Research Article

The Centre of the Spaces of Banach Lattice-Valued Continuous Functions on the Generalized Alexandroff Duplicate

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Received 12 December 2010; Accepted 13 February 2011

Academic Editor: Yong Zhou

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We characterize the centre of the Banach lattice of Banach lattice $E$-valued continuous functions on the Alexandroff duplicate of a compact Hausdorff space $K$ in terms of the centre of $C(K,E)$, the space of $E$-valued continuous functions on $K$. We also identify the centre of $CD_0(Q,E) = C(Q,E) + c_0(Q,E)$ whose elements are the sums of $E$-valued continuous and discrete functions defined on a compact Hausdorff space $Q$ without isolated points, which was given by Alpay and Ercan (2000).

1. Preliminaries and Definitions

Throughout the paper, our terminology is mainly standard and a background on Riesz spaces and Banach lattices may be obtained from [1] or [2]. In order to avoid trivial cases, we assume that all topological spaces are nonempty and all Banach lattices are nonzero.

The centre of a Banach lattice $E$, denoted by $Z(E)$, is the lattice of the linear operators, $T : E \to E$ for which there exists a real number $\lambda > 0$ such that $|Tx| \leq \lambda |x|$ for all $x \in E$. The operator norm of a central operator $T$ is the minimum of those $\lambda$ with this property. It is well known that $Z(E)$ equipped with the operator norm is an AM-space with order unit. The order unit is identity operator $I$.

For a given locally compact Hausdorff space $K$ and a Banach lattice $E$, $C_0(K,E)$ denotes the space of all continuous functions $f$ from $K$ into $E$ which vanish at infinity; that is, there exists a compact set $A \subset K$ such that $\|f(k)\| < \varepsilon$ for each $\varepsilon > 0$ and $k \in K \setminus A$. We consider this space to be normed by

$$\|f\| = \sup\{\|f(k)\| : k \in K\},$$

(1.1)
and ordered by

\[ f \geq g \iff f(k) \geq g(k), \quad \forall k \in K. \] (1.2)

One can show that \( C_0(K, E) \) is a Banach lattice with these definitions.

Ercan and Wickstead [3] showed that the centre of \( C_0(K, E) \) is isometrically Riesz isomorphic to \( C^b(K, Z(E)_e) \) the space of all functions \( f \) from \( K \) into \( Z(E) \) such that \( f \) is norm bounded, continuous, and \( f(k_\alpha)(e) \to f(k)(e) \) in \( E \) for each \( e \in E \) whenever \( k_\alpha \to k \) in \( K \). Here, \( Z(E) \) is given the strong operator topology.

If \( K \) is a compact Hausdorff space, then \( C_0(K, E) = C(K, E) \), where \( C(K, E) \) is the space of continuous functions \( f : K \to E \). Hence, the centre of \( C(K, E) \) can also be identified with \( C^b(K, Z(E)_e) \). We will use this identification in the sequel.

If \( K \) is a discrete topological space, then \( C_0(K, E) \) is the space of \( E \)-valued bounded functions \( f \) on \( K \) such that the set

\[ \{ k \in K : \varepsilon < \| f(k) \| \} \] (1.3)

is finite for each \( \varepsilon > 0 \), and we will write \( c_0(K, E) \) in this case.

Let \( \Sigma \) and \( \Gamma \) be compact Hausdorff and locally compact Hausdorff topologies on a nonempty set \( K \), respectively, such that \( \Sigma \) is coarser than \( \Gamma \). These topologies on \( K \) will be denoted by \( K_\Sigma \) and \( K_\Gamma \). The compact Hausdorff topology on \( K \times \{0,1\} \) generated by the open base \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \), where

\[ \mathcal{A}_1 = \{ H \times \{1\} : H \text{ is } \Gamma\text{-open} \} , \]
\[ \mathcal{A}_2 = \{ G \times \{0,1\} \setminus M \times \{1\} : G \text{ is } \Sigma\text{-open}, M \text{ is } \Gamma\text{-compact} \} \] (1.4)

is called generalized Alexandroff duplicate of \( K \) and denoted by \( K_{\Sigma,\Gamma} \otimes \{0,1\} \) (see [4]). When \( \Gamma \) is discrete topology on \( K \), the compact Hausdorff topological space \( K_{\Sigma,\Gamma} \otimes \{0,1\} \) will be denoted by \( A(K) \). The space \( A(K) \) was first considered by Engelking [5]. For \( K = [0,1] \) under the usual metric topology, \( A(K) \) was constructed by Alexandroff and Urysohn [6] as an example of a compact Hausdorff space containing a discrete dense subspace. This space is called the Alexandroff duplicate.

Note that \( K \times \{0\} \) is a closed subspace of \( K_{\Sigma,\Gamma} \otimes \{0,1\} \) and the map \( k \to (k, 0) \) is a homeomorphism between \( K_\Sigma \) and \( K \times \{0\} \).

In [4], it is not proved that \( K_{\Sigma,\Gamma} \otimes \{0,1\} \) is a compact Hausdorff space. We give the proof here for the benefit of the reader.

**Theorem 1.1.** \( K_{\Sigma,\Gamma} \otimes \{0,1\} \) is a compact Hausdorff space.

**Proof.** Consider an open cover \( \{O_i\}_{i \in I} \) of \( K_{\Sigma,\Gamma} \otimes \{0,1\} \). By replacing each set in the cover by a union of basic open neighborhoods of all points in the set, we can assume that the cover is formed by basic open neighborhoods of the form

\[ \{ H_\alpha \times \{1\} \}_{\alpha \in \Lambda} \cup \{ G_\gamma \times \{0,1\} \setminus M_\gamma \times \{1\} \}_{\gamma \in \Omega} \] (1.5)
where \( H_s \) is a \( \Gamma \)-open set, \( G_s \) is a \( \Sigma \)-open set, and \( M_t \) is a \( \Gamma \)-compact set. It is easy to see that \( \{G_t \times \{0\}\}_{t \in \Omega} \) is an open cover of \( K \times \{0\} \), thus there is a finite subcover \( G_{t_1} \times \{0\}, \ldots, G_{t_n} \times \{0\} \). Then,

\[
G_{t_1} \times \{0,1\} \setminus M_{t_1} \times \{1\} \cup \cdots \cup G_{t_n} \times \{0,1\} \setminus M_{t_n} \times \{1\} \tag{1.6}
\]

misses only finitely many \( \Gamma \)-compact sets \( M_{t_1} \times \{1\}, \ldots, M_{t_n} \times \{1\} \).

As \( M_{t_j} \) \((j = 1, 2, \ldots, n)\) is compact, we have that \( M_{t_j} \times \{1\} \subset \cup H_{\alpha} \times \{1\} \). So, \( M_{t_j} \times \{1\} \subset \cup_{\alpha=1}^{n} H_{\alpha} \times \{1\} \). Hence, if we add the corresponding open sets from the cover, then we obtain a finite cover of the entire space \( K_{\Sigma,\Gamma} \times \{0,1\} \). Therefore, \( K_{\Sigma,\Gamma} \times \{0,1\} \) is compact.

To show that \( K_{\Sigma,\Gamma} \times \{0,1\} \) is Hausdorff, it is enough to show that \( (k,0) \) and \( (k,1) \) can be separated. Let \( V \) be a \( \Gamma \)-open neighborhood of \( k \) such that \( cl\Gamma(V) \) (closure of \( V \) in \( K_{\Gamma} \)) is compact. Then, \( K_{\Sigma,\Gamma} \times \{0,1\} \setminus (cl\Gamma(V) \times \{1\}) \) and \( V \times \{1\} \) are the separating open sets of \( (k,0) \) and \( (k,1) \), respectively. This completes the proof.

If \( K_{\Sigma} \) is a compact Hausdorff space without isolated points and \( K_{\Gamma} \) is a discrete topological space, then \( C(K_{\Sigma},E) \cap c_{0}(K_{\Gamma},E) = \{0\} \) and \( CD_{0}(K_{\Sigma},E) = C(K_{\Sigma},E) \oplus c_{0}(K_{\Gamma},E) \) is a Banach lattice under the pointwise ordering and supremum norm of the sums \( f + d \), where \( f \in C(K_{\Sigma},E) \) and \( d \in c_{0}(K_{\Gamma},E) \). We refer to [7-9] for more detailed information on these spaces. In [4], it is showed that \( CD_{0}(K_{\Sigma},E) \) is isometrically Riesz isomorphic to \( C(A(K),E) \), where \( A(K) \) is the Alexandroff duplicate of \( K \). We will use this identification in the sequel to characterize the centre of the space \( CD_{0}(K_{\Sigma},E) \).

2. Main Results

Let \( \Sigma \) and \( \Gamma \) be compact Hausdorff and locally compact Hausdorff topologies on \( K \), respectively, such that \( \Sigma \) is coarser than \( \Gamma \), and let \( E \) be a Banach lattice. Then \( C^{b}(K_{\Sigma},Z(E)_s) \) denotes the set of all norm bounded and continuous functions \( f \) from \( K \) into \( Z(E) \) such that \( r_{a}f(k_{s})(e) \rightarrow r_{f}(k)(e) \) in \( E \) for each \( e \in E \) whenever \( (k_{s},r_{a}) \rightarrow (k,r) \) in \( K_{\Sigma,\Gamma} \times \{0,1\} \).

We consider the vector space \( C^{b}(K_{\Sigma},Z(E)_s) \times C^{b}(K_{\Sigma},Z(E)_s) \) equipped with coordinatewise algebraic operations, the order

\[
0 \leq (f,d) \iff 0 \leq f(k)(e), \quad 0 \leq f(k)(e) + d(k)(e) \quad \text{for each} \quad k \in K,
\]

and the norm

\[
\|(f,d)\| = \max\{\|f(k) + rd(k)\| : (k,r) \in K \times \{0,1\}\}. \tag{2.2}
\]

The norm defined on \( C^{b}(K_{\Sigma},Z(E)_s) \times C^{b}(K_{\Sigma},Z(E)_s) \) makes it a Banach space. This is clear, as this norm is equivalent to standard products norms (we have, e.g., \((1/2)\max\{\|f\|,\|d\|\} \leq \|(f,d)\| \leq (\|f\| + \|d\|)\)). This has no relation to Banach lattices, but it is just a property of Banach spaces. The space \( C^{b}(K_{\Sigma},Z(E)_s) \times C^{b}(K_{\Sigma},Z(E)_s) \) is a lattice. This is proved by computing \( \|(f,d)\| = (\|f\|, |f| + |d| - |f|) \), where the absolute values on the right-hand side are pointwise. The norm defined on \( C^{b}(K_{\Sigma},Z(E)_s) \times C^{b}(K_{\Sigma},Z(E)_s) \) is a Riesz norm. This is obvious from definitions. Therefore, the space \( C^{b}(K_{\Sigma},Z(E)_s) \times C^{b}(K_{\Sigma},Z(E)_s) \) is a Banach lattice.
Actually, the space $C^b(K_{\Sigma}, Z(E)_s) \times C^b(K_{\Sigma}, Z(E)_s)$ is isometrically Riesz isomorphic to $C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E))$ the space of norm bounded, continuous functions $f$ from $K \times [0, 1]$ into $Z(E)$ such that $f(k, r)(e) \to f(k, r)(e)$ in $E$ for each $e \in E$ whenever $(k, r) \to (k, r)$ in $K_{\Sigma, \Gamma} \otimes [0, 1]$ as the following shows.

**Theorem 2.1.** $C^b(K_{\Sigma}, Z(E)_s) \times C^b(K_{\Sigma}, Z(E)_s)$ and $C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s)$ are isometrically Riesz isomorphic spaces.

**Proof.** Define the map

$$\pi : C^b(K_{\Sigma}, Z(E)_s) \times C^b(K_{\Sigma}, Z(E)_s) \to C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s),$$

by

$$\pi(f, d)(k, r)(e) = f(k)(e) + rd(k)(e),$$

for each $(k, r) \in K \times [0, 1]$ and $e \in E$.

Let $(k_\alpha, r_\alpha) \to (k, r)$ in $K_{\Sigma, \Gamma} \otimes [0, 1]$. Then, $k_\alpha \to k$ in $K_{\Sigma}$ so that $f(k_\alpha)(e) \to f(k)(e)$ and $r_\alpha d(k_\alpha)(e) \to rd(k)(e)$ in $E$ for each $e \in E$. Hence, $f(k_\alpha)(e) + r_\alpha d(k_\alpha)(e) \to f(k)(e) + rd(k)(e)$ in $E$ for each $e \in E$ so that the map $\pi$ is well defined. It follows immediately that $\pi$ is an isometry, as $\pi(f, d)$ agrees with $f + d$ on $K \times [1]$ and with $f$ on $K \times \{0\}$. It is obvious that $\pi(f, d) \geq 0 \iff (f, d) \geq 0$.

It remains to show that $\pi$ is onto. Let $h \in C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s)$ be given. Define

$$f(k)(e) = h(k, 0)(e), \quad d(k)(e) = h(k, 1)(e) - h(k, 0)(e),$$

for each $k \in K$ and $e \in E$. The norm boundedness of $f$ and $d$ follows directly from the norm boundedness of $h$. If $k_\alpha \to k$ in $K_{\Sigma}$, then $(k_\alpha, 0) \to (k, 0)$ in $K_{\Sigma, \Gamma} \otimes [0, 1]$ so that

$$f(k_\alpha)(e) = h(k_\alpha, 0)(e) \to h(k, 0)(e) = f(k)(e),$$

in $E$ for each $e \in E$, hence $f \in C^b(K_{\Sigma}, Z(E)_s)$.

To show that $d \in C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s)$, let $(k_\alpha, r_\alpha) \to (k, r) \in K_{\Sigma, \Gamma} \otimes [0, 1]$. We now examine the possibilities.

Suppose first that $r = 1$. Then, $(r_\alpha)$ is eventually 1. As $(k_\alpha, 0) \to (k, 0)$ in $K_{\Sigma, \Gamma} \otimes [0, 1]$, we have $r_\alpha d(k_\alpha)(e) \to rd(k)(e)$ in $E$ for each $e \in E$ in this possibility.

Suppose now that $(k_\alpha, r_\alpha) \to (k, 0)$ and assume that $r_\alpha d(k_\alpha)(e)$ does not converge to zero in $E$. Then, there is a subnet $(r_{\alpha})$ of $(r_\alpha)$ such that $r_{\alpha} = 1$ and $e < \|d(k_{\alpha})(e)\|$ for each $\beta$ and for some $e > 0$. On the other hand, since $(k_{\alpha}, 1) \to (k, 0)$ and $(k_{\alpha}, 0) \to (k, 0)$ in $K_{\Sigma, \Gamma} \otimes [0, 1]$, we have $h(k_{\alpha}, 1)(e) \to h(k, 0)(e)$ and $h(k_{\alpha}, 0)(e) \to h(k, 0)(e)$ so that $d(k_{\alpha})(e) = h(k_{\alpha}, 1)(e) - h(k_{\alpha}, 0)(e) \to 0$. This contradiction shows that $d \in C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s)$. It is clear that $\pi(f, d) = h$, and this completes the proof.

Since $Z(C(K_{\Sigma, \Gamma}, E))$ and $Z(C(K_{\Sigma, \Gamma} \otimes [0, 1], E))$ can be identified with $C^b(K_{\Sigma}, Z(E)_s)$ and $C^b(K_{\Sigma, \Gamma} \otimes [0, 1], Z(E)_s)$, respectively, we immediately have the following from the previous theorem.
Corollary 2.2. \(Z(C(K \otimes \{0,1\}, E) \text{ and } Z(C(K_{\Sigma}, E)) \times C^b(K_{\Sigma}, Z(E)), \) are isometrically Riesz isomorphic spaces.

Let \(K_{\Sigma} \) be a discrete topology, and let \(E \) be a Banach lattice. The set of all bounded functions \(f : K \rightarrow Z(E) \) such that the set \(\{k : \varepsilon < \|f(k)\| \text{ for all } e \in E\} \) is finite will be denoted by \(c_0(K_{\Sigma}, Z(E))\).

Lemma 2.3. Let \(K_{\Sigma} \) be a compact Hausdorff space, and let \(\Gamma \) be a discrete topology on \(K\). Then, \(C^b(K_{\Sigma}, Z(E)) = c_0(K_{\Gamma}, Z(E))\).

Proof. Let \(f \in c_0(K_{\Gamma}, Z(E)). \) Suppose that \(f \notin C^b(K_{\Sigma}, Z(E))\). Then, there exists a net \((k_n, 1) \rightarrow (k, 0) \in A(K)\) and \(\varepsilon < \|f(k_n)(e)\|\) for some subnet \((k_{n_\eta})\) of \((k_n)\), \(\varepsilon > 0\), and for each \(e \in E\). So, \((k_{n_\eta})\) has finite range which is a contradiction. Conversely, assume that \(f \in C^b(K_{\Sigma}, Z(E))\) but \(f \notin c_0(K_{\Gamma}, Z(E))\). Then, there exist some \(e \in E\) and a sequence \((k_n)\) such that \(\varepsilon < \|f(k_n)(e)\|\) for each \(n\) and \(k_n \neq k_m\) whenever \(n \neq m\). Then, there exists a subnet \((k_{n_\eta})\) of \(k_n\) such that \((k_{n_\eta}, 1) \rightarrow (k, 0)\) so that \(f(k_{n_\eta})(e) \rightarrow 0\) which is impossible and this completes the proof.

By Theorem 2.1 and the previous lemma, we have the following.

Theorem 2.4. Let \(K_{\Sigma} \) be a compact Hausdorff space, and let \(\Gamma \) be a discrete topology on \(K\). Then, \(C^b(A(K), Z(E)) \times c_0(K_{\Gamma}, Z(E))\) and \(C^b(K_{\Sigma}, Z(E)) \times c_0(K_{\Gamma}, Z(E))\) are isometrically Riesz isomorphic spaces.

As the centre of \(CD_0(K_{\Sigma}, E)\) can be identified with \(C^b(A(K), Z(E))\), we immediately have Theorem 3.1 of [8] as follows.

Corollary 2.5. Let \(K_{\Sigma} \) be a compact Hausdorff space without isolated points, and let \(\Gamma \) be a discrete topology on \(K\). Then, the centre of \(CD_0(K_{\Sigma}, E) \times Z(C(K_{\Sigma}, E)) \times c_0(K_{\Gamma}, Z(E))\) isometrically Riesz isomorphic spaces.

Note that in the corollary above, if all the operators \(T \in Z(E)\) are norm attaining; that is, there exists some \(e \in E\) with \(\|e\| = 1\) such that \(\|T\| = \|T(e)\|\), then \(c_0(K_{\Gamma}, Z(E))\) can be replaced by \(c_0(K_{\Gamma}, Z(E))\).

References

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