Research Article

A Sharp Double Inequality between Harmonic and Identric Means

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We find the greatest value \( p \) and the least value \( q \) in \((0, 1/2)\) such that the double inequality

\[
H(pa + (1 - p)b, pb + (1 - p)a) < I(a, b) < H(qa + (1 - q)b, qb + (1 - q)a)
\]

holds for all \( a, b > 0 \) with \( a \neq b \). Here, \( H(a, b) \) and \( I(a, b) \) denote the harmonic and identric means of two positive numbers \( a \) and \( b \), respectively.

1. Introduction

The classical harmonic mean \( H(a, b) \) and identric mean \( I(a, b) \) of two positive numbers \( a \) and \( b \) are defined by

\[
H(a, b) = \frac{2ab}{a + b}, \quad \text{(1.1)}
\]

\[
I(a, b) = \begin{cases} 
\frac{1}{e} \left( \frac{b}{a} \right)^{1/(b-a)}, & a \neq b, \\
1, & a = b,
\end{cases} \quad \text{(1.2)}
\]

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for \( H \) and \( I \) can be found in the literature [1–17].

Let \( M_p(a, b) = [(a^p + b^p)/2]^{1/p} \), \( L(a, b) = (a - b)/(\log a - \log b) \), \( G(a, b) = \sqrt{ab} \), \( A(a, b) = (a + b)/2 \), and \( P(a, b) = (a - b)/[4 \arctan(\sqrt{a/b} - \pi)] \) be the \( p \)th power, logarithmic,
geometric, arithmetic, and Seiffert means of two positive numbers \( a \) and \( b \) with \( a \neq b \), respectively. Then it is well-known that

\[
\min\{a, b\} < H(a, b) = M_{1/3}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\
< P(a, b) < I(a, b) < A(a, b) = M_1(a, b) < \max\{a, b\}
\]

for all \( a, b > 0 \) with \( a \neq b \).

Long and Chu [18] answered the question: what are the greatest value \( p \) and the least value \( q \) such that \( M_p(a, b) < A^p(a, b)G^q(a, b)H^{1-p/q}(a, b) < M_q(a, b) \) for all \( a, b > 0 \) with \( a \neq b \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta < 1 \).

In [19], the authors proved that the double inequality

\[
\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \leq 2/\pi \) and \( \beta \geq 5/6 \).

The following sharp bounds for \( I, \ (LI)^{1/2}, \) and \( (L + I)/2 \) in terms of power means are presented in [20]:

\[
M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b), \quad M_0(a, b) < \sqrt{L(a, b)I(a, b)} < M_{1/2}(a, b), \\
M_{2/(1+\log 2)}(a, b) < \frac{L(a, b) + I(a, b)}{2} < M_{1/2}(a, b)
\]

for all \( a, b > 0 \) with \( a \neq b \).

Alzer and Qiu [21] proved that the inequalities

\[
\alpha A(a, b) + (1 - \alpha)G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta)G(a, b)
\]

hold for all positive real numbers \( a \) and \( b \) with \( a \neq b \) if and only if \( \alpha \leq 2/3 \) and \( \beta \geq 2/e = 0.73575 \), and so forth.

For fixed \( a, b > 0 \) with \( a \neq b \) and \( x \in [0, 1/2] \), let

\[
f(x) = H(xa + (1-x)b, xb + (1-x)a).
\]

Then it is not difficult to verify that \( f(x) \) is continuous and strictly increasing in \([0, 1/2]\). Note that \( f(0) = H(a, b) < I(a, b) \) and \( f(1/2) = A(a, b) > I(a, b) \). Therefore, it is natural to ask what are the greatest value \( p \) and the least value \( q \) in \((0, 1/2)\) such that the double inequality \( H(pa + (1-p)b, pb + (1-p)a) < I(a, b) < H(qa + (1-q)b, qb + (1-q)a) \) holds for all \( a, b > 0 \) with \( a \neq b \). The main purpose of this paper is to answer these questions. Our main result is Theorem 1.1.
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**Theorem 1.1.** If \( p, q \in (0, 1/2) \), then the double inequality

\[
H(pa + (1-p)b, pb + (1-p)a) < I(a, b) < H(qa + (1-q)b, qb + (1-q)a)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq (1 - \sqrt{1-2/e})/2 \) and \( q \geq (6 - \sqrt{6})/12 \).

**2. Proof of Theorem 1.1**

**Proof of Theorem 1.1.** Let \( \lambda = (6 - \sqrt{6})/12 \) and \( \mu = (1 - \sqrt{1-2/e})/2 \). Then from the monotonicity of the function \( f(x) = H(xa + (1-x)b, xb + (1-x)a) \) in \([0,1/2]\) we know that to prove inequality (1.8) we only need to prove that inequalities

\[
\begin{align*}
I(a, b) &< H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a), \\
I(a, b) &> H(\mu a + (1-\mu)b, \mu b + (1-\mu)a),
\end{align*}
\]

hold for all \( a, b > 0 \) with \( a \neq b \).

Without loss of generality, we assume that \( a > b \). Let \( t = a/b > 1 \) and \( r \in (0, 1/2) \), then from (1.1) and (1.2) one has

\[
\log H(ra + (1-r)b, rb + (1-r)a) - \log I(a, b) = \log \left\{ r(1-r)t^2 + \left[ r^2 + (1-r)^2 \right] t + r(1-r) \right\}
\]

\[
- \log(t+1) - \frac{t \log t}{t-1} + 1 + \log 2.
\]

Let

\[
g(t) = \log \left\{ r(1-r)t^2 + \left[ r^2 + (1-r)^2 \right] t + r(1-r) \right\}
\]

\[
- \log(t+1) - \frac{t \log t}{t-1} + 1 + \log 2.
\]

Then simple computations lead to

\[
g(1) = 0, \quad \lim_{t \to +\infty} g(t) = \log[r(1-r)] + 1 + \log 2,
\]

\[
g'(t) = \frac{g_1(t)}{(t-1)^2}.
\]
where

\[ g_1(t) = \log t - \frac{(t - 1) \left[ (2r^2 - 2r + 1)t^2 + 4r(1-r)t + 2r^2 - 2r + 1 \right]}{(t + 1)[r(1-r)t^2 + (2r^2 - 2r + 1)t + r(1-r)]}, \]  

(2.8)

\[ g_1(1) = 0, \]  

(2.9)

\[ \lim_{t \to +\infty} g_1(t) = +\infty, \]  

(2.10)

\[ g'_1(t) = \frac{g_2(t)}{t(t + 1)^2[r(1-r)t^2 + (2r^2 - 2r + 1)t + r(1-r)]^2}, \]  

(2.11)

where

\[ g_2(t) = r^2(1-r)^2t^5 + (2r^4 - 4r^3 - 2r^2 + 4r - 1)t^5 - (17r^4 - 34r^3 + 25r^2 - 8r + 1)t^4 \]
\[ + 4(7r^4 - 14r^3 + 13r^2 - 6r + 1)t^3 - (17r^4 - 34r^3 + 25r^2 - 8r + 1)t^2 \]
\[ + (2r^4 - 4r^3 - 2r^2 + 4r - 1)t + r^2(1-r)^2, \]  

(2.12)

\[ g_2(1) = 0, \]  

(2.13)

\[ \lim_{t \to +\infty} g_2(t) = +\infty, \]  

(2.14)

\[ g'_2(t) = 6r^2(1-r)^2t^5 + 5(2r^4 - 4r^3 - 2r^2 + 4r - 1)t^4 - 4(17r^4 - 34r^3 + 25r^2 - 8r + 1)t^3 \]
\[ + 12(7r^4 - 14r^3 + 13r^2 - 6r + 1)t^2 - 2(17r^4 - 34r^3 + 25r^2 - 8r + 1)t \]
\[ + 2r^4 - 4r^3 - 2r^2 + 4r - 1, \]  

(2.15)

\[ g'_2(1) = 0, \]  

(2.16)

\[ \lim_{t \to +\infty} g'_2(t) = +\infty, \]  

(2.17)

\[ g''_2(t) = 30r^2(1-r)^2t^4 + 20(2r^4 - 4r^3 - 2r^2 + 4r - 1)t^4 - 12(17r^4 - 34r^3 + 25r^2 - 8r + 1)t^3 \]
\[ + 24(7r^4 - 14r^3 + 13r^2 - 6r + 1)t^2 - 2(17r^4 - 34r^3 + 25r^2 - 8r + 1), \]  

(2.18)

\[ g''_2(1) = -2\left(24r^2 - 24r + 5\right), \]  

(2.19)

\[ \lim_{t \to +\infty} g''_2(t) = +\infty, \]  

(2.20)

\[ g'''_2(t) = 120r^2(1-r)^2t^3 + 60(2r^4 - 4r^3 - 2r^2 + 4r - 1)t^3 - 24(17r^4 - 34r^3 + 25r^2 - 8r + 1)t^2 \]
\[ - 24(7r^4 - 14r^3 + 13r^2 - 6r + 1)t + 24(7r^4 - 14r^3 + 13r^2 - 6r + 1), \]  

(2.21)

\[ g'''_2(1) = -12\left(24r^2 - 24r + 5\right), \]  

(2.22)

\[ \lim_{t \to +\infty} g'''_2(t) = \infty, \]  

(2.23)

\[ g^{(4)}_2(t) = 360r^2(1-r)^2t^2 + 120(2r^4 - 4r^3 - 2r^2 + 4r - 1)t \]
\[ - 24(17r^4 - 34r^3 + 25r^2 - 8r + 1), \]  

(2.24)
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\[
\begin{align*}
g_2^{(4)}(1) & = 48\left(4r^4 - 8r^3 - 10r^2 + 14r - 3 \right), \\
\lim_{t \to +\infty} g_2^{(4)}(t) & = +\infty, \\
g_2^{(5)}(t) & = 720r^2(1-r)^2t + 120\left(2r^4 - 4r^3 - 2r^2 + 4r - 1 \right), \\
g_2^{(5)}(1) & = 120\left(8r^4 - 16r^3 + 4r^2 + 4r - 1 \right).
\end{align*}
\] (2.25) (2.26) (2.27) (2.28)

We divide the proof into two cases.

**Case 1** \((r = \lambda = (6 - \sqrt{6})/12)\). Then (2.19), (2.22), (2.25), and (2.28) lead to

\[
\begin{align*}
g_2''(1) & = 0, \\
g_2'''(1) & = 0, \\
g_2^{(4)}(1) & = \frac{13}{3} > 0, \\
g_2^{(5)}(1) & = \frac{65}{3} > 0.
\end{align*}
\] (2.29) (2.30) (2.31) (2.32)

From (2.27) we clearly see that \(g_2^{(5)}(t)\) is strictly increasing in \([1, +\infty)\), then inequality (2.32) leads to the conclusion that \(g_2^{(5)}(t) > 0\) for \(t \in [1, +\infty)\), hence \(g_2^{(4)}(t)\) is strictly increasing in \([1, +\infty)\).

It follows from inequality (2.31) and the monotonicity of \(g_2^{(4)}(t)\) that \(g_2'''(t)\) is strictly increasing in \([1, +\infty)\). Then (2.30) implies that \(g_2''(t) > 0\) for \(t \in [1, +\infty)\), so \(g_2''(t)\) is strictly increasing in \([1, +\infty)\).

From (2.29) and the monotonicity of \(g_2''(t)\) we clearly see that \(g_2'(t)\) is strictly increasing in \([1, +\infty)\).

From (2.5), (2.7), (2.9), (2.11), (2.13), (2.16), and the monotonicity of \(g_2'(t)\) we conclude that

\[
g(t) > 0
\] (2.33)

for \(t \in (1, +\infty)\).

Therefore, inequality (2.1) follows from (2.3) and (2.4) together with inequality (2.33).

**Case 2** \((r = \mu = (1 - \sqrt{1 - 2/e})/2)\). Then (2.19), (2.22), (2.25), and (2.28) lead to

\[
\begin{align*}
g_2''(1) & = -\frac{2}{e}(5e - 12) < 0, \\
g_2'''(1) & = -\frac{12}{e}(5e - 12) < 0, \\
g_2^{(4)}(1) & = -\frac{48}{e^2}\left(3e^2 - 7e - 1 \right) < 0, \\
g_2^{(5)}(1) & = \frac{120}{e^2}\left(2 + 2e - e^2 \right) > 0.
\end{align*}
\] (2.34) (2.35) (2.36) (2.37)
From (2.27) and (2.37) we know that $g_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$. Then (2.26) and (2.36) lead to the conclusion that there exists $t_1 > 1$ such that $g_2^{(4)}(t) < 0$ for $t \in [1, t_1)$ and $g_2^{(4)}(t) > 0$ for $t \in (t_1, +\infty)$, hence $g_2'''(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

It follows from (2.23) and (2.35) together with the piecewise monotonicity of $g_2'''(t)$ that there exists $t_2 > t_1 > 1$ such that $g_2''(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$. Then (2.20) and (2.34) lead to the conclusion that there exists $t_3 > t_2 > 1$ such that $g_2'(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

From (2.16) and (2.17) together with the piecewise monotonicity of $g_2'(t)$ we clearly see that there exists $t_4 > t_3 > 1$ such that $g_2'(t) < 0$ for $t \in (1, t_4)$ and $g_2'(t) > 0$ for $t \in (t_4, +\infty)$. Therefore, $g_2(t)$ is strictly decreasing in $[1, t_4]$ and strictly increasing in $[t_4, +\infty)$. Then (2.11)–(2.14) lead to the conclusion that there exists $t_5 > t_4 > 1$ such that $g_1(t)$ is strictly decreasing in $[1, t_5]$ and strictly increasing in $[t_5, +\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $g_1(t)$ that there exists $t_6 > t_5 > 1$ such that $g(t)$ is strictly decreasing in $[1, t_6]$ and strictly increasing in $[t_6, +\infty)$.

Note that (2.6) becomes

$$
\lim_{t \to +\infty} g(t) = \log[r(1 - r)] + 1 + \log 2 = 0
$$

for $r = \mu = (1 - \sqrt{1 - 2/e})/2$.

From (2.5) and (2.38) together with the piecewise monotonicity of $g(t)$ we clearly see that

$$
g(t) < 0
$$

for $t \in (1, +\infty)$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with inequality (2.39).

Next, we prove that the parameter $\lambda = (6 - \sqrt{6})/12$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $r < \lambda = (6 - \sqrt{6})/12$, then (2.19) leads to $g_2''(1) = -2(24r^2 - 24r + 5) < 0$. From the continuity of $g_2''(t)$ we know that there exists $\delta > 0$ such that

$$
g_2''(t) < 0
$$

for $t \in (1, 1 + \delta)$.

It follows from (2.3)–(2.5), (2.7), (2.9), (2.11), (2.13), and (2.16) that $I(a, b) > H(ra + (1 - r)b, rb + (1 - r)a)$ for $a/b \in (1, 1 + \delta)$.

Finally, we prove that the parameter $\mu = (1 - \sqrt{1 - 2/e})/2$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $(1 - \sqrt{1 - 2/e})/2 = \mu < r < 1/2$, then (2.6) leads to $\lim_{t \to +\infty} g(t) > 0$. Hence, there exists $T > 1$ such that

$$
g(t) > 0
$$

for $t \in (T, +\infty)$.
Therefore, \( H(ra + (1 - r)b, rb + (1 - r)a) > I(a/b) \) for \( a/b \in (T, +\infty) \), follows from (2.3) and (2.4) together with inequality (2.41).

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## References


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