Research Article

A Fixed Point Approach to Superstability of Generalized Derivations on Non-Archimedean Banach Algebras

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We investigate the superstability of generalized derivations in non-Archimedean algebras by using a version of fixed point theorem via Cauchy functional equation.

1. Introduction

A functional equation (\( \xi \)) is \textit{superstable} if every approximately solution of (\( \xi \)) is an exact solution of it.

The stability of functional equations was first introduced by Ulam [1] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940.

Given a metric group \( (\cdot, \rho) \), a number \( \varepsilon > 0 \), and a mapping \( f : G \to G \) which satisfies the inequality \( \rho(f(x \cdot y), f(x) \cdot f(y)) \leq \varepsilon \) for all \( x, y \) in \( G \), does there exist an automorphism \( a \) of \( G \) and a constant \( k > 0 \), depending only on \( G \) such that \( \rho(a(x), f(x)) \leq k\varepsilon \) for all \( x \in G ? \)

If the answer is affirmative, we would call the equation \( a(x \cdot y) = a(x) \cdot a(y) \) of automorphism is stable. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences \( \|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^q), \ (\varepsilon > 0, \ p \in [0, 1)) \). In 1991, Gajda [4] answered the question for the case
Suppose that \((G,+)\) is an abelian group, \(X\) is a Banach space \(\varphi : G \times G \rightarrow [0, \infty)\) which satisfies

\[
\bar{\varphi}(x, y) = \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^nx, 2^ny) < \infty,
\]

for all \(x, y \in G\). If \(f : G \rightarrow X\) is a mapping with

\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y),
\]

for all \(x, y \in G\), then there exists a unique mapping \(T : G \rightarrow X\) such that \(T(x+y) = T(x)+T(y)\) and \(\|f(x) - T(x)\| \leq \bar{\varphi}(x, x)\) for all \(x, y \in G\).

In 1949, Bourgin [8] proved the following result, which is sometimes called the superstability of ring homomorphisms: suppose that \(A\) and \(B\) are Banach algebras with unit. If \(f : A \rightarrow B\) is a surjective mapping such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,
\]

\[
\|f(xy) - f(x)f(y)\| \leq \delta,
\]

for some \(\varepsilon \geq 0\), \(\delta \geq 0\) and for all \(x, y \in A\), then \(f\) is a ring homomorphism.

Badora [9] and Miura et al. [10] proved the Ulam-Hyers stability and the Isac and Rassias-type stability of derivations [11] (see also [12, 13]); Savadkouhi et al. [14] have contributed works regarding the stability of ternary Jordan derivations. Jung and Chang [15] investigated the stability and superstability of higher derivations on rings. Recently, Ansari-Piri and Anjidani [16] discussed the superstability of generalized derivations on Banach algebras. In this paper, we investigate the superstability of generalized derivations on non-Archimedean Banach algebras by using the fixed point methods.

2. Preliminaries

In 1897, Hensel [17] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [18, 19].

A non-Archimedean field is a field \(\mathbb{K}\) equipped with a function (valuation) \(|\cdot|\) from \(\mathbb{K}\) into \([0, \infty)\) such that \(|r| = 0\) if and only if \(r = 0\), \(|rs| = |r||s|\), and \(|r + s| \leq \max\{|r|, |s|\}\) for all \(r, s \in \mathbb{K}\) (see [20, 21]).

**Definition 2.1.** Let \(X\) be a vector space over a scalar field \(\mathbb{K}\) with a non-Archimedean nontrivial valuation \(|\cdot|\). A function \(\|\cdot\| : X \rightarrow \mathbb{R}\) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) \(\|x\| = 0\) if and only if \(x = 0\),

(NA2) \(\|rx\| = |r|\|x\|\) for all \(r \in \mathbb{K}\) and \(x \in X\),

(NA3) \(\|x + y\| \leq \max\{\|x\|, \|y\|\}\) for all \(x, y \in X\) (the strong triangle inequality).
A sequence \( \{x_m\} \) in a non-Archimedean space is Cauchy if and only if \( \{x_{m+1} - x_m\} \) converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent. A non-Archimedean normed algebra is a non-Archimedean normed space \( A \) with a linear associative multiplication, satisfying \( \|xy\| \leq \|x\|\|y\| \) for all \( x, y \in A \). A non-Archimedean complete normed algebra is called a non-Archimedean Banach algebra (see [22]).

**Example 2.2.** Let \( p \) be a prime number. For any nonzero rational number \( x = (a/b)p^n \) such that \( a \) and \( b \) are integers not divisible by \( p \), define the \( p \)-adic absolute value \( |x|_p := p^{-n} \). Then, \( |\cdot| \) is a non-Archimedean norm on \( \mathbb{Q} \). The completion of \( \mathbb{Q} \) with respect to \( |\cdot| \) is denoted by \( \mathbb{Q}_p \), which is called the \( p \)-adic number field.

**Definition 2.3.** Let \( X \) be a nonempty set and \( d : X \times X \to [0, \infty] \) satisfy the following properties:

\[
\begin{align*}
(D_1) & \quad d(x, y) = 0 \text{ if and only if } x = y, \\
(D_2) & \quad d(x, y) = d(y, x) \text{ (symmetry)}, \\
(D_3) & \quad d(x, z) \leq \max\{d(x, y), d(y, z)\} \text{ (strong triangle in equality)},
\end{align*}
\]

for all \( x, y, z \in X \). Then, \( (X, d) \) is called a non-Archimedean generalized metric space. \( (X, d) \) is called complete if every \( d \)-Cauchy sequence in \( X \) is \( d \)-convergent.

**Definition 2.4.** Let \( A \) be a non-Archimedean algebra. An additive mapping \( D : A \to A \) is said to be a ring derivation if \( D(xy) = D(x)y + xD(y) \) for all \( x, y \in A \). An additive mapping \( H : A \to A \) is said to be a generalized ring derivation if there exists a ring derivation \( D : A \to A \) such that

\[
H(xy) = xH(y) + D(x)y,
\]

for all \( x, y \in A \).

We need the following fixed point theorem (see [23, 24]).

**Theorem 2.5 (non-Archimedean alternative Contraction Principle).** Suppose that \( (X, d) \) is a non-Archimedean generalized complete metric space and \( \Lambda : X \to X \) is a strictly contractive mapping; that is,

\[
d(\Lambda x, \Lambda y) \leq Ld(x, y), \quad (x, y \in X),
\]

for some \( L < 1 \). If there exists a nonnegative integer \( k \) such that \( d(\Lambda^{k+1}x, \Lambda^kx) < \infty \) for some \( x \in X \), then the followings are true:

(a) the sequence \( \{\Lambda^nx\} \) converges to a fixed point \( x^* \) of \( \Lambda \),

(b) \( x^* \) is a unique fixed point of \( \Lambda \) in

\[
X^* = \{ y \in X \mid d(\Lambda^kx, y) < \infty \},
\]

(c) if \( y \in X^* \), then

\[
d(y, x^*) \leq d(\Lambda y, y).
\]
3. Non-Archimedean Superstability of Generalized Derivations

Hereafter, we will assume that $A$ is a non-Archimedean Banach algebra with unit over a non-Archimedean field $\mathbb{K}$.

**Theorem 3.1.** Let $\varphi : A \times A \to [0, \infty)$ be a function. Suppose that $f, g : A \to A$ are mappings such that $g$ is additive and

$$
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad (3.1)
$$

$$
\|f(xy) - xf(y) - g(x)y\| \leq \varphi(x, y), \quad (3.2)
$$

for all $x, y \in A$. If there exists a natural number $k \in \mathbb{K}$ and $0 < L < 1$,

$$
|k|^{-1}\varphi(kx, ky), |k|^{-1}\varphi(kx, y), |k|^{-1}\varphi(x, ky) \leq L\varphi(x, y), \quad (3.3)
$$

for all $x, y \in A$. Then, $f$ is a generalized ring derivation and $g$ is a ring derivation.

**Proof.** By induction on $i$, we prove that

$$
\|f(ix) - if(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((i - 1)x, x)\}, \quad (3.4)
$$

for all $x \in A$ and $i \geq 2$. Let $x = y$ in (3.1). Then,

$$
\|f(2x) - 2f(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x)\}, \quad n \in \mathbb{N}_0, \ x \in A. \quad (3.5)
$$

This proves (3.4) for $i = 2$. Let (3.4) holds for $i = 1, 2, \ldots, j$. Replacing $x$ by $jx$ and $y$ by $x$ in (3.1) for each $n \in \mathbb{N}_0$, and for all $x \in A$, we get

$$
\|f((j + 1)x) - f(jx) - f(x)\| \leq \max\{\varphi(0, 0), \varphi(jx, x)\}. \quad (3.6)
$$

Since

$$
f((j + 1)x) - f(jx) - f(x) = f((j + 1)x) - (j + 1)f(x) + (j + 1)f(x) - f(jx) - f(x) \quad (3.7)
$$

$$
= f((j + 1)x) - (j + 1)f(x) +jf(x) - f(jx),
$$

for all $x \in A$, it follows from induction hypothesis and (3.7) that

$$
\|f((j + 1)x) - (j + 1)f(x)\|
\leq \max\{\|f((j + 1)x) - f(jx) - f(x)\|, \|jf(x) - f(jx)\|\} \quad (3.8)
\leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi(jx, x)\},
$$
for all $x \in A$. This proves (3.4) for all $i \geq 2$. In particular,

$$\|f(kx) - kf(x)\| \leq \psi(x), \quad (3.9)$$

for all $x \in A$ where

$$\psi(x) = \max\{\psi(0,0), \psi(x,x), \psi(2x,x), \ldots, \psi((k-1)x,x)\} \quad (x \in A). \quad (3.10)$$

Let $X$ be the set of all functions $r : A \to A$. We define $d : X \times X \to [0, \infty]$ as follows:

$$d(r, s) = \inf\{\alpha > 0 : \|r(x) - s(x)\| \leq \alpha \psi(x) \forall x \in A\}. \quad (3.11)$$

It is easy to see that $d$ defines a generalized complete metric on $X$. Define $J : X \to X$ by $J(r)(x) = k^{-1}r(kx)$. Then, $J$ is strictly contractive on $X$, in fact, if

$$\|r(x) - s(x)\| \leq \alpha \psi(x), \quad (x \in A), \quad (3.12)$$

then by (3.3),

$$\|J(r)(x) - J(s)(x)\| = |k|^{-1}\|r(kx) - s(kx)\| \leq \alpha |k|^{-1} \psi(kx) \leq L \psi(x), \quad (x \in A). \quad (3.13)$$

It follows that

$$d(J(r), J(s)) \leq L d(r, s) \quad (r, s \in X). \quad (3.14)$$

Hence, $J$ is a strictly contractive mapping with Lipschitz constant $L$. By (3.9),

$$\|J(f)(x) - f(x)\| = \|k^{-1}f(kx) - f(x)\|, \quad (3.15)$$

$$|k|^{-1}\|f(kx) - kf(x)\| \leq |k|^{-1} \psi(x) \quad (x \in A).$$

This means that $d(J(f), f) \leq 1/|k|$. By Theorem 2.5, $J$ has a unique fixed point $h : A \to A$ in the set

$$U = \{r \in X : d(r, J(f)) < \infty\}, \quad (3.16)$$

and for each $x \in A$,

$$h(x) = \lim_{m \to \infty} J^{m} (f(x)) = \lim_{m \to \infty} k^{-m}f(k^{m}x). \quad (3.17)$$
Therefore,

\[
\|h(x + y) - h(x) - h(y)\| \\
= \lim_{m \to \infty} |k|^m \|f(k^m(x + y)) - f(k^m x) - f(k^m y)\| \\
\leq \lim_{m \to \infty} |k|^m \max\{\varphi(0,0), \varphi(k^n x, k^n y)\} \\
\leq \lim_{m \to \infty} L^m \varphi(x, y) = 0,
\] (3.18)

for all \(x, y \in A\). This shows that \(h\) is additive.

Replacing \(x\) by \(k^n x\) in (3.2) to get

\[
\|f(k^n xy) - k^n xf(y) - g(k^n x) y\| \leq \varphi(k^n x, y),
\] (3.19)

and so

\[
\left\| \frac{f(k^n xy)}{k^n} - xf(y) - \frac{g(k^n x)}{k^n} y \right\| \leq \frac{1}{|k|^n} \varphi(k^n x, y) \leq L^n \varphi(x, y),
\] (3.20)

for all \(x, y \in A\) and all \(n \in \mathbb{N}\). By taking \(n \to \infty\), we have

\[
h(xy) = xf(y) + \lim_{n \to \infty} \frac{g(k^n x)}{k^n} y,
\] (3.21)

for all \(x, y \in A\).

Fix \(m \in \mathbb{N}\). By (3.21), we have

\[
xf(k^m y) = h(k^m xy) - \lim_{n \to \infty} \left( \frac{g(k^n x)}{k^n} (k^m y) \right)
\]

\[
= k^m xf(y) + \lim_{n \to \infty} \left( \frac{g(k^n k^m x)}{k^n} y \right) - k^m \lim_{n \to \infty} \left( \frac{g(k^n x)}{k^n} y \right)
\]

\[
= k^m xf(y) + k^n \lim_{n \to \infty} \left( \frac{g(k^n k^m x)}{k^n} y \right) - k^m \lim_{n \to \infty} \left( \frac{g(k^n x)}{k^n} y \right)
\]

\[
= k^m xf(y),
\] (3.22)

for all \(x, y \in A\). Then, \(xf(y) = x(f(k^m y)/k^m)\) for all \(x, y \in A\) and each \(m \in \mathbb{N}\), and so by taking \(m \to \infty\), we have \(xf(y) = xh(y)\). Now, we obtain \(h = f\), since \(A\) is with unit. Replacing \(y\) by \(k^n y\) in (3.2), we obtain

\[
\|f(k^n(xy)) - xf(k^n y) - k^n g(x) y\| \leq \varphi(x, k^ny),
\] (3.23)
Abstract and Applied Analysis

and hence,

\[
\left\| \frac{f(k^nxy)}{k^n} - x \frac{f(k^ny)}{k^n} - g(x)y \right\| \leq \frac{1}{|k^n|} \varphi(x,k^ny) \leq L^n \varphi(x,y),
\]

(3.24)

for all \( x, y \in A \) and each \( n \in \mathbb{N} \). Letting \( n \) tends to infinite, we have

\[
f(xy) = xf(y) + g(x)y.
\]

(3.25)

Now, we show that \( g \) is a ring derivation. By (3.25), we get

\[
g(xy)z = f(xy)z - xyf(z)
\]

\[
= xf(yz) + g(x)yz - xyf(z)
\]

(3.26)

\[
= (xg(y) + g(x)y)z,
\]

for all \( x, y, z \in A \). Therefore, we have \( g(xy) = xg(y) + g(x)y. \)

\[\square\]

The proof of following theorem is similar to that in Theorem 3.1, hence it is omitted.

**Theorem 3.2.** Let \( \varphi : A \times A \to [0, \infty) \) be a function. Suppose that \( f, g : A \to A \) are mappings such that \( g \) is additive and

\[
\left\| f(x + y) - f(x) - f(y) \right\| \leq \varphi(x,y),
\]

\[
\left\| f(xy) - xf(y) - g(x)y \right\| \leq \varphi(x,y),
\]

(3.27)

for all \( x, y \in A \). If there exists a natural number \( k \in \mathbb{N} \) and \( 0 < L < 1 \),

\[
|k|\varphi(k^{-1}x, k^{-1}y), |k|\varphi(k^{-1}x, y), |k|\varphi(x, k^{-1}y) \leq L \varphi(x,y),
\]

(3.28)

for all \( x, y \in A \). Then, \( f \) is a generalized ring derivation and \( g \) is a ring derivation.

The following results are immediate corollaries of Theorems 3.1 and 3.2 and Example 2.3.

**Corollary 3.3.** Let \( A \) be a non-Archimedean Banach algebra over \( \mathbb{Q}_p \), \( \varepsilon > 0 \), and \( p_1, p_2 \in (1, \infty) \). Suppose that \( f, g : A \to A \) are mappings such that \( g \) is additive and

\[
\left\| f(x + y) - f(x) - f(y) \right\| \leq \varepsilon (\|x\|^{p_1} \|y\|^{p_2}),
\]

\[
\left\| f(xy) - xf(y) - g(x)y \right\| \leq \varepsilon (\|x\|^{p_1} \|y\|^{p_2}),
\]

(3.29)

for all \( x, y \in A \). Then, \( f \) is a generalized ring derivation and \( g \) is a ring derivation.
Corollary 3.4. Let $A$ be a non-Archimedean Banach algebra over $\mathbb{Q}_p$, $\varepsilon > 0$ and $p_1, p_2, p_1 + p_2 \in (-\infty, 1)$. Suppose that $f, g : A \to A$ are mappings such that $g$ is additive and

$$
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^{p_1} \|y\|^{p_2}),
$$

$$
\|f(xy) - xf(y) - g(x)y\| \leq \varepsilon (\|x\|^{p_1} \|y\|^{p_2}),
$$

(3.30)

for all $x, y \in A$. Then, $f$ is a generalized ring derivation and $g$ is a ring derivation.

References


