Research Article
An Inequality of Meromorphic Vector Functions and Its Application

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Firstly, an inequality for vector-valued meromorphic functions is established which extend a corresponding inequality of Milloux for meromorphic scalar-valued function (1946). As an application, the relationship between the characteristic function of a vector-valued meromorphic function $f$ and its derivative $f'$ is studied, results are obtained to extend some related results for meromorphic scalar-valued function of Weitsman (1969) and Singh and Gopalakrishna (1971).

1. Introduction of Vector-Valued Meromorphic Function

In 1980s, Ziegler [1] established Nevanlinna’s theory for the vector-valued meromorphic function in finite dimensional spaces. After Ziegler some works related to vector-valued meromorphic function were done in 1990s [2–4]. In this section, we shall introduce the following fundamental notations and results of vector-valued Nevanlinna theory which were quoted from Ziegler [1].

We denote by $\mathbb{C}^n$ the usual $n$ dimensional complex Euclidean space with the coordinates $w = (w_1, w_2, \ldots, w_n)$, the Hermitian scalar product
\begin{equation}
\langle v, w \rangle = v_1 \overline{w}_1 + v_2 \overline{w}_2 + \cdots + v_n \overline{w}_n, \quad (v, w \in \mathbb{C}^n),
\end{equation}
and the distance
\begin{equation}
||v - w|| = + (v - w, v - w)^{1/2}.
\end{equation}
Let
\[ w_1 = f_1(z), \quad w_2 = f_2(z), \ldots, w_n = f_n(z) \] (1.3)
be \( n \geq 1 \) complex valued functions of the complex variable \( z \), which are meromorphic and not all constant in the Gaussian plane \( \mathbb{C}^1 = \mathbb{C} \), or in a finite disc
\[ \mathbb{C}_R = \{|z| < R\} \subset \mathbb{C}, \quad 0 < R < +\infty. \] (1.4)
Thus in \( \mathbb{C}_R, \ 0 < R \leq +\infty \) (we put \( \mathbb{C}_0 = \mathbb{C} \)), a vector-valued meromorphic function
\[ f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \] (1.5)
is given, which does not reduce to the constant zero vector \( 0 = (0,0,\ldots,0) \). The \( j \)th derivative \( j = 1,2,\ldots \) of \( f(z) \) are defined by
\[ f^{(j)}(z) = \left( f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_n^{(j)}(z) \right). \] (1.6)

For such a function, the notations “zero,” “pole,” and “multiplicity” are defined as in the scalar case \( n = 1 \) of only one meromorphic function \( f_1(z) \). More explicitly, in the punctured vicinity of each point \( z_0 \in \mathbb{C}_R \), the vector function \( w = f(z) \) can developed into a Laurent series
\[ f(z) = c_{k_0}(z - z_0)^{k_0} + c_{k_0+1}(z - z_0)^{k_0+1} + \cdots, \] (1.7)
where the coefficients are vectors
\[ c_k = (c_k^1, c_k^2, \ldots, c_k^n) \in \mathbb{C}^n, \quad c_{k_0} \neq (0,0,\ldots,0). \] (1.8)

In order to introduce the Nevanlinna theory of vector-valued meromorphic function, we will denote by “\( \infty \)” the ideal element of the Aleksandrov one-point compactification of \( \mathbb{C}^n \) (the two real infinities will be denoted by \( +\infty \) and \( -\infty \), resp.). Now, if \( k_0 < 0 \) in the above Laurent expansion, then \( z_0 \) will be called a pole or an \( \infty \)-point of \( f(z) \) of multiplicity \( -k_0 \); in such a point \( z_0 \) at least one of the meromorphic component functions \( f_j(z) \) has a pole of this multiplicity in the ordinary sense of function theory, so that in \( z_0 \) itself \( f(z) \) is not defined. If \( k_0 > 0 \) in Laurent expansion, then \( z_0 \) is called a zero of \( f(z) \) of multiplicity \( k_0 \); in such a point \( z_0 \), all component functions \( f_j(z) \) vanish, each with at least this multiplicity.
Let \( n(r, f) \) or \( n(r, \infty) \) denote the number of poles of \( f(z) \) in \( |z| \leq r \) and \( n(r, a) \) denote the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), counting with multiplicities. Define the volume function associated with vector-valued meromorphic function \( f(z) \),

\[
V(r, \infty) = V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \| f(\xi) \| dx \wedge dy, \quad \xi = x + iy
\]

\[
V(r, a) = V(r, \frac{1}{f-a}) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \| f(\xi) - a \| dx \wedge dy, \quad \xi = x + iy
\]  \hspace{1cm} (1.9)

and the counting function of finite or infinite \( a \)-points by

\[
N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt, \\
N(r, \infty) = n(0, \infty) \log r + \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt, \\
N(r, a) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt,
\]

respectively. Next, we define

\[
m(r, \infty) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \| f(re^{i\theta}) \| d\theta, \\
m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\| f(re^{i\theta}) - a \|} d\theta, \\
T(r, f) = m(r, f) + N(r, f).
\]  \hspace{1cm} (1.11)

Let \( \overline{n}(r, f) \) or \( \overline{n}(r, \infty) \) denote the number of poles of \( f(z) \) in \( |z| \leq r \) and \( \overline{n}(r, a) \) denote the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), ignoring multiplicities. Define the counting function of finite or infinite \( a \)-points by

\[
\overline{N}(r, f) = \overline{n}(0, f) \log r + \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt, \\
\overline{N}(r, \infty) = \overline{n}(0, \infty) \log r + \int_0^r \frac{\overline{n}(t, \infty) - \overline{n}(0, \infty)}{t} dt, \\
\overline{N}(r, a) = \overline{n}(0, a) \log r + \int_0^r \frac{\overline{n}(t, a) - \overline{n}(0, a)}{t} dt
\]  \hspace{1cm} (1.12)

respectively.
If \( f(z) \) is a vector-valued meromorphic function in the whole complex plane, then the order and the lower order of \( f(z) \) are defined by

\[
\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},
\]

(1.13)

\[
\mu(f) = \liminf_{r \to -\infty} \frac{\log T(r, f)}{\log r}.
\]

We call the vector-valued meromorphic function \( f \) admissible if

\[
\limsup_{r \to +\infty} \frac{T(r, f)}{\log r} = +\infty.
\]

Definition 1.1. For a meromorphic function \( f(z) \) (vector-valued or scalar-valued), we denote by \( S(r, f) \) any quantity such that

\[
S(r, f) = o(T(r, f)), \quad r \to +\infty
\]

without restriction if \( f(z) \) is of finite order and otherwise except possibly for a set of values of \( r \) of finite linear measure.

Definition 1.1 quoted from [2]. In [1], Ziegler established the following first main theorem, logarithmic derivative lemma, and deficient values theorem for meromorphic vector function.

Theorem A. Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be a meromorphic vector function in \( \mathbb{C}_R \). Then for \( 0 < r < R \leq +\infty \), \( a \in \mathbb{C}^n \), \( f(z) \neq a \), then

\[
T(r, f) = V(r, a) + N(r, a) + m(r, a) + O(1).
\]

(1.16)

Theorem B. Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be a nonconstant meromorphic vector function in \( \mathbb{C} \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| \frac{f'(re^{i\theta})}{f(re^{i\theta}) - a} \right\| d\theta = S(r, f), \quad a \in \mathbb{C}^n.
\]

(1.17)

By the second main theorem, Ziegler [1] studies the following deficiency theorem for meromorphic vector function. For any vector \( a \in \mathbb{C}^n \), we define the number \( \delta(a) = \delta(a, f) \) by putting

\[
\delta(a) = \delta(a, f) = \liminf_{r \to +\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{V(r, a) + N(r, a)}{T(r, f)},
\]

(1.18)

\[
\delta(\infty) = \delta(\infty, f) = \liminf_{r \to +\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N(r, f)}{T(r, f)}.
\]
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and \( \Theta(a) = \Theta(a, f) \) by putting

\[
\Theta(a) = \Theta(a, f) = 1 - \lim_{r \to +\infty} \sup_{r > 0} \frac{V(r, a) + N(r, a)}{T(r, f)},
\]

(1.19)

\[
\Theta(\infty) = \Theta(\infty, f) = 1 - \lim_{r \to +\infty} \frac{N(r, f)}{T(r, f)},
\]

\[
\Theta(z) = \Theta(z, f) = 1 - \lim_{r \to +\infty} \frac{V(r, z) + N(r, z)}{T(r, f)},
\]

(1.20)

2. A Fundamental Inequality of Meromorphic Vector Function

For meromorphic scalar-valued function \( f(z) \), Milloux [5] has proved the following theorem.

**Theorem D.** If \( f(z) \) is a nonconstant meromorphic scalar-valued function in Gaussian complex plane \( \mathbb{C} \) and if \( a_i, i = 1, 2, \ldots, q \), are distinct elements of \( \mathbb{C} \) (where \( q \) is any positive integer), then

\[
qT(r, f) \leq T(r, f') + \sum_{i=1}^{q} N(r, a_i) + S(r, f).
\]

(2.1)

For some alternative proofs of Theorem D, see [6] or [7]. It is natural to consider whether there exists a similar results if meromorphic scalar-valued function \( f(z) \) is replaced by meromorphic vector-valued function \( f(z) \). In this section, the main contribution is to extend Theorem D to vector-valued meromorphic function by referring the method of [1, 7].

**Theorem 2.1.** Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be an admissible meromorphic vector function in \( \mathbb{C} \) and if \( a^{[j]}_i, j = 1, 2, \ldots, q \), are distinct elements of \( \mathbb{C}^n \) (where \( q \) is any positive integer), then

\[
qT(r, f) \leq T(r, f') + \sum_{j=1}^{q} \left( N(r, a^{[j]}_i) + V(r, a^{[j]}_i) \right) + S(r, f).
\]

(2.2)

**Proof.** Put

\[
F(z) = \sum_{j=1}^{q} \frac{1}{\| f(z) - a^{[j]}_i \|}.
\]

(2.3)

We can get

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log^+ F(re^{i\theta}) d\theta \leq m(r, 0, f') + \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ \left\{ F(re^{i\theta}) \left\| f'(re^{i\theta}) \right\| \right\} d\theta.
\]

(2.4)
Let for the moment \( \mu \in [1, 2, \ldots, q] \) be fixed. Then we get in every point where

\[
\| f(z) - a^{[\mu]} \| \leq \frac{\delta}{2q} \leq \frac{\delta}{4},
\]

the inequality

\[
\| f(z) - a^{[\nu]} \| \geq \| a^{[\mu]} - a^{[\nu]} \| - \| f(z) - a^{[\mu]} \| \geq \frac{3\delta}{4},
\]

for \( \mu \neq \nu \). Therefore, the set of points on \( \partial \mathbb{C}_r \), which is determined by (2.6) is either empty or any two such sets for different \( \mu \) have empty intersection. In any case,

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int \| f(z) - a^{[\mu]} \| < \delta/2q, |z|=r \log^+ F(re^{i\theta}) d\theta
\]

\[
\geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int \| f(z) - a^{[\mu]} \| < \delta/2q, |z|=r \log^+ \frac{1}{\| f(re^{i\theta}) - a^{[\mu]} \|} d\theta.
\]

Because of

\[
\frac{1}{2\pi} \int \| f(z) - a^{[\mu]} \| < \delta/2q, |z|=r \log^+ \frac{1}{\| f(re^{i\theta}) - a^{[\mu]} \|} d\theta
\]

\[
= m(r, a^{[\mu]}) \geq \frac{1}{2\pi} \int \| f(z) - a^{[\mu]} \| < \delta/2q, |z|=r \log^+ \frac{1}{\| f(re^{i\theta}) - a^{[\mu]} \|} d\theta
\]

\[
\geq m(r, a^{[\mu]}) - \log^+ \frac{2q}{\delta},
\]

it follows that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \geq \sum_{\mu=1}^{q} m(r, a^{[\mu]}) - \log^+ \frac{2q}{\delta},
\]

so that by (2.4)

\[
\sum_{\mu=1}^{q} m(r, a^{[\mu]}) \leq m(r, 0, f') + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\{ F(re^{i\theta}) \| f'(re^{i\theta}) \| \right\} d\theta + \log^+ \frac{2q}{\delta},
\]
Thus by Theorem B, we have

\[ \sum_{\mu=1}^{q} m\left(r, a^{[\mu]} \right) \leq m(r, 0, f') + S(r, f). \tag{2.12} \]

It follows from Theorem A that

\[ m(r, 0, f') + N(r, 0, f') + V(r, 0, f') = T(r, f') + O(1). \tag{2.13} \]

Thus from (2.12) and (2.13), we deduce

\[ \sum_{\mu=1}^{q} m\left(r, a^{[\mu]} \right) \leq T(r, f') - N(r, 0, f') + S(r, f). \tag{2.14} \]

Adding \( \sum_{\mu=1}^{q} N(r, a^{[\mu]}) \) to both sides,

\[ \sum_{\mu=1}^{q} T\left(r, \frac{1}{f - a^{[\mu]}} \right) \leq T(r, f') + \sum_{\mu=1}^{q} N\left(r, a^{[\mu]} \right) - N(r, 0, f') + S(r, f) \]

\[ = T(r, f') + \sum_{\mu=1}^{q} N\left(r, a^{[\mu]} \right) - N_0(r, 0, f') + S(r, f), \tag{2.15} \]

where \( N_0(r, 0, f') \) is formed with the zeros of \( f' \) which are not zeros of any of \( f - a^{[\mu]}, (i = 1, 2, \ldots, q) \). Since \( N_0(r, 0, f') \geq 0 \), we have

\[ \sum_{\mu=1}^{q} T\left(r, \frac{1}{f - a^{[\mu]}} \right) \leq T(r, f') + \sum_{\mu=1}^{q} N\left(r, a^{[\mu]} \right) + S(r, f). \tag{2.16} \]

Since

\[ T\left(r, \frac{1}{f - a^{[\mu]}} \right) + V\left(r, a^{[\mu]} \right) = T(r, f) + O(1), \tag{2.17} \]

it follows that

\[ qT(r, f) \leq T(r, f') + \sum_{j=1}^{q} \left( N\left(r, a^{[j]} \right) + V\left(r, a^{[j]} \right) \right) + S(r, f). \tag{2.18} \]
3. Characteristic Function of Derivative of Meromorphic Vector Function

Let \( f(z) \) be a meromorphic scalar-valued function in \( \mathbb{C} \). The characteristic function of derivative of \( f(z) \) with \( \sum_a \delta(a) = 2 \) has been studied by Edrei [8], Shan and Singh [9], Singh and Gopalakrishna [7], Singh and Kulkarni [10] and Weitsman [11]. For example, Edrei [8] and Weitsman [11] have proved the following theorem.

**Theorem E.** Let \( f(z) \) be a transcendental meromorphic scalar-valued function of finite order and assume \( \sum_a \delta(a) = \eta \geq 1 \) and \( \delta(\infty) = 2 - \eta \). Then

\[
T(r, f') - \eta T(r, f), \quad r \to +\infty.
\]  

(3.1)

If \( \sum_a \delta(a) = 2 \) is replaced by \( \sum_a \Theta(a) = 2 \), Singh and Gopalakrishna [7] and Singh and Kulkarni [10] have proved the following theorem.

**Theorem F.** Let \( f(z) \) be a transcendental meromorphic scalar-valued function of finite order and assume \( \sum_a \Theta(a) = 2 \). Then

\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty),
\]

(3.2)

\[
\lim_{r \to +\infty} \frac{N(r, a)}{T(r, f)} = 1 - \Theta(a)
\]

for every \( a \in \mathbb{C} \cup \{\infty\} \).

It is natural to consider whether there exists a similar results if meromorphic scalar-valued function \( f(z) \) is replaced by meromorphic vector-valued function \( f(z) \). In this section, the main purpose is to extend the above theorems to vector-valued meromorphic function by referring the method of [1, 7].

**Theorem 3.1.** Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be an admissible meromorphic vector function of finite order in \( \mathbb{C} \) and assume \( \sum_a \Theta(a) = 2 \). Then

\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty),
\]

(3.3)

\[
\lim_{r \to +\infty} \frac{N(r, f)}{T(r, f)} = 1 - \Theta(\infty), \quad \lim_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a)
\]

for every \( a \in \mathbb{C}^n \).
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Proof. Now, basic estimates in vector-valued Nevanlinna theory \([1]\) or \([4]\) yields

\[
T(r, f') = m(r, f') + N(r, f') \\
= m\left( r, \frac{f'}{f} \right) + N(r, f') \\
\leq m\left( r, \frac{f'}{f} \right) + m(r, f) + N(r, f) + N(r, f) \\
\leq T(r, f) + N(r, f) + m\left( r, \frac{f'}{f} \right) .
\]

By Theorem B and the above inequality, we have

\[
\limsup_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \Theta(\infty). \tag{3.5}
\]

Let \( \{a^{[j]}\} \) be a sequence of distinct vector in \( \mathbb{C}^n \) containing all the vector of \( \delta(a^{[j]}) > 0 \).

From Theorem 2.1, for any positive integer \( q \), we have

\[
qT(r, f) \leq T(r, f') + \sum_{j=1}^{q} \left( N(r, a^{[j]}) + V(r, a^{[j]}) \right) + S(r, f) . \tag{3.6}
\]

Hence

\[
q \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{j=1}^{q} \limsup_{r \to +\infty} \frac{N(r, a^{[j]}) + V(r, a^{[j]})}{T(r, f)} + \limsup_{r \to +\infty} \frac{S(r, f)}{T(r, f)} \\
= \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{j=1}^{q} \left\{ 1 - \Theta(a^{[j]}) \right\} + \limsup_{r \to +\infty} \frac{S(r, f)}{T(r, f)} . \tag{3.7}
\]

Thus

\[
\liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \geq \sum_{j=1}^{q} \Theta(a^{[j]}) . \tag{3.8}
\]

Since \( q \) was arbitrary, we have

\[
2 - \Theta(\infty) = \sum_{a \in \mathbb{C}^n} \Theta(a) \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} . \tag{3.9}
\]
This and (3.5) yield
\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty). \tag{3.10}
\]

Let \( a \in \mathbb{C}^n \cup \{\infty\} \) and \( \{a^{[i]}\}_{i=1}^{+\infty} \) an infinite sequence of distinct elements of \( \mathbb{C}^n \cup \{\infty\} \) which includes every \( b \in \mathbb{C}^n \cup \{\infty\} \) satisfying \( b \neq a \) and \( \Theta(b) > 0 \). Then
\[
\sum_{i=1}^{+\infty} \Theta(a^{[i]}) = \sum_{b \in \mathbb{C}^n \cup \{\infty\}, b \neq a} \Theta(b) = 2 - \Theta(a). \tag{3.11}
\]

Let \( q \) be any integer \( \geq 3 \). From Generalized Second Main Theorem (see [1], Page 126), we have
\[
(q - 2) T(r, f) = \sum_{i=1}^{q-1} \left( N(r, a^{[i]}) + V(r, a^{[i]}) \right) + \bar{N}(r, f) + S(r, f). \tag{3.12}
\]
Hence
\[
q - 2 \leq \sum_{i=1}^{q-1} \left\{ 1 - \Theta(a^{[i]}) \right\} + \liminf_{r \to +\infty} \frac{\bar{N}(r, f)}{T(r, f)}. \tag{3.13}
\]
Thus
\[
\sum_{i=1}^{q-1} \Theta(a^{[i]}) - 1 \leq \liminf_{r \to +\infty} \frac{\bar{N}(r, f)}{T(r, f)}. \tag{3.14}
\]
Since this holds for all \( q \geq 3 \), letting \( q \to +\infty \) and combining (3.11), we get
\[
1 - \Theta(\infty) = \sum_{i=1}^{+\infty} \Theta(a^{[i]}) - 1 \leq \liminf_{r \to +\infty} \frac{\bar{N}(r, f)}{T(r, f)} \leq \limsup_{r \to +\infty} \frac{\bar{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty). \tag{3.15}
\]
So
\[
\lim_{r \to +\infty} \frac{\bar{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty). \tag{3.16}
\]

For every \( a \in \mathbb{C}^n \), Let \( q \) be any integer \( \geq 3 \). From Generalized Second Main Theorem (see [1], Page 126), we have
\[
(q - 2) T(r, f) = \sum_{i=1}^{q-2} \left( N(r, a^{[i]}) + V(r, a^{[i]}) \right) + \left( \bar{N}(r, a) + V(r, a) \right) + \bar{N}(r, f) + S(r, f). \tag{3.17}
\]
\[ q - 2 \leq \sum_{i=1}^{q-2} \{1 - \Theta\left(a^{[i]}\right)\} + (1 - \Theta(\infty)) + \liminf_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)}. \] 

(3.18)

Thus

\[ \sum_{i=1}^{q-2} \Theta\left(a^{[i]}\right) + \Theta(\infty) - 1 \leq \liminf_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)}. \] 

(3.19)

Since this holds for all \( q \geq 3 \), letting \( q \to +\infty \) and combining (3.11), we get

\[
1 - \Theta(a) = \sum_{i=1}^{+\infty} \Theta\left(a^{[i]}\right) - 1 \leq \liminf_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} \\
\leq \limsup_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a).
\]

(3.20)

So

\[
\lim_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a).
\]

(3.21)

From Theorem 3.1, we have the following corollary

**Corollary 3.2.** Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be an admissible meromorphic vector function of finite order in \( \mathbb{C} \) and assume \( \sum_{a \in \mathbb{C}^n} \Theta(a) = 2 \). Then

\[ T(r, f') \sim 2T(r, f), \quad r \to +\infty. \]

(3.22)

**Corollary 3.3.** Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be an admissible meromorphic vector function of finite order in \( \mathbb{C} \) and assume \( \sum_{a \in \mathbb{C}^n} \Theta(a) = \eta \geq 1 \) and \( \delta(\infty) = 2 - \eta \). Then

\[ T(r, f') \sim \eta T(r, f), \quad r \to +\infty. \]

(3.23)

**Corollary 3.4.** Let \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \) be an admissible meromorphic vector function of finite order in \( \mathbb{C} \) and assume \( \sum_{a} \delta(a) = 2 \). Then

\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty),
\]

\[
\lim_{r \to +\infty} \frac{N(r, f)}{T(r, f)} = 1 - \delta(\infty) \quad \text{and} \quad \lim_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} = 1 - \delta(a)
\]

for every \( a \in \mathbb{C}^n \).
Proof. Since $\delta(a) \leq \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$ and Theorem C, it follows that, if $\sum_a \Theta(a) = 2$, then $\sum_a \delta(a) = 2$ and $\delta(a) = \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$. Hence

$$
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty)
$$

follows by Theorem 3.1.

Now, for every $a \in \mathbb{C}^n$,

$$
\lim_{r \to +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a) = 1 - \delta(a).
$$

Further

$$
\overline{N}(r, a) \leq N(r, a).
$$

Hence

$$
1 - \delta(a) \leq \lim_{r \to +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} \\
\leq \lim_{r \to +\infty} \inf \frac{N(r, a) + V(r, a)}{T(r, f)} \\
\leq \lim_{r \to +\infty} \sup \frac{N(r, a) + V(r, a)}{T(r, f)} \\
= 1 - \delta(a).
$$

Similarly,

$$
\lim_{r \to +\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty) = 1 - \delta(\infty).
$$

Further

$$
\overline{N}(r, \infty) \leq N(r, \infty).
$$

Hence

$$
1 - \delta(\infty) \leq \lim_{r \to +\infty} \frac{\overline{N}(r, \infty)}{T(r, f)} \leq \lim_{r \to +\infty} \inf \frac{N(r, \infty)}{T(r, f)} \leq \lim_{r \to +\infty} \sup \frac{N(r, \infty)}{T(r, f)} = 1 - \delta(\infty).
$$

From Corollary 3.4, we have the following corollary.
Corollary 3.5. Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}} \delta(a) = 2$. Then

$$T(r, f') \sim 2T(r, f), \quad r \to +\infty.$$  \hfill (3.32)

Corollary 3.6. Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}} \delta(a) = \eta \geq 1$ and $\delta(\infty) = 2 - \eta$. Then

$$T(r, f') \sim \eta T(r, f), \quad r \to +\infty.$$  \hfill (3.33)

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References
