Research Article

A Note on the Modified $q$-Bernoulli Numbers and Polynomials with Weight $\alpha$

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A systemic study of some families of the modified $q$-Bernoulli numbers and polynomials with weight $\alpha$ is presented by using the $p$-adic $q$-integration $\mathbb{Z}_p$. The study of these numbers and polynomials yields an interesting $q$-analogue related to Bernoulli numbers and polynomials.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(\alpha \log q)$ for $|x|_p \leq 1$.

The $q$-number $[x]_q$ is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (1.1)$$

see [1–10].
We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by \( f \in UD(\mathbb{Z}_p) \), if the difference quotients
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]
have a limit \( l = f'(a) \) as \( (x, y) \to (a, a) \), c.f. [11].

For \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{ f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable functions} \} \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x,
\]
(see [3]).

From (1.3), we can easily derive the following:
\[
q^n I_q(f_n) = I_q(f) + (q-1) \sum_{l=0}^{n-1} f(l)q^l + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^lf'(l),
\]
where \( f_n(x) = f(x + n) \), (see [5, 12]).

In [1, 2], Carlitz defined a set of numbers \( B_{k, q} \) inductively by
\[
B_{0, q} = 1, \quad (qB_q + 1)^k - B_{k, q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}
\]
with the usual convention about replacing \( B_k^q \) by \( B_{k, q} \).

These numbers are the \( q \)-extension of ordinary Bernoulli numbers. But they do not remain finite when \( q = 1 \). So, Carlitz modified (1.5) as follows:
\[
\beta_{0, q} = 1, \quad q(q\beta + 1)^k - \beta_{k, q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}
\]
with the usual convention of replacing \( \beta_k \) by \( \beta_{k, q} \).

In [1], Carlitz also considered the extension of Carlitz’s \( q \)-Bernoulli numbers as follows:
\[
\beta_{0, q}^h = \frac{h}{[h]_q}, \quad q^h(q\beta^h + 1)^k - \beta_{k, q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}
\]
with the usual convention of replacing \( (\beta^h)^k \) by \( \beta_{k, q}^h \).

In this paper, we construct the modified \( q \)-Bernoulli numbers with weight \( \alpha \), which are different Carlitz’s \( q \)-Bernoulli numbers, by using \( p \)-adic \( q \)-integral equation. From (1.4),
we derive some interesting identities and relations on the modified $q$-Bernoulli numbers and polynomials.

## 2. The Modified $q$-Bernoulli Numbers and Polynomials with Weight $\alpha$

In this section, we assume $\alpha \in \mathbb{Q}$. Now, we define the modified $q$-Bernoulli numbers with weight $\frac{\alpha}{\log q}$ as follows:

$$
\tilde{B}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} q^n x^n d\mu_q(x) \\
= \frac{1}{(1 - q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha}{[al]_q}.
$$

Thus, by (2.1), we have

$$
\tilde{B}_{n,q}^{(\alpha)} = \frac{1}{(1 - q^n)^n} \log q - n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am} [m]_{q}^{n-1}.
$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, one has

$$
\tilde{B}_{n,q}^{(\alpha)} = \frac{\alpha}{(1 - q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{[al]_q}.
$$

Let us define the generating function of the modified $q$-Bernoulli numbers with weight $\alpha$ as follows:

$$
F_q^{(\alpha)}(t) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)} \frac{t^n}{n!}.
$$

Then, by (2.3) and (2.4), we get

$$
F_q^{(\alpha)}(t) = \frac{q - 1}{\log q} e^{(1/(1-q^t))} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am} e^{[m]_q e^t}.
$$
In the viewpoint of (2.1), we define the modified $q$-Bernoulli numbers with weight $\alpha$ as follows:

$$
\tilde{B}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} q^{-y} [x + y]_q^n \, d\mu_q(y)
$$

$$
= \sum_{l=0}^{n \alpha} \binom{n}{l} [x]_{q^{\alpha}}^{n-1} q^{alx} \tilde{B}_{l,q}^{(\alpha)}
$$

$$
= \left([x]_{q^{\alpha}} + q^{ax} \tilde{B}_{q}^{(\alpha)}\right)^n, \quad \text{for } n \in \mathbb{Z}_+,
$$

with the usual convention of replacing $(\tilde{B}_{q}^{(\alpha)})^n$ by $	ilde{B}_{n,q}^{(\alpha)}$.

From (2.6), we note that

$$
\tilde{B}_{n,q}^{(\alpha)}(x) = \frac{\alpha}{(1 - q^{\alpha})^n} \sum_{l=0}^{n \alpha} \binom{n}{l} (-1)^l q^{alx} \frac{l}{[al]_q}
$$

$$
= \frac{1}{(1 - q^{\alpha})^n} \frac{q - 1}{\log q} - n \frac{\alpha}{[\alpha]_q} q^{ax} \sum_{m=0}^{\infty} q^{am}[m + x]_{q^{\alpha}}^{n-1}.
$$

Therefore, by (2.7), we obtain the following theorem.

**Theorem 2.2.** For $n \in \mathbb{Z}_+$, one has

$$
\tilde{B}_{n,q}^{(\alpha)}(x) = \frac{\alpha}{(1 - q^{\alpha})^n} \sum_{l=0}^{n \alpha} \binom{n}{l} (-1)^l q^{alx} \frac{l}{[al]_q}
$$

$$
= \frac{1}{(1 - q^{\alpha})^n} \frac{q - 1}{\log q} - n \frac{\alpha}{[\alpha]_q} q^{ax} \sum_{m=0}^{\infty} q^{am}[m + x]_{q^{\alpha}}^{n-1}.
$$

Let $F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)}(x) (t^n/n!)$ be the generating function of the modified $q$-Bernoulli polynomials with weight $\alpha$.

Then, by (2.7), we get

$$
F_q^{(\alpha)}(t,x) = \frac{q - 1}{\log q} e^{(1/(1-q^{\alpha}))t} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am}[m + x]_{q^{\alpha}} e^{[m + x]_{q^{\alpha}} t}.
$$

Therefore, by (2.9), we obtain the following corollary

**Corollary 2.3.** Let $F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)}(x) (t^n/n!)$. Then one has

$$
F_q^{(\alpha)}(t,x) = \frac{q - 1}{\log q} e^{(1/(1-q^{\alpha}))t} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am}[m + x]_{q^{\alpha}} e^{[m + x]_{q^{\alpha}} t}.
$$

In particular, $F_q^{(\alpha)}(t,0) = F_q^{(\alpha)}(t)$. 

From Corollary 2.3, we can derive the following equation:

\[ F^*_q(t, 1) - F^*_q(t) = t \frac{\alpha}{[\alpha]_q}. \]  

(2.11)

By (2.5) and (2.11), we get

\[ \tilde{B}^{(a)}_{0,q} = \frac{q - 1}{\log q}, \quad \tilde{B}^{(a)}_{n,q}(1) - \tilde{B}^{(a)}_{n,q} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \]  

(2.12)

Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.4.** For \( n \in \mathbb{Z}_+ \), one has

\[ \tilde{B}^{(a)}_{0,q} = \frac{q - 1}{\log q}, \quad \tilde{B}^{(a)}_{n,q}(1) - \tilde{B}^{(a)}_{n,q} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \]  

(2.13)

By using (2.6), we obtain the following corollary.

**Corollary 2.5.** For \( n \in \mathbb{Z}_+ \), one has

\[ \tilde{B}^{(a)}_{0,q} = \frac{q - 1}{\log q}, \quad \left( q^a \tilde{B}^{(a)}_{n,q} + 1 \right)^n - \tilde{B}^{(a)}_{n,q} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \]  

(2.14)

with the usual convention of replacing \( (\tilde{B}^{(a)}_{q})^n \) by \( \tilde{B}^{(a)}_{n,q} \).

From (1.4), we can derive the following equation:

\[ \int_{\mathbb{Z}_p} f(x + n)q^{-x} \, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)q^{-x} \, d\mu_q(x) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} f'(l). \]  

(2.15)

Thus, by (1.6), (2.6), and (2.15), we get

\[ \tilde{B}^{(a)}_{m,q}(n) - \tilde{B}^{(a)}_{m,q} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^a}^{m-1} q^al, \quad n \in \mathbb{N}, \ m \in \mathbb{Z}_+. \]  

(2.16)

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.6.** For \( n \in \mathbb{N}, m \in \mathbb{Z}_+ \), one has

\[ \tilde{B}^{(a)}_{m,q}(n) - \tilde{B}^{(a)}_{m,q} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^a}^{m-1} q^al. \]  

(2.17)
From (2.6), we note that
\[
\tilde{B}^{(a)}_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n q^{-y} d\mu_q(y)
\]
\[
= \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} [x + y]_q^n
\]
\[
= \frac{1 - q}{1 - q^d} \sum_{a=0}^{d-1} \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} [a + x + dy]_q^n
\]
\[
= \left[ d \right]_q^n \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} \left[ \frac{a + x}{d} + y \right]_q^n q^{-dy} d\mu_q(y)
\]
\[
= \left[ d \right]_q^n \sum_{a=0}^{d-1} \tilde{B}^{(a)}_{n,q} \left( \frac{x + a}{d} \right),
\]
(2.18)

Therefore, by (2.18), we obtain the following distribution relation for the modified \(q\)-Bernoulli polynomials with weight \(\alpha\).

**Theorem 2.7.** For \(d \in \mathbb{N}, n \in \mathbb{Z}_+,\) one has
\[
\tilde{B}^{(a)}_{n,q}(x) = \left[ d \right]_q^n \sum_{a=0}^{d-1} \tilde{B}^{(a)}_{n,q} \left( \frac{x + a}{d} \right).
\]
(2.19)

To derive the relation of reflection symmetry of the modified \(q\)-Bernoulli polynomials with weight \(\alpha\), we evaluate the following \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p:\)
\[
\tilde{B}^{(a)}_{n,q^{-1}}(1 - x) = \int_{\mathbb{Z}_p} [1 - x + x_1]_q^n q^{-x_1} d\mu_{q^{-1}}(x_1)
\]
\[
= \frac{1}{(1 - q^{-a})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{al-1} \frac{al}{[al]_q}
\]
\[
= \frac{(-1)^n}{q} \frac{q^m}{(1 - q^a)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{al}{[al]_q}
\]
\[
= q^{an-1}(-1)^n \tilde{B}^{(a)}_{n,q^{-1}}(x).
\]
(2.20)

Therefore, by (2.20), we obtain the following reflection symmetry relation of the modified \(q\)-Bernoulli polynomials with weight \(\alpha\).

**Theorem 2.8.** For \(n \in \mathbb{Z}_+,\) one has
\[
\tilde{B}^{(a)}_{n,q^{-1}}(1 - x) = q^{an-1}(-1)^n \tilde{B}^{(a)}_{n,q}(x).
\]
(2.21)
From (1.3), we note that
\[
\frac{1}{q} \int_{\mathbb{Z}_q} [1 - x]^n q^{-x} d\mu_q(x) = (-1)^n q^{\alpha n - 1} \int_{\mathbb{Z}_q} [x - 1]^n q^{-x} d\mu_q(x)
\]
\[= (-1)^n q^{\alpha n - 1} \tilde{B}_{n,q}^{(a)}(-1) = \tilde{B}_{n,q}^{(a)}(2),
\]
(2.22)

and, by (2.6), we get
\[
\tilde{B}_{n,q}^{(a)}(2) = \left( q^{2\alpha} \tilde{B}_{q}^{(a)} + [2]_q \right)^n = \left( q^{\alpha} \left( q^{\alpha} \tilde{B}_{q}^{(a)} + 1 \right) + 1 \right)^n
\]
\[= \sum_{l=0}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) q^{\alpha l} \left( q^{\alpha} \tilde{B}_{q}^{(a)} + 1 \right)^l
\]
\[= \tilde{B}_{0,q}^{(a)} + n q^{\alpha} \left( q^{\alpha} \tilde{B}_{q}^{(a)} + 1 \right)^1 + \sum_{l=2}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) q^{\alpha l} \left( q^{\alpha} \tilde{B}_{q}^{(a)} + 1 \right)^l
\]
\[= \frac{(q - 1)}{\log q} + n q^{\alpha} \left( \frac{\alpha}{[\alpha] q} + \tilde{B}_{1,q}^{(a)} \right) + \sum_{l=2}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) q^{\alpha l} \tilde{B}_{l,q}^{(a)}
\]
\[= n q^{\alpha} \frac{\alpha}{[\alpha] q} + \sum_{l=0}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) q^{\alpha l} \tilde{B}_{l,q}^{(a)}.
\]
(2.23)

Let \( n \in \mathbb{N} \) with \( n \geq 2 \). Then, by (2.12) and (2.23), we obtain the following theorem.

**Theorem 2.9.** For \( n \in \mathbb{N} \) with \( n \geq 2 \), one has
\[
\tilde{B}_{n,q}^{(a)}(2) - n q^{\alpha} \frac{\alpha}{[\alpha] q} = \left( q^{\alpha} \tilde{B}_{q}^{(a)} + 1 \right)^n = \tilde{B}_{n,q}^{(a)}.
\]
(2.24)

In particular,
\[
\frac{1}{q} \int_{\mathbb{Z}_q} [1 - x]^n q^{-x} d\mu_q(x) = \tilde{B}_{n,q}^{(a)}(2) = \frac{n}{q} \frac{\alpha}{[\alpha] q} + \tilde{B}_{n,q}^{(a)}.
\]
(2.25)

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**References**


