Research Article

Approximate Best Proximity Pairs in Metric Space

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Let A and B be nonempty subsets of a metric space X and also T : A ∪ B → A ∪ B and T(A) ⊆ B, T(B) ⊆ A. We are going to consider element x ∈ A such that d(x, Tx) ≤ d(A, B) + ε for some ε > 0. We call pair (A, B) an approximate best proximity pair. In this paper, definitions of approximate best proximity pair for a map and two maps, their diameters, T-minimizing a sequence are given in a metric space.

1. Introduction

Let X be a metric space and A and B nonempty subsets of X, and d(A, B) is distance of A and B. If d(x0, y0) = d(A, B), then the pair (x0, y0) is called a best proximity pair for A and B and put

\[ \text{prox}(A, B) := \{ (x, y) \in A \times B : d(x, y) = d(A, B) \} \]  \hspace{1cm} (1.1)

as the set of all best proximity pair (A, B). Best proximity pair evolves as a generalization of the concept of best approximation. That reader can find some important result of it in [1–4].

Now, as in [5] (see also [4, 6–11]), we can find the best proximity points of the sets A and B, by considering a map T : A ∪ B → A ∪ B such that T(A) ⊆ B and T(B) ⊆ A. Best proximity pair also evolves as a generalization of the concept of fixed point of mappings. Because if A ∩ B ≠ ∅, every best proximity point is a fixed point of T.

We say that the point x ∈ A ∪ B is an approximate best proximity point of the pair (A, B), if d(x, Tx) ≤ d(A, B) + ε, for some ε > 0.

In the following, we introduce a concept of approximate proximity pair that is stronger than proximity pair.
Definition 1.1. Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \) and \( T : A \cup B \to A \cup B \) a map such that \( T(A) \subseteq B, T(B) \subseteq A \). Let

\[
P^n_T(A,B) = \{ x \in A \cup B : d(x,Tx) \leq d(A,B) + \varepsilon \text{ for some } \varepsilon > 0 \}. \tag{1.2}\]

We say that the pair \((A,B)\) is an approximate best proximity pair if \(P^n_T(A,B) \neq \emptyset\).

Example 1.2. Suppose that \( X = \mathbb{R}^2 \), \( A = \{ (x,y) \in X : (x-y)^2 + y^2 \leq 1 \} \), and \( B = \{ (x,y) \in X : (x+y)^2 + y^2 \leq 1 \} \) with \( T(x,y) = (-x,y) \) for \((x,y) \in X\). Then \( d((x,y),T(x,y)) \leq d(A,B) + \varepsilon \) for some \( \varepsilon > 0 \). Hence \( P^n_T(A,B) \neq \emptyset \).

2. Approximate Best Proximity

In this section, we will consider the existence of approximate best proximity points for the map \( T : A \cup B \to A \cup B \), such that \( T(A) \subseteq B, T(B) \subseteq A \), and its diameter.

Theorem 2.1. Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \), and

\[
\lim_{n \to \infty} d\left(T^n x, T^{n+1} x\right) = d(A,B) \quad \text{for some } x \in A \cup B. \tag{2.1}\]

Then the pair \((A,B)\) is an approximate best proximity pair.

Proof. Let \( \varepsilon > 0 \) be given and \( x \in A \cup B \) such that \( \lim_{n \to \infty} d(T^n x, T^{n+1} x) = d(A,B) \); then there exists \( N_0 > 0 \) such that

\[
\forall n \geq N_0 : d\left(T^n x, T^{n+1} x\right) < d(A,B) + \varepsilon. \tag{2.2}\]

If \( n = N_0 \), then \( d(T^{N_0}(x), T(T^{N_0}(x))) < d(A,B) + \varepsilon \), and \( T^{N_0}(x) \in P^n_T(A,B) \) and \( P^n_T(A,B) \neq \emptyset \).

Theorem 2.2. Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \) and

\[
d(Tx,Ty) \leq \alpha d(x,y) + \beta [d(x,Tx) + d(y,Ty)] + \gamma d(A,B) \tag{2.3}\]

for all \( x, y \in A \cup B \), where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + \gamma < 1 \). Then the pair \((A,B)\) is an approximate best proximity pair.

Proof. If \( x \in A \cup B \), then

\[
d(Tx, T^2 x) \leq \alpha d(x,Tx) + \beta [d(x,Tx) + d(Tx,T^2 x)] + \gamma d(A,B). \tag{2.4}\]
Therefore,
\[ d(Tx, T^2x) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B). \] (2.5)

Now if \( k = (\alpha + \beta) / (1 - \beta) \), then
\[ d(Tx, T^2x) \leq kd(x, Tx) + (1 - k)d(A, B) \] (2.6)

also
\[ d(T^2x, T^3x) \leq k^2d(x, Tx) + (1 - k^2)d(A, B). \] (2.7)

Therefore,
\[ d(T^n x, T^{n+1}x) \leq k^n d(x, Tx) + (1 - k^n)d(A, B), \] (2.8)

and so
\[ d(T^n x, T^{n+1}x) \to d(A, B) \text{ as } n \to \infty. \] (2.9)

Therefore, by Theorem 2.1, \( P_T^n(A, B) \neq \emptyset \); then pair \((A, B)\) is an approximate best proximity pair.

**Definition 2.3.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \). We say that the sequence \( \{z_n\} \subseteq A \cup B \) is \( T \)-minimizing if

\[ \lim_{n \to \infty} d(z_n, Tz_n) = d(A, B). \] (2.10)

**Theorem 2.4.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \), suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \). If \( \{T^n x\} \) is a \( T \)-minimizing for some \( x \in A \cup B \), then \((A, B)\) is an approximate best pair proximity.

**Proof.** Since
\[ \lim_{n \to \infty} d(T^n x, T^{n+1}x) = d(A, B) \text{ for some } x \in A \cup B, \] (2.11)

therefore, by Theorem 2.1, \( P_T^n(A, B) \neq \emptyset \); then pair \((A, B)\) is an approximate best proximity pair.
Theorem 2.5. Let $A$ and $B$ be nonempty subsets of a normed space $X$ such that $A \cup B$ is compact. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, $T$ is continuous and

$$\|Tx - Ty\| \leq \|x - y\|,$$  \hspace{1cm} (2.12)

where $(x, y) \in A \times B$. Then $P^a_T(A, B)$ is nonempty and compact.

Proof. Since $A \cup B$ compact, there exists a $z_0 \in A \cup B$ such that

$$\|z_0 - Tz_0\| = \inf_{z \in A \cup B} \|z - Tz\|.$$

If $\|z_0 - Tz_0\| > d(A, B)$, then $\|Tz_0 - T^2z_0\| < \|z_0 - Tz_0\|$ which contradict to the definition of $z_0$, $(Tz_0 \in A \cup B$ and by ($\ast$) $\|Tz_0 - T(Tz_0)\| \geq \|z_0 - Tz_0\|$). Therefore, $\|z_0 - Tz_0\| = d(A, B) \leq d(A, B) + \varepsilon$ for some $\varepsilon > 0$ and $z_0 \in P^a_T(A, B)$. Therefore, $P^a_T(A, B)$ is nonempty.

Also, if $\{z_n\} \subseteq P^a_T(A, B)$, then $\|z_n - Tz_n\| < d(A, B) + \varepsilon$, for some $\varepsilon > 0$, and by compactness of $A \cup B$, there exists a subsequence $z_{n_k}$ and a $z_0 \in A \cup B$ such that $z_{n_k} \to z_0$ and so

$$\|z_0 - Tz_0\| = \lim_{k \to \infty} \|z_{n_k} - Tz_{n_k}\| < d(A, B) + \varepsilon$$  \hspace{1cm} (2.13)

for some $\varepsilon > 0$, hence $P^a_T(A, B)$ is compact. \hfill $\Box$

Example 2.6. If $A = [-3, -1], B = [1, 3]$, and $T : A \cup B \to A \cup B$ such that

$$T(x) = \begin{cases} 
\frac{1 - x}{2}, & x \in A, \\
-\frac{1 - x}{2}, & x \in B,
\end{cases}$$  \hspace{1cm} (2.14)

then $P^a_T(A, B)$ is compact, and we have

$$P^a_T(A, B) = \{x \in A \cup B : d(x, Tx) < d(A, B) + \varepsilon \text{ for some } \varepsilon > 0\}$$

$$= \{x \in A \cup B : d(x, Tx) < 2 + \varepsilon \text{ for some } \varepsilon > 0\}$$  \hspace{1cm} (2.15)

$$= \{1, -1\}.$$

That is compact.

In the following, by $\text{diam}(P^a_T(A, B))$ for a set $P^a_T(A, B) \neq \emptyset$, we will understand the diameter of the set $P^a_T(A, B)$. 

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**Definition 2.7.** Let \( T : A \cup B \to A \cup B \) be a continuous map such that \( T(A) \subseteq B \), \( T(B) \subseteq A \) and \( \epsilon > 0 \). We define diameter \( P^a_T(A,B) \) by

\[
\text{diam}(P^a_T(A,B)) = \sup \{ d(x,y) : x,y \in P^a_T(A,B) \}. \tag{2.16}
\]

**Theorem 2.8.** Let \( T : A \cup B \to A \cup B \), such that \( T(A) \subseteq B \), \( T(B) \subseteq A \) and \( \epsilon > 0 \). If there exists an \( \alpha \in [0,1] \) such that for all \( (x,y) \in A \times B \)

\[
d(Tx,Ty) \leq ad(x,y), \tag{2.17}
\]

then

\[
\text{diam}(P^a_T(A,B)) \leq \frac{2\epsilon}{1-\alpha} + \frac{2d(A,B)}{1-\alpha}. \tag{2.18}
\]

**Proof.** If \( x,y \in P^a_T(A,B) \), then

\[
d(x,y) \leq d(x,Tx) + d(Tx,Ty) + d(Ty,y) \leq \epsilon_1 + ad(x,y) + 2d(A,B) + \epsilon_2.
\]

Put \( \epsilon = \text{Max}\{\epsilon_1,\epsilon_2\} \), therefore, \( d(x,y) \leq 2\epsilon/(1-\alpha) + (2d(A,B))/\alpha \). Hence \( \text{diam}(P^a_T(A,B)) \leq 2\epsilon/(1-\alpha) + (2d(A,B))/\alpha \). \( \square \)

### 3. Approximate Best Proximity for Two Maps

In this section, we will consider the existence of approximate best proximity points for two maps \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \), and its diameter.

**Definition 3.1.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X,d) \) and let \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \) two maps such that \( T(A) \subseteq B \), \( S(B) \subseteq A \). A point \( (x,y) \in A \times B \) is said to be an approximate-pair fixed point for \( (T,S) \) in \( X \) if there exists \( \epsilon > 0 \)

\[
d(Tx,Sy) \leq d(A,B) + \epsilon. \tag{3.1}
\]

We say that the pair \( (T,S) \) has the approximate-pair fixed property in \( X \) if \( P^a_{(T,S)}(A,B) \neq \emptyset \), where

\[
P^a_{(T,S)}(A,B) = \{ (x,y) \in A \times B : d(Tx,Sy) \leq d(A,B) + \epsilon \text{ for some } \epsilon > 0 \}. \tag{3.2}
\]

**Theorem 3.2.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X,d) \) and let \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \) two maps such that \( T(A) \subseteq B \), \( S(B) \subseteq A \). If, for every \( (x,y) \in A \times B \),

\[
d(T^n(x),S^n(y)) \to d(A,B), \tag{3.3}
\]

then \( (T,S) \) has the approximate-pair fixed property.
Proof. For $\epsilon > 0$, Suppose $(x, y) \in A \times B$. Since

$$d(T^n(x), S^n(y)) \rightarrow d(A, B),$$

then $d(T(T^{n-1}(x), S(S^{n-1}(y))) < d(A, B) + \epsilon$ for every $n \geq n_0$. Put $x_0 = T^{n_0-1}(x)$ and $y_0 = S^{n_0-1}(y))$. Hence $d(T(x_0), S(y_0)) \leq d(A, B) + \epsilon$ and $P^a_{(T, S)}(A, B) \neq \emptyset$. \hfill $\square$

**Theorem 3.3.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and let $T : A \cup B \rightarrow A \cup B$ and $S : A \cup B \rightarrow A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$ and, for every $(x, y) \in A \times B$,

$$d(Tx, Sy) \leq \alpha d(x, y) + \beta [d(x, Tx) + d(y, Sy)] + \gamma d(A, B),$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$. Then if $x$ is an approximate fixed point for $T$, or $y$ is an approximate fixed point for $S$, then $P^a_{(T, S)}(A, B) \neq \emptyset$.

**Proof.** If $(x, y) \in A \times B$, then

$$d(Tx, S(Tx)) \leq \alpha d(x, Tx) + \beta [d(x, Tx) + d(Tx, S(Tx))] + \gamma d(A, B).$$

Therefore,

$$d(Tx, S(Tx)) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B).$$

(3.7)

Now if $k = (\alpha + \beta)/(1 - \beta)$, then

$$d(Tx, S(Tx)) \leq kd(x, Tx) + (1 - k)d(A, B) \tag{\star}$$

also

$$d(Sy, T(Sy)) \leq kd(y, Sy) + (1 - k)d(A, B). \tag{\star\star}$$

If $x$ is an approximate fixed point for $T$, then there exists a $\epsilon > 0$ and by (\star)

$$d(Tx, S(Tx)) \leq kd(x, Tx) + (1 - k)d(A, B)$$

$$\leq k(d(A, B) + \epsilon) + (1 - k)d(A, B)$$

$$= d(A, B) + k\epsilon$$

$$< d(A, B) + \epsilon.$$
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And \((x, Tx) \in P_{(T,S)}^a(A, B)\); also if \(y\) is an approximate fixed point for \(S\), then there exists a \(\epsilon > 0\) and by (**)

\[
d(Sy, T(Sy)) \leq kd(y, Sy) + (1 - k)d(A, B)
\]

\[
\leq k(d(A, B) + \epsilon) + (1 - k)d(A, B)
\]

\[
= d(A, B) + k\epsilon
\]

\[
< d(A, B) + \epsilon.
\]

And \((y, Sy) \in P_{(T,S)}^a(A, B)\). Therefore, \(P_{(T,S)}^a(A, B) \neq \emptyset\).

**Theorem 3.4.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and let \(T : A \cup B \to A \cup B\)

and \(S : A \cup B \to A \cup B\) be two continuous maps such that \(T(A) \subseteq B\), \(S(B) \subseteq A\). If, for every

\((x, y) \in A \times B\),

\[
d(Tx, Sy) \leq ad(x, y) + \gamma d(A, B),
\]

where \(\alpha, \gamma \geq 0\) and \(\alpha + \gamma = 1\), also let \(\{x_n\}\) and \(\{y_n\}\) be as follows:

\[
x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n \quad \text{for some} \quad (x_1, y_1) \in A \times B, \quad n \in N.
\]

If \(\{x_n\}\) has a convergent subsequence in \(A\), then there exists a \(x_0 \in A\) such that \(d(x_0, Tx_0) = d(A, B)\).

**Proof.** We have

\[
d(x_{n+1}, y_{n+1}) = d(Tx_n, Sy_n)
\]

\[
\leq ad(x_n, y_n) + \gamma d(A, B)
\]

\[
\leq \cdots
\]

\[
\leq a^{n+1}d(x_0, y_0) + (1 + \alpha + \cdots + a^n)\gamma d(A, B).
\]

If \(\{x_{n_k}\}_{k \geq 1}\) converges to \(x_1 \in A\), that is, \(x_{n_k} \to x_1\), then

\[
d(x_{n_{k+1}}, y_{n_{k+1}}) \leq a^{n+1}d(x_0, y_0) + (1 + \alpha + \cdots + a^n)\gamma d(A, B).
\]

Since \(T\) is continuous, then

\[
d(x_{n_{k+1}}, Tx_{n_k}) \longrightarrow \frac{\gamma}{1 - \alpha}d(A, B) = d(A, B).
\]

Therefore, \(d(x_1, Tx_1) = d(A, B)\).  \(\square\)
Definition 3.5. Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be continuous maps such that $T(A) \subseteq B$ and $S(B) \subseteq A$. We define diameter $P_{(T,S)}^a(A,B)$ by

$$\text{diam}\left(P_{(T,S)}^a(A,B)\right) = \sup \{d(x,y) : d(Tx,Ty) \leq \varepsilon + d(A,B) \text{ for some } \varepsilon > 0\}.$$  \hfill (3.15)

Example 3.6. Suppose $A = \{(x,0) : 0 \leq x \leq 1\}$, $B = \{(x,1) : 0 \leq x \leq 1\}$, $T(x,0) = T(x,1) = (1/2,1)$, and $S(x,1) = S(x,0) = (1/2,0)$. Then $d(T(x,0),S(y,1)) = 1$ and $\text{diam}(P_{(T,S)}^a(A,B)) = \text{diam}(A \times B) = \sqrt{2}$.

Theorem 3.7. Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be continuous maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If there exists a $k \in [0,1]$,

$$d(x,Tx) + d(Sy,y) \leq kd(x,y),$$  \hfill (3.16)

then

$$\text{diam}\left(P_{(T,S)}^a(A,B)\right) \leq \frac{\varepsilon}{1-k} + \frac{d(A,B)}{1-k} \text{ for some } \varepsilon > 0.$$  \hfill (3.17)

Proof. If $(x,y) \in P_{(T,S)}^a(A,B)$, then

$$d(x,y) \leq d(x,Tx) + d(Tx,Sy) + d(Sy,y)$$

$$\leq \varepsilon + kd(x,y) + d(A,B).$$  \hfill (3.18)

Therefore, $d(x,y) \leq \varepsilon/(1-k) + (d(A,B))/(1-k)$. Then $\text{diam}(P_{(T,S)}^a(A,B)) \leq \varepsilon/(1-k) + (d(A,B))/(1-k)$.  

References


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