Research Article

Fuzzy Stability of a Functional Equation Deriving from Quadratic and Additive Mappings

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1. Introduction and Preliminaries

A classical question in the theory of functional equations is “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”. Such a problem, called a stability problem of the functional equation, was formulated by Ulam [1] in 1940. In the next year, Hyers [2] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [3] for additive mappings, and by Rassias [4] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5–15].

In 1984, Katsaras [16] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, Bag and Samanta [17], following Cheng and Mordeson [18], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. In 2008, Mirmostafaee and Moslehian [20] obtained a fuzzy version of stability for the Cauchy functional equation

\[ f(x + y) - f(x) - f(y) = 0. \] (1.1)
In the same year, they [21] proved a fuzzy version of stability for the quadratic functional equation

\[ f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0. \]  

(1.2)

We call a solution of (1.1) an additive mapping and a solution of (1.2) is called a quadratic mapping. Now, we consider the functional equation

\[ f(2x + y) + f(2x - y) + 2f(x) - f(x + y) - f(x - y) - 2f(2x) = 0, \]  

(1.3)

which is called a functional equation deriving from quadratic and additive mappings. We call a solution of (1.3) a general quadratic mapping. In 2008, Najati and Moghimi [22] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping \( A \) and a quadratic mapping \( Q \) to prove the existence of a general quadratic mapping \( F \) which is close to the given mapping \( f \). In their processing, \( A \) is approximate to the odd part \( (f(x) - f(-x))/2 \) of \( f \), and \( Q \) is close to the even part \( (f(x) + f(-x))/2 - f(0) \) of it, respectively.

In this paper, we get a general stability result of the functional equation deriving from quadratic and additive mappings (1.3) in the fuzzy normed linear space. To do it, we introduce a Cauchy sequence \( \{J_n f(x)\} \), starting from a given mapping \( f \), which converges to the desired mapping \( F \) in the fuzzy sense. As we mentioned before, in previous studies of stability problem of (1.3), they attempted to get stability theorems by handling the odd and even part of \( f \), respectively. According to our proposal in this paper, we can take the desired approximate solution \( F \) at once. Therefore, this idea is a refinement with respect to the simplicity of the proof.

2. Fuzzy Stability of the Functional Equation (1.3)

We use the definition of a fuzzy normed space given in [17] to exhibit a reasonable fuzzy version of stability for the functional equation deriving from quadratic and additive mappings in the fuzzy normed linear space.

**Definition 2.1** (see [17]). Let \( X \) be a real linear space. A function \( N : X \times \mathbb{R} \to [0, 1] \) (the so-called fuzzy subset) is said to be a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

(1) \( N(x, c) = 0 \) for \( c \leq 0 \),

(2) \( x = 0 \) if and only if \( N(x, c) = 1 \) for all \( c > 0 \),

(3) \( N(cx, t) = N(x, t/|c|) \) if \( c \neq 0 \),

(4) \( N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\} \),

(5) \( N(x, \cdot) \) is a non-decreasing function on \( \mathbb{R} \) and \( \lim_{t \to -\infty} N(x, t) = 1 \).

The pair \((X, N)\) is called a fuzzy normed linear space. Let \((X, N)\) be a fuzzy normed linear space. Let \( \{x_n\} \) be a sequence in \( X \). Then, \( \{x_n\} \) is said to be convergent if there exists \( x \in X \) such that \( \lim_{n \to \infty} N(x_n - x, t) = 1 \) for all \( t > 0 \). In this case, \( x \) is called the limit of the sequence \( \{x_n\} \), and we denote it by \( X - \lim_{n \to \infty} x_n = x \). A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \) and all \( p > 0 \) we have \( N(x_{n+p} - x_n, t) > 1 - \varepsilon \). It is known that every convergent sequence in a fuzzy normed space
is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be \textit{complete}, and the fuzzy normed space is called \textit{a fuzzy Banach space}.

Let $(X, N)$ be a fuzzy normed space and $(Y, N')$ a fuzzy Banach space. For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x, y) := f(2x + y) + f(2x - y) + 2f(x) - f(x + y) - f(x - y) - 2f(2x), \quad (2.1)$$

for all $x, y \in X$. Recall $Df \equiv 0$ means that $f$ is a general quadratic mapping. For given $q > 0$, the mapping $f$ is called \textit{a fuzzy q-almost general quadratic mapping} if

$$N'(Df(x, y), t + s) \geq \min \{N(x, s^q), N(y, t^q)\}, \quad (2.2)$$

for all $x, y \in X \setminus \{0\}$ and all $s, t \in [0, \infty)$. Now, we get the general stability result in the fuzzy normed linear setting.

\textbf{Theorem 2.2.} Let $q$ be a positive real number with $q \neq 1/2, 1$. And let $f$ be a fuzzy $q$-almost general quadratic mapping from a fuzzy normed space $(X, N)$ into a fuzzy Banach space $(Y, N')$. Then, there is a unique general quadratic mapping $F : X \to Y$ such that

$$N'(F(x) - f(x), t) \geq \sup_{0 < r < 1} N \left( x, \frac{t^q}{\left( (7 + 2p + 2p^q + 4p) / (|4 - 2p|3p) + (5 + 2 : 2p + 3p) / (2|2 - 2p|) \right)^q} \right), \quad (2.3)$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

\textit{Proof.} We will prove the theorem in three cases, $q > 1, 1/2 < q < 1$, and $0 < q < 1/2$.

\textbf{Case 1.} Let $q > 1$. We define a mapping $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2} (4^{-n}(f(2^nx) + f(-2^nx) - 2f(0)) + 2^{-n} (f(2^nx) - f(-2^nx))) + f(0), \quad (2.4)$$

for all $x \in X$. Then, $J_0 f(x) = f(x), J_1 f(0) = f(0)$, and

$$J_i f(x) - J_{i+1} f(x) = \frac{Df(2^i x/3, 2^{i+1} x/3)}{4^{i+1}} - \frac{Df(2^i x/3, 2^{i+1} x/3)}{2 \cdot 4^{i+1}} - \frac{Df(2^i x/3, 2^{i+1} x/3)}{2 \cdot 4^{i+1}}$$

$$- \frac{Df(2^i x/3, 2^{i+2} x/3)}{2 \cdot 4^{i+1}} + \frac{Df(-2^i x/3, -2^{i+1} x/3)}{4^{i+1}} - \frac{Df(-2^i x/3, -2^{i+1} x/3)}{2 \cdot 4^{i+1}}$$
\[
\begin{align*}
&- \frac{Df(-2^j x/3, -2^{j+1} x)}{2^{-j+1}} - \frac{Df(-2^j x/3, -2^{j+2} x/3)}{2^{-j+1}} + \frac{Df(2^{j+1} x, 2^j x)}{2^{j+2}} \\
&- \frac{Df(2^j x, 3 \cdot 2^j x)}{2^{j+2}} + \frac{Df(2^j x, 2^j x)}{2^{j+2}} + \frac{Df(2^j x, -2^{j+1} x)}{2^{j+2}} ,
\end{align*}
\]

(2.5)

for all \( x \in X \setminus \{0\} \) and \( j \geq 0 \). Together with (N3), (N4), and (2.2), this equation implies that if \( n + m > m \geq 0 \), then

\[
N'(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{7 + 2^p + 3^p + 4^p}{4 \cdot 3^p} \left( \frac{2^p}{4} \right) j + \frac{5 + 2 \cdot 2^p + 3^p}{4} \left( \frac{2^p}{2} \right)^j \right) t^p \}
\]

\[
\geq \min \bigcup_{j=m}^{n+m-1} \{ N' \left( J_j f(x) - J_{j+1} f(x), \left( \frac{7 + 2^p + 3^p + 4^p}{4j+1 \cdot 3^p} \frac{2^p}{j+1} + \frac{5 + 2 \cdot 2^p + 3^p}{2j+2} \right) \right) \}
\]

\[
\geq \min \bigcup_{j=m}^{n+m-1} \{ N' \left( \frac{Df(2^j x/3, 2^j x/3)}{2^-j+1}, \frac{2^p}{2^j} \right) \}
\]

\[
\begin{align*}
&N' \left( - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( - \frac{Df(2^j x/3, 2^j x)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( - \frac{Df(2^j x/3, 2^{j+2} x/3)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( - \frac{Df(-2^j x/3, -2^{j+1} x/3)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( - \frac{Df(-2^j x/3, -2^j x)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( - \frac{Df(-2^j x/3, -2^{j+2} x/3)}{2^{-j+1}}, \frac{2^p}{2^j+1} \right), \\
&N' \left( \frac{Df(2^{j+1} x, 2^j x)}{2^{j+2}}, \frac{2^p}{2^{j+2}} \right), \\
&N' \left( - \frac{Df(2^j x, 3 \cdot 2^j x)}{2^{j+2}}, \frac{2^p}{2^{j+2}} \right), \\
&N' \left( \frac{Df(2^j x, 2^j x)}{2^{j+2}}, \frac{2^p}{2^{j+2}} \right), \\
&N' \left( \frac{Df(2^j x, -2^{j+1} x)}{2^{j+2}}, \frac{2^p}{2^{j+2}} \right) \}
\end{align*}
\]
\[
\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N\left(\frac{2}{3}, \frac{2}{3}t\right), N\left(\frac{2^{i+1}x, 2^{i+1}t}{3}, N\left(\frac{2^{i+2}x, 2^{i+2}t}{3}\right)\right) \right\} \right\} = N(x, t),
\]
(2.6)

for all \(x \in X \setminus \{0\}\) and \(t > 0\). Let \(\varepsilon > 0\) be given. Since \(\lim_{t \to \infty} N(x, t) = 1\), there is \(t_0 > 0\) such that
\[
N(x, t_0) \geq 1 - \varepsilon.
\]
(2.7)

We observe that for some \(\tilde{t} > t_0\), the series \(\sum_{j=0}^{\infty} \left( \frac{7 + 2^p + 3^p + 4^p}{4^{i+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+1}} \right)^{2i+1+2} < c\),
(2.8)

for each \(m \geq n_0\) and \(n > 0\). By (N5) and (2.6) we have
\[
N' \left( J_m f(x) - J_{n+m} f(x), c \right) \geq N' \left( J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{7 + 2^p + 3^p + 4^p}{4^{i+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+1}} \right)^{2i+1+2} \right) \geq N \left( x, \tilde{t} \right) \geq N(x, t_0) \geq 1 - \varepsilon,
\]
(2.9)

for all \(x \in X \setminus \{0\}\). Recall \(J_n f(0) = f(0)\) for all \(n \geq 0\). Thus, \(\{J_n f(x)\}\) becomes a Cauchy sequence for all \(x \in X\). Since \((Y, N')\) is complete, we can define a mapping \(F : X \to Y\) by
\[
F(x) := N' - \lim_{n \to \infty} J_n f(x),
\]
(2.10)

for all \(x \in X\). Moreover, if we put \(m = 0\) in (2.6), we have
\[
N' \left( f(x) - J_n f(x), t \right) \geq \left( X, \frac{t^q}{\left( \sum_{j=0}^{n-1} \left( \frac{7 + 2^p + 3^p + 4^p}{4^{i+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+1}} \right)^{2i+1+2} \right)^q} \right),
\]
(2.11)
for all \( x \in X \). Next, we will show that \( F \) is a general quadratic mapping. Using (N4), we have

\[
N'(DF(x, y), t) \geq \min \left\{ N' \left( F(2x + y) - J_n f(2x + y), \frac{t}{16} \right), \right.
N' \left( -F(x + y) + J_n f(x + y), \frac{t}{16} \right), \left. N' \left( -F(x - y) + J_n f(x - y), \frac{t}{16} \right), N' \left( 2F(x) - 2J_n f(x), \frac{t}{8} \right), N' \left( -2F(2x) + 2J_n f(2x), \frac{t}{8} \right), \right.
N' \left( F(2x - y) + J_n f(2x - y), \frac{t}{16} \right), \left. N' \left( DJ_n f(x, y), \frac{t}{2} \right) \right\},
\]

(2.12)

for all \( x, y \in X \setminus \{0\} \) and \( n \in \mathbb{N} \). The first six terms on the right hand side of (2.12) tend to 1 as \( n \to \infty \) by the definition of \( F \) and (N2), and the last term holds

\[
N' \left( DJ_n f(x, y), \frac{t}{2} \right) \geq \min \left\{ N' \left( \frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left( \frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), \right.
N' \left( \frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right), \left. N' \left( \frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right) \right\},
\]

(2.13)

for all \( x, y \in X \setminus \{0\} \). By (N3) and (2.2), we obtain

\[
N' \left( \frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right) = N' \left( \frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{2 \cdot 4^n t}{8} \right) \geq \min \left\{ N \left( \pm 2^n x, \left( \frac{4^n t}{8} \right)^q \right), N \left( \pm 2^n y, \left( \frac{4^n t}{8} \right)^q \right) \right\}
\]

(2.14)

\[
\geq \min \left\{ N \left( x, 2^{(2q-1)n-3q} t^q \right), N \left( y, 2^{(2q-1)n-3q} t^q \right) \right\},
\]

for all \( x, y \in X \setminus \{0\} \) and \( n \in \mathbb{N} \). Since \( q > 1 \), together with (N5), we can deduce that the last term of (2.12) also tends to 1 as \( n \to \infty \). It follows from (2.12) that

\[
N'(DF(x, y), t) = 1,
\]

(2.15)

for all \( x, y \in X \setminus \{0\} \) and \( t > 0 \). Since \( DF(0, 0) = 0 \), \( DF(x, 0) = 0 \) and \( DF(0, y) = 0 \) for all \( x, y \in X \setminus \{0\} \), this means that \( DF(x, y) = 0 \) for all \( x, y \in X \) by (N2).
Now, we approximate the difference between $f$ and $F$ in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since $F$ is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that $N'(F(x) - J_n f(x), t - t') \geq 1 - \varepsilon$. By (2.11), we have

\[
N'(F(x) - f(x), t) \geq \min \left\{ N'(F(x) - J_n f(x), t - t'), N'(J_n f(x) - f(x), t') \right\}
\]

\[
\geq \min \left\{ 1 - \varepsilon, N\left( x, \frac{t^q}{\left( \sum_{j=0}^{n-1} \left( (7 + 2^p + 3^p + 4^p) / (4^{j+1} \cdot 3^p) + (5 + 2 \cdot 2^p + 3^p) / (2^{j+2}) \right)^q \right)} \right) \right\}
\]

\[
\geq \min \left\{ 1 - \varepsilon, N\left( x, \frac{t^q}{\left( (7 + 2^p + 3^p + 4^p) / (4 - 2^p) 3^p + (5 + 2 \cdot 2^p + 3^p) / (2(2 - 2^p)) \right)^q} \right) \right\}.
\]

(2.16)

Because $0 < \varepsilon < 1$ is arbitrary and $F(0) = f(0)$, we get (2.3) in this case.

Finally, to prove the uniqueness of $F$, let $F' : X \rightarrow Y$ be another general quadratic mapping satisfying (2.3). Then, by (2.5), we get

\[
F(x) - J_n F(x) = \sum_{j=0}^{n-1} (J_j F(x) - J_{j+1} F(x)) = 0,
\]

(2.17)

\[
F'(x) - J_n F'(x) = \sum_{j=0}^{n-1} (J_j F'(x) - J_{j+1} F'(x)) = 0,
\]

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.3), this implies that

\[
N'(F(x) - F'(x), t) = N'(J_n F(x) - J_n F'(x), t)
\]

\[
\geq \min \left\{ N' \left( J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left( J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}
\]

\[
\geq \min \left\{ N' \left( \frac{(F - f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left( \frac{(f - F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right),
\]

\[
N' \left( \frac{(F - f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left( \frac{(f - F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right),
\]

\[
N' \left( \frac{(F - f)(2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left( \frac{(f - F')(2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right),
\]

(2.18)
\[
N' \left( \frac{(F - f)(-2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), \quad N' \left( \frac{(f' - F')(2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right)
\geq \sup_{t \in \mathbb{N}} \left( x, \frac{2^{(q-1)n-2}t^q}{((7 + 2p + 3p + 4p)(4 - 2p)3^n + (5 + 2 \cdot 2p + 3p)/2(2 - 2p))^q} \right),
\]

for all \( x \in X \) and \( n \in \mathbb{N} \). Observe that for \( q = 1/p \), the last term of the above inequality tends to 1 as \( n \to \infty \) by (N5). This implies that \( N'(F(x) - F'(x), t) = 1 \), and so we get
\[
F(x) = F'(x),
\]
for all \( x \in X \) by (N2).

Case 2. Let \( 1/2 < q < 1 \), and let \( J_n f : X \to Y \) be a mapping defined by
\[
J_n f(x) = \frac{1}{2} \left( 4^{-n}(f(2^n x) + f(-2^n x) - 2f(0)) + 2^n (f(\frac{x}{2^n}) - f(-\frac{x}{2^n})) + f(0) \right),
\]
for all \( x \in X \). Then, we have \( J_0 f(x) = f(x), J_1 f(0) = f(0) \), and
\[
J_j f(x) - J_{j+1} f(x) = \frac{Df(2^j x/3, 2^j x/3)}{2^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}}
\]
\[
- \frac{Df(2^j x/3, -2^j x/3)}{2 \cdot 4^{j+1}} - \frac{Df(-2^j x/3, -2^j x/3)}{2 \cdot 4^{j+1}}
\]
\[
- 2^{j-1} \left( Df(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}) - Df(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}) + Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}) \right)
\]
\[
+ Df(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}),
\]
for all \( x \in X \) and \( j \geq 0 \). If \( n + m > m \geq 0 \), then we have
\[
N' \left( f_m f(x) - f_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{7 + 2p + 3p + 4p}{4 \cdot 3^n} \left( \frac{2^p}{4} \right)^j + \frac{5 + 2 \cdot 2p + 3p}{2 \cdot 2p} \left( \frac{2}{2^p} \right)^j \right) t^p \right)
\]
\[
\geq \min \left\{ \min_{j=m}^{n+m-1} \left( N' \left( \frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}}, 2 \cdot 2^{j+1} t^p \right) \right) \right\},
\]
Define the limit $F$ for all $x \in Y$. Moreover, putting $\frac{(2^{j-1}x - 2^{j+1}x/3) - 2^{j}t}{2^{j+1}}$, we have

$$N'(f) = N(x, t),$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.6) of the previous case, we can define the limit $F(x) := N' - \lim_{n \to \infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space $Y$. Moreover, putting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_n f(x), t) \geq \min \left\{ \min \left\{ N \left( \frac{x}{2^{j+1}}, \frac{t}{2^{j+1}} \right), N \left( \frac{x}{2^j}, \frac{t}{2^j} \right), N \left( \frac{3x}{2^{j+1}}, \frac{3t}{2^{j+1}} \right), N \left( \frac{2^{j+1}x}{3}, \frac{2^{j+1}t}{3} \right), N \left( \frac{2^{j+2}x}{3}, \frac{2^{j+2}t}{3} \right) \right\} \right\}$$

(2.22)
for all $x \in X$ and $t > 0$. To prove that $F$ is a general quadratic mapping, we have enough to show that the last term of (2.12) in Case 1 tends to 1 as $n \to \infty$. By (N3) and (2.2), we get

$$N\left( D_{n}f(x, y), \frac{t}{2} \right) \geq \min \left\{ N\left( D_{n}f(x, y), \frac{2^{n}Df(2^{n}x, 2^{n}y) - 2^{n}Df(-2^{n}x, -2^{n}y)}{2 \cdot 4^{n}}, \frac{t}{8} \right), N\left( D_{n}f(x, y), \frac{2^{n-1}Df(-x, -y)}{2 \cdot 4^{n}}, \frac{t}{8} \right) \right\} \geq \min \left\{ N\left( x, 2^{(2q-1)n-4q}q \right), N\left( y, 2^{(2q-1)n-4q}q \right) \right\},$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $1/2 < q < 1$. Hence, together with the similar argument after (2.12), we can say that $DF(x, y) = 0$ for all $x, y \in X$. Recall that in Case 1, (2.3) follows from (2.11). By the same reasoning, we get (2.3) from (2.23) in this case. Now, to prove the uniqueness of $F$, let $F'$ be another general quadratic mapping satisfying (2.3). Then, together with (N4), (2.3), and (2.17), we have

$$N\left( F(x) - F'(x), t \right) = N\left( J_{n}F(x) - J_{n}F'(x), t \right) \geq \min \left\{ N\left( J_{n}F(x) - J_{n}F'(x), \frac{t}{2} \right), N\left( J_{n}f(x) - J_{n}f'(x), \frac{t}{2} \right) \right\} \geq \min \left\{ N\left( (f - f)(2^{n}x) - (f - f)(2^{n}x), \frac{2^{n}Df(2^{n}x, 2^{n}y)}{2 \cdot 4^{n}}, \frac{t}{8} \right), N\left( (f - f)(2^{n}x) - (f - f)(2^{n}x), \frac{2^{n}Df(-x, -y)}{2 \cdot 4^{n}}, \frac{t}{8} \right) \right\} \geq \min \left\{ N\left( x, 2^{(2q-1)n-4q}q \right), N\left( y, 2^{(2q-1)n-4q}q \right) \right\},$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n \to \infty} 2^{(2q-1)n-4q}q = \lim_{n \to \infty} 2^{(1-q)n-2q} = \infty$ in this case, both terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This implies that $N\left( F(x) - F'(x), t \right) = 1$, and so $F(x) = F'(x)$ for all $x \in X$ by (N2).
Finally, we take $0 < q < 1/2$ and define $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2} \left( 4^n (f(2^{-n} x) + f(-2^{-n} x) - 2 f(0)) + 2^n \left( f \left( \frac{x}{2^n} \right) - f \left( -\frac{x}{2^n} \right) \right) \right) + f(0), \quad (2.26)$$

for all $x \in X$. Then, we have $J_0 f(x) = f(x), J_1 f(0) = f(0)$, and

$$J_j f(x) - J_{j+1} f(x) = -4^j D f \left( \frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}} \right) - 4^j D f \left( \frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}} \right)$$

$$+ \frac{4^j}{2} D f \left( \frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}} \right) + \frac{4^j}{2} D f \left( \frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}} \right)$$

$$+ \frac{4^j}{2} D f \left( \frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}} \right) + \frac{4^j}{2} D f \left( \frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}} \right)$$

$$- 2^{j-1} \left( D f \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) - D f \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) + D f \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right)$$

$$+ D f \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right), \quad (2.27)$$

which implies that if $n + m > m \geq 0$, then

$$N^j \left( J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{7 + 2^p + 3^p + 4^p}{2^p \cdot 3^p} + \frac{4}{2^p} \right)^i \right) \geq \min \sum_{j=m}^{n+m-1} \left( \min \left\{ N^j \left( -4^j D f \left( \frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}} \right), \frac{2 \cdot 4^j p^p}{2(j+1)^p \cdot 3^p} \right), \right. \right.$$}

$$N^j \left( -4^j D f \left( \frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}} \right), \frac{2 \cdot 4^j p^p}{2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( x/(3 \cdot 2^{j+1}), x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 2^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( x/(3 \cdot 2^{j+1}), x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 3^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( x/(3 \cdot 2^{j+1}), x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 4^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( -x/(3 \cdot 2^{j+1}), -x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 2^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( -x/(3 \cdot 2^{j+1}), -x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 3^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}

$$N^j \left( \frac{4^j D f \left( -x/(3 \cdot 2^{j+1}), -x/(3 \cdot 2^{j+1}) \right)}{2}, \frac{4^j (1 + 4^p)^p}{2 \cdot 2(j+1)^p \cdot 3^p} \right), \right.$$}
for all \( x \in X \setminus \{0\} \) and \( t > 0 \). Similar to the previous cases, it leads us to define the mapping 
\( F : X \to Y \) by \( F(x) := N' - \lim_{n \to \infty} J_n f(x) \). Putting \( m = 0 \) in the above inequality, we have

\[
N'(f(x) - J_n f(x), t) = N(\frac{x}{t}) \geq N(\frac{x}{t}) = \frac{x}{t},
\]

for all \( x \in X \). Notice that

\[
N'(D J_n f(x,y), \frac{t}{2}) \geq \min \left\{ N'\left( \frac{4^n}{2} D f\left( \frac{y}{2^n}, \frac{t}{8} \right) \right), N'\left( \frac{4^n}{2} D f\left( \frac{y}{2^n}, \frac{t}{8} \right) \right) \right\}
\]

for all \( x, y \in X \setminus \{0\} \) and \( t > 0 \). Since \( 0 < q < 1/2 \), both terms on the right-hand side tend to 1 as \( n \to \infty \), which implies that the last term of (2.12) tends to 1 as \( n \to \infty \). Therefore, we can say that \( DF \equiv 0 \). Moreover, using the similar argument after (2.12) in Case 1, we get...
Remark 2.3. Consider a mapping \( f : X \to Y \) satisfying (2.2) for all \( x, y \in X \setminus \{0\} \) and a real number \( q < 0 \). Take any \( t > 0 \). If we choose a real number \( s \) with \( 0 < 2s < t \), then we have

\[
N'(Df(x,y), t) \geq N'(Df(x,y), 2s) \geq \min\{N(x,s^t), N(y,s^t)\},
\]

for all \( x, y \in X \setminus \{0\} \). Since \( q < 0 \), we have \( \lim_{s \to 0^+} s^t = \infty \). This implies that

\[
\lim_{s \to 0^+} N(x,s^t) = \lim_{s \to 0^+} N(y,s^t) = 1,
\]

and so

\[
N'(Df(x,y), t) = 1,
\]

for all \( t > 0 \) and \( x, y \in X \setminus \{0\} \). Since \( DF(0,0) = 0 \), \( DF(x,0) = 0 \), and \( DF(0,y) = 0 \) for all \( x, y \in X \setminus \{0\} \), this means that \( DF(x,y) = 0 \) for all \( x, y \in X \setminus \{0\} \) by (N2). In other words, \( f \) is itself a general quadratic mapping if \( f \) is a fuzzy \( q \)-almost general quadratic mapping for the case \( q < 0 \).
We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let \((X, \| \cdot \|)\) be a normed linear space. Then, we can define a fuzzy norm \(N_X\) on \(X\) by

\[
N_X(x, t) = \begin{cases} 
0, & t \leq \|x\|, \\
1, & t > \|x\|,
\end{cases}
\]

(2.35)

where \(x \in X\) and \(t \in \mathbb{R}\) [21]. Suppose that \(f : X \to Y\) is a mapping into a Banach space \((Y, || \cdot ||)\) such that

\[
\||Df(x, y)|| \leq \|x\|^p + \|y\|^p,
\]

(2.36)

for all \(x, y \in X\), where \(p > 0\) and \(p \neq 1, 2\). Let \(N_Y\) be a fuzzy norm on \(Y\). Then, we get

\[
N_Y(Df(x, y), t + s) = \begin{cases} 
0, & t + s \leq \|Df(x, y)\|, \\
1, & t + s > \|Df(x, y)\|,
\end{cases}
\]

(2.37)

for all \(x, y \in X\) and \(s, t \in \mathbb{R}\). Consider the case \(N_Y(Df(x, y), t + s) = 0\). This implies that

\[
\|x\|^p + \|y\|^p \geq \||Df(x, y)|| \geq t + s,
\]

(2.38)

and so, either \(\|x\|^p \geq t\) or \(\|y\|^p \geq s\) in this case. Hence, for \(q = 1/p\), we have

\[
\min\{N_X(x, s^q), N_X(y, t^q)\} = 0,
\]

(2.39)

for all \(x, y \in X\) and \(s, t > 0\). Therefore, in every case,

\[
N_Y(Df(x, y), t + s) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}
\]

(2.40)

holds. It means that \(f\) is a fuzzy \(q\)-almost general quadratic mapping, and by Theorem 2.2, we get the following stability result.

**Corollary 2.4.** Let \((X, \| \cdot \|)\) be a normed linear space, and let \((Y, || \cdot ||)\) be a Banach space. If \(f : X \to Y\) satisfies

\[
\||Df(x, y)|| \leq \|x\|^p + \|y\|^p,
\]

(2.41)

for all \(x, y \in X\), where \(p > 0\) and \(p \neq 1, 2\), then there is a unique general quadratic mapping \(F : X \to Y\) such that

\[
\||F(x) - f(x)|| \leq \left(\frac{2(7 + 2^p + 3^p + 4^p)}{3^p|4 - 2^p|} + \frac{5 + 2 \cdot 2 + 3^p}{2 - 2^p}\right)\|x\|^p,
\]

(2.42)

for all \(x \in X\).
References


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