors of $T$ corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$. So, $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. By Proposition 1.4, $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ which implies that $|\lambda_1| = |\lambda_2| = 1$. Suppose that $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{X}$. Then

$$
0 = \|\lambda_1 x + \lambda_2 y\|^2 - \|x + y\|^2 = 2 \text{Re} \left( \lambda_1 \overline{\lambda_2} - 1 \right) \langle x, y \rangle. \quad (1.18)
$$
Replacing $y$ by $iy$, we obtain $\text{Im}(\lambda_1 \overline{\lambda}_2 - 1) \langle x, y \rangle = 0$; consequently,

$$\left( \frac{\lambda_1}{\lambda_2} - 1 \right) \langle x, y \rangle = \left( \lambda_1 \overline{\lambda}_2 - 1 \right) \langle x, y \rangle = 0. \quad (1.19)$$

But $\lambda_1 \neq \lambda_2$, and so $\langle x, y \rangle = 0$.

Recall that an operator $T$ is power bounded if there exists some constant $c > 0$ such that $\|T^n\| \leq c$ for all $n = 1, 2, 3, \ldots$.

**Proposition 1.6.** Let $X$ be a normed space and $x \in X$. If there is a constant $c > 0$ such that $\|T^n x\| \leq c$ for all $n \geq 1$, then $Qx = 0$. In particular, if $T$ is power bounded, then $Q = 0$.

**Proof.** Since the sequence $\{\|T^n x\|\}_n$ is bounded, an argument similar to the proof of the Proposition 1.4 shows that $Qx = 0$. \(\square\)

## 2. Supercyclicity and Hypercyclicity

We begin this section with a useful lemma.

**Lemma 2.1.** Let $X$ be a normed space. For nonnegative integers $k, n$, if

$$P_k(n) = x_0 + x_1 n + x_2 n^2 + \cdots + x_k n^k \quad (2.1)$$

is a polynomial in $n$ with coefficients in $X$ of degree $k$, then the sequence $\{\|P_k(n)\|\}_n$ is eventually increasing.

**Proof.** We prove the lemma by induction on $k$, the degree of the polynomial $P_k(n)$. For $k = 1$, let $P_1(n) = x_0 + x_1 n$. It is easily seen that for every $n \geq 1$

$$\|P_1(n + 1)\| \leq \frac{1}{2} (\|P_1(n)\| + \|P_1(n + 2)\|). \quad (2.2)$$

Since $\lim_{n \to \infty} \|P_1(n)\| = +\infty$, there is a positive integer $i$ such that

$$\|P_1(i)\| < \|P_1(i + 1)\|. \quad (2.3)$$

This fact coupled with (2.2) implies that

$$0 < \|P_1(i + 1)\| - \|P_1(i)\| \leq \|P_1(n + 1)\| - \|P_1(n)\| \quad (2.4)$$

for every $n \geq i$. Therefore, the sequence $\{\|P_1(n)\|\}_{n \geq i}$ is increasing. Suppose that $\{\|P_k(n)\|\}_n$ is eventually increasing and let

$$P_{k+1}(n) = x_0 + x_1 n + \cdots + x_{k+1} n^{k+1}, \quad (2.5)$$
where \( x_{k+1} \neq 0 \). Since
\[
\lim_{n \to \infty} \left\| x_1 + x_2(n + 1)^+ \cdots + x_{k+1}(n+1)^k \right\| = +\infty,
\] (2.6)

using the induction hypothesis there is a positive integer \( j \) such that for every \( n \geq j \)
\[
\left\| x_1 + x_2(n + 1)^+ \cdots + x_{k+1}(n+1)^k \right\| \geq \max \left\{ 2\|x_0\|, \left\| x_1 + x_2n^+ + \cdots + x_{k+1}n^k \right\| \right\}.
\] (2.7)

Therefore,
\[
\frac{\|P_{k+1}(n+1)\|}{\|P_{k+1}(n)\|} \geq \frac{(n+1)\left\| x_1 + x_2(n + 1)^+ \cdots + x_{k+1}(n+1)^k \right\| - \|x_0\|}{n\|x_1 + x_2n^+ + \cdots + x_{k+1}n^k\| + \|x_0\|} \geq 1
\] (2.8)

for every \( n \geq j \). Hence, the sequence \( (\|p_{k+1}(n)\|)_{n \geq j} \) is increasing.

\begin{proof}
Let \( \tilde{\mathcal{K}} \) be the completion of \( \mathcal{K} \) and \( \tilde{T}, \tilde{A}, \text{ and } \tilde{Q} \) the extensions of \( T, A, \text{ and } Q \) on \( \tilde{\mathcal{K}} \), respectively. Thus, \( \tilde{T} = \tilde{A} + \tilde{Q} \) where \( \tilde{A} \) is an isometry and \( \tilde{Q} \) is a nilpotent operator; moreover, \( \tilde{A} \tilde{Q} = \tilde{Q} \tilde{A} \). Also, note that the supercyclicity of the operator \( T \) implies the supercyclicity of \( \tilde{T} \). So we can assume, without loss of generality, that \( \mathcal{K} \) is a Banach space.

As we have seen in the proof of Proposition 1.4, if \( x \in \mathcal{K} \) then
\[
\|T^n x\| = \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|,
\] (2.9)

and so by Lemma 2.1, the sequence \( (\|T^n x\|)_n \) is eventually increasing. Suppose that \( x_0 \) is a supercyclic vector for \( T \). Thus, for any \( x \in \mathcal{K} \) there is a sequence \( (n_i)_i \) of positive integers and a sequence \( (\alpha_i)_i \) of scalars such that \( \alpha_i T^{n_i} x_0 \to x \). Moreover, since the sequence \( (\|T^n x_0\|)_n \) is eventually increasing, we have \( \|\alpha_i T^{n_i} x_0\| \leq \alpha_i T^{n_i+1} x_0 \) for large \( i \). So letting \( i \to \infty \), we conclude that \( \|x\| \leq \|Tx\|, \) for all \( x \in \mathcal{K} \). On the other hand, the supercyclicity of \( T \) implies that it has a dense range and so is invertible. Thus, in light of Proposition 1.1 we see that \( A \) is invertible. It is easy to see that
\[
T^{-1} = A^{-1} + P,
\] (2.10)

where
\[
P = \sum_{k=1}^{N-1} (-1)^k A^{-(k+1)} Q^k.
\] (2.11)
Since $P^N = 0$, by a similar argument the sequence $(\|T^{-n}x\|)_n$ is eventually increasing for every $x \in X$. But $T^{-1}$ is also supercyclic (see [1, Theorem 1.12]); therefore,

$$\|x\| \leq \|T^{-1}x\|$$

(2.12)

for every $x \in X$. Thus, $T$ is an isometry which implies that it is not a supercyclic operator.

To show that the operator $T$ is not weakly hypercyclic, note that

$$\|T^n x\| = \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{n-k} x^* \right\|$$

(2.13)

for every $x^* \in X^*$ and every positive integer $n$. If $\ker Q^* \neq \{0\}$, then there is a nonzero $x^* \in X^*$ such that $\|T^n x^*\| = \|A^* x^*\| \leq \|x^*\|$ because $\|A^*\| = \|A\| = 1$. Now, suppose that $x_0$ is a weakly hypercyclic vector for $T$. Since orb$(T, x_0)$ is weakly dense in $X$ and $x^*$ is nonzero, the set $\{x^*(T^n x_0) : n \geq 0\}$ is dense in $C$. But

$$\|x^*(T^n x_0)\| = \|(T^n x^*)(x_0)\| \leq \|T^n x^*\|||x_0|| \leq ||x^*|||x_0||$$

(2.14)

for all $n \geq 0$, which is a contradiction. If $\ker Q^* = \{0\}$, then $Q^* = 0$ and so $T = A$ is not a weakly hypercyclic operator. \qed

We remark that there are Banach space isometries which are also weakly supercyclic. Indeed, the unweighted bilateral weighted shift on the space $l^p(\mathbb{Z})$ where $p > 2$ is weakly supercyclic (see [1, Corollary 10.32]). However, the question that whether an isometry plus a nonzero nilpotent which commute with each other, are weakly supercyclic or not is still an open question.

The following examples show that the commutativity of $A$ and $Q$ is essential in the preceding theorem.

**Example 2.3.** Let $(e_n)_{n=-\infty}^{\infty}$ be the standard orthonormal basis for $l^2(\mathbb{Z})$. Define the isometric operator $A$ by $A e_n = e_{n+1}$ for all $n \in \mathbb{Z}$ and the weighted shift operator $Q$ by $Q e_n = \omega_n e_{n+1}$, where $\omega_{2n} = 0$ for all integers $n$, $\omega_{2n-1} = 1/(2n-1)^2$ for all $n \geq 1$, and $\omega_{2n-1} = 1/(2-2n)$ for all $n \leq 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Moreover, since $1 \leq \inf_n (1 + \omega_n) \leq \sup_n (1 + \omega_n) \leq 2$, the weighted shift operator $T = A + Q$ is invertible. To see that $T$ is supercyclic by Theorem 3.4 of [6], it is sufficient to show that

$$\lim_{n \to \infty} \prod_{j=1}^{n} (1 + \omega_j) \prod_{j=1}^{n} \frac{1}{1 + \omega_j} = 0.$$  

(2.15)
But $\prod_{j=1}^{\infty}(1 + w_j)$ is finite, because $\sum_{j=1}^{\infty} w_j < \infty$. Furthermore, $\prod_{j=1}^{\infty}1/(1 + w_j) = 0$, because

$$\sum_{j=1}^{\infty}\left(1 - \frac{1}{1 + w_j}\right) = \sum_{j=1}^{\infty} \frac{w_{-j}}{1 + w_{-j}} = \sum_{j=1}^{\infty} \frac{w_{-(2j-1)}}{1 + w_{-(2j-1)}}$$

$$= \sum_{j=1}^{\infty} \frac{1/(2j-1)}{1 + 1/(2j-1)} = \sum_{j=1}^{\infty} \frac{1}{2j} = \infty \quad (2.16)$$

(see [7, pages 299 and 300]). Therefore, (2.15) holds.

**Example 2.4.** Consider the isometric operator $A$ on $l^2(\mathbb{Z})$ defined by $Ae_n = e_{n-1}$ and the weighted shift operator $Q$ defined by $Qe_n = w_ne_{n-1}$, where $w_{2n} = 0$ for all $n \in \mathbb{Z}$, $w_{2n-1} = 1/(2n - 1)$, for $n \geq 1$, and $w_{2n-1} = 1/(2n - 1)^2$ for $n \leq 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Also, since

$$(1 + w_1)(1 + w_2) \cdots (1 + w_n) \geq w_1 + w_2 + \cdots + w_n \quad (2.17)$$

for all $n \geq 1$, and $\sum_{n=1}^{\infty} w_n = \infty$, we conclude that

$$\lim_{n \to \infty} (1 + w_1)(1 + w_2) \cdots (1 + w_n) = \infty. \quad (2.18)$$

Furthermore,

$$\lim_{n \to \infty} (1 + w_{-1})(1 + w_{-2}) \cdots (1 + w_{-n}) < \infty, \quad (2.19)$$

because

$$\sum_{n=1}^{\infty} w_{-n} < \infty. \quad (2.20)$$

Hence, using Corollary 10.27 of [1], we observe that the operator $A + Q$ is weakly hypercyclic.

## 3. A Co-isometry Plus a Nilpotent

From now on, we assume that $\mathcal{H}$ is a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Recall that the unilateral shift operator $S : \mathcal{H} \to \mathcal{H}$ is given by $Se_n = e_{n+1}$ for all $n$ and the backward shift operator $B : \mathcal{H} \to \mathcal{H}$ is defined by $Be_0 = 0$ and $Be_n = e_{n-1}$ for all $n \geq 1$. It is known that the operator $B$ is supercyclic (see [1, page 9]). It follows that a co-isometry can be supercyclic. In this section, we give sufficient conditions such that the sum of a co-isometry and a nilpotent is supercyclic on $\mathcal{H}$.

**Theorem 3.1.** Suppose that $A$ is a co-isometric operator on a Hilbert space $\mathcal{H}$. Then $A$ is supercyclic if and only if $\cap_{n \geq 0} A^n \mathcal{H} = \{0\}$.

**Proof.** First assume that $\cap_{n \geq 0} A^n \mathcal{H} = \{0\}$. Then by the von Neumann-Wold decomposition, $A^* = S^m$ for some positive integer $m$ (see [8]). Therefore, $A = B^m$ which is a positive
Hence, $g$ is a nonnegative integer $n$.

**Theorem 3.2.** Suppose that $\mathcal{H}$ is a supercyclic vector for $A|_M$ which is impossible.\[\square\]

To prove the next theorem, we need the supercyclicity criterion due to H. N. Salas (see [10], or more generally [11]).

**Supercyclicity Criterion**

Suppose that $X$ is a separable Banach space and $T$ is a bounded operator on $X$. If there is an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ and two dense sets $Y$ and $Z$ of $X$ such that

1. there exists a function $S : Z \rightarrow Z$ satisfying $TSx = x$ for all $x \in Z$,
2. $\|T^{n_k}x\| \cdot \|S^m y\| \rightarrow 0$ for every $x \in Y$ and $y \in Z$,

then $T$ is supercyclic.

**Theorem 3.2.** Suppose that $A$ is a co-isometry on a Hilbert space $\mathcal{H}$ such that $\cap_{n \geq 0} A^n \mathcal{H} = (0)$. If $AQ = QA$, then the operator $T = A + Q$ satisfies the supercyclicity criterion.

**Proof.** By Corollary 1.2, the operator $T^*$ is bounded below and so is left invertible. Consequently, $T$ is a right invertible operator. Let $x \in \cap_{n \geq 0} T^n \mathcal{H}$. For every $i \geq 0$, there is a vector $x_{N+i}$ in $\mathcal{H}$ such that $T^{N+i} x_{N+i} = x$. Since $Q^N = 0$, we have

\[
x = T^{N+i} x_{N+i} = \sum_{k=0}^{N+i} \binom{N+i}{k} A^k Q^N x_{N+i-k} = \sum_{k=i+1}^{N+i} \binom{N+i}{k} A^k Q^N x_{N+i-k}
\]

which implies that $x \in A^{N+i} \mathcal{H}$. Hence, $x \in \cap_{n \geq 0} T^n \mathcal{H} = (0)$ and so the operator $T$ admits a complete set of eigenvectors that is, $\mathcal{H} = \bigvee_{\mu \in \mathbb{B}} \ker(T - \mu)$ for every positive real number $r$, where $\mathbb{B} = \{ z \in \mathbb{C} : |z| < r \}$ (see [12], Part (A) of the lemma). Since $T^*$ is bounded below, $TT^*$ is invertible. Take $S = T^* (TT^*)^{-1}$. Choose $r > 0$ so that $r < 1/\|S\|$, and let

\[
Y = Z = \operatorname{span}\{ \ker(T - \mu) : \mu \in \mathbb{D}_r \}.
\]
Proposition 3.3. Suppose that

\[ \|T^n h\| \|S^n h\| \leq |\mu|^n \|S\| \|h\| \leq (r\|S\|)^n \|h\| \to 0 \]  

(3.4)

as \( n \to \infty \). Finally, \( T^n S^n h = h \) for every \( h \in \mathcal{L} \) and every \( n \geq 0 \). Thus, the operator \( T \) satisfies the supercyclicity criterion.

The Hilbert-Schmidt class, \( C_2(\mathcal{L}) \), is the class of all bounded operators \( S \) defined on a Hilbert space \( \mathcal{L} \), satisfying

\[ \|S\|_2^2 = \sum_{n=1}^{\infty} \|Se_n\|^2 < +\infty, \]  

(3.5)

where \( \| \cdot \| \) is the norm on \( \mathcal{L} \) induced by its inner product. We recall that \( C_2(\mathcal{L}) \) is a Hilbert space equipped with the inner product \( \langle S, T \rangle = \text{tr}(ST^*) \) in which \( \text{tr}(ST^*) \) denotes the trace of \( ST^* \). Furthermore, \( C_2(\mathcal{L}) \) is an ideal of the algebra of all bounded operators on \( \mathcal{L} \), see [8]. For any bounded operator \( B \) on a Hilbert space \( \mathcal{L} \), the left multiplication operator \( LB \) and the right multiplication operator \( RB \) on \( C_2(\mathcal{L}) \) are defined by \( LB(S) = BS \) and \( RB(S) = SB \) for all \( S \in C_2(\mathcal{L}) \). It is known that an operator \( B \) satisfies the supercyclicity criterion if and only if \( LB \) is supercyclic on the space \( C_2(\mathcal{L}) \) (see [13, page 37]). In the following proposition, we see that an operator \( T \) may satisfy the supercyclicity criterion although \( RT \) is not a supercyclic operator on \( C_2(\mathcal{L}) \).

**Proposition 3.3.** Suppose that \( \mathcal{L} \) is a Hilbert space and \( A \in \mathcal{B}(\mathcal{L}) \) is a co-isometry such that \( \cap_{n \geq 0} A^n \mathcal{L} = \{0\} \) and \( AQ = QA \). Then the operator \( T = A + Q \) satisfies the supercyclicity criterion but the operator \( RT \) is not supercyclic on \( C_2(\mathcal{L}) \).

**Proof.** By Theorem 3.2, the operator \( T \) satisfies the supercyclicity criterion. Moreover, for every \( S \in C_2(\mathcal{L}) \) we have

\[
\|RA(S)\|_2^2 = \|SA\|_2^2 = \|(SA)^*\|_2^2 = \|A*S^*\|_2^2 = \sum_{n=1}^{\infty} \|A*S^*e_n\|^2 = \sum_{n=1}^{\infty} \|S^*e_n\|^2 = \|S\|_2^2,
\]  

(3.6)

which implies that \( RA \) is an isometry. Also, if \( S \in C_2(\mathcal{L}) \), then \( R_A^N(S) = 0 \). Since \( RT(S) = R_A(S) + R_Q(S) \), Theorem 2.2 implies that \( RT \) is not supercyclic. \( \square \)

The proof of the following proposition is similar to the proof of the second part of Theorem 2.2, and we omit it.

**Proposition 3.4.** Suppose that \( \mathcal{L} \) is a normed space and \( A \in \mathcal{B}(\mathcal{L}) \) is a co-isometry such that \( AQ = QA \). Then the operator \( T = A + Q \) is not weakly hypercyclic.

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References


