Research Article

Nearly Jordan ∗-Homomorphisms between Unital C∗-Algebras

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Received 26 February 2011; Revised 7 April 2011; Accepted 10 April 2011

1. Introduction

The stability of functional equations was first introduced by Ulam [1] in 1940. More precisely, he proposed the following problem: given a group $G_1$, a metric group $(G_2, d)$ and a positive number $\varepsilon$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$. As mentioned above, when this problem has a solution, we say that the homomorphisms from $G_1$ to $G_2$ are stable. In 1941, Hyers [2] gave a partial solution of Ulam’s problem for the case of approximate additive mappings under the assumption that $G_1$ and $G_2$ are Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. This phenomenon of stability is called the Hyers-Ulam-Aoki-Rassias stability.

During the last decades, several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [8–10].

Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [11–19] and references therein.

Jun and Lee [20] proved the following: Let $X$ and $Y$ be Banach spaces. Denote by $\phi : X - \{0\} \times Y - \{0\} \to [0, \infty)$ a function such that $\phi(x, y) = \sum_{n=0}^{\infty} 3^{-n} \phi(3^n x, 3^n y) < \infty$ for all $x, y \in X - \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\left\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right\| \leq \phi(x, y)$$

for all $x, y \in X - \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\left\| f(x) - f(0) - T(x) \right\| \leq \frac{1}{3} \left( \phi(x, -x) + \phi(-x, 3x) \right)$$

for all $x \in X - \{0\}$.

Recently, C. Park and W. Park [21] applied the Jun and Lee’s result to the Jensen’s equation in Banach modules over a $C^*$-algebra. Johnson (Theorem 7.2 of [22]) also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras. Suppose that $U$ and $B$ are Banach $*$-algebras which satisfy the conditions of (Theorem 3.1 of [22]). Then for each positive $e$ and $K$, there is a positive $\delta$ such that if $T ∈ L(U, B)$ with $\|T\| < K$, $\|T^*\| < \delta$ and $\|T(x^*)^* - T(x)\| < \delta \|x\|(x ∈ U)$, then there is a $*$-homomorphism $T' : U \to B$ with $\|T - T'\| < e$. Here $L(U, B)$ is the space of bounded linear maps from $U$ into $B$, and $T'(x, y) = T(xy) - T(x)T(y)(x, y ∈ U)$. See [22] for details. Throughout this paper, let $A$ be a unital $C^*$-algebra with unit $e$, and $B$ a unital $C^*$-algebra. Let $U(A)$ be the set of unitary elements in $A$, $A_{sa} := \{x ∈ Ax = x^*\}$, and $I_1(A_{sa}) = \{v ∈ A_{sa} \mid \|v\| = 1, v ∈ Inv(A)\}$. In this paper, we prove that every almost unital almost linear mapping $h : A \to B$ is a Jordan homomorphism when $h(3^nu + 3^nu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u ∈ U(A)$, all $y ∈ A$, and all $n = 0, 1, 2, \ldots$, and that for a unital $C^*$-algebra $A$ of real rank zero (see [23]), every almost unital almost linear continuous mapping $h : A \to B$ is a Jordan homomorphism when $h(3^nu + 3^nu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u ∈ I_1(A_{sa})$, all $y ∈ A$, and all $n = 0, 1, 2, \ldots$. Furthermore, we investigate the Hyers-Ulam-Aoki-Rassias stability of Jordan $*$-homomorphisms between unital $C^*$-algebras by using the fixed point methods.

Note that a unital $C^*$-algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [23]). We denote the algebraic center of algebra $A$ by $Z(A)$.

2. Jordan $*$-Homomorphisms on Unital $C^*$-Algebras

By a following similar way as in [24], we obtain the next theorem.
Theorem 2.1. Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u)$$  \hspace{1cm} (2.1)

for all $u \in U(A), y \in A$, and all $n = 0, 1, 2, \ldots$ If there exists a function $\phi : (A - \{0\})^2 \times A \to [0, \infty)$ such that $\tilde{\phi}(x, y, z) = \sum_{n=0}^{\infty} 3^{-n} \phi(3^n x, 3^n y, 3^n z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$ and that

$$\left\| \frac{2f}{2} \left( \frac{ux + uy}{2} \right) - u f(x) - \mu f(y) + f(u^*) + f(u^*) \right\| \leq \phi(x, y, u),$$  \hspace{1cm} (2.2)

for all $\mu \in \mathbb{T}$ and all $x, y \in A, u \in (U(A) \cup \{0\})$. If $\lim_n (f(3^n e)/3^n) = U(B) \cap Z(B)$, then the mapping $f : A \to B$ is a Jordan $*$-homomorphism.

Proof. Put $u = 0, \mu = 1$ in (2.2), it follows from of [20, Theorem 1] that there exists a unique additive mapping $T : A \to B$ such that

$$\left\| f(x) - T(x) \right\| \leq \frac{1}{3} \left( \tilde{\phi}(x, -x, 0) + \tilde{\phi}(-x, 3x, 0) \right)$$  \hspace{1cm} (2.3)

for all $x \in A - \{0\}$. This mapping is given by

$$T(x) = \lim_n \frac{f(3^n x)}{3^n}$$  \hspace{1cm} (2.4)

for all $x \in A$. By the same reasoning as the proof of [24, Theorem 1], $T$ is $\mathbb{C}$-linear and $*$-preserving. It follows from (2.1) that

$$T(uy + yu) = \lim_n \frac{f(3^n uy + 3^n yu)}{3^n} = \lim_n \frac{f(3^n u)f(y) + f(y)f(3^n u)}{3^n} = T(u)f(y) + f(y)T(u)$$  \hspace{1cm} (2.5)

for all $u \in U(A), y \in A$. Since $T$ is additive, then by (2.5), we have

$$3^n T(uy + yu) = T(u(3^n y) + (3^n y)u) = T(u)f(3^n y) + f(3^n y)T(u)$$  \hspace{1cm} (2.6)

for all $u \in U(A)$ and all $y \in A$. Hence,

$$T(uy + yu) = \lim_n \left[ T(u)f(3^n y)/3^n + f(3^n y)/3^n T(u) \right] = T(u)T(y) + T(y)T(u)$$  \hspace{1cm} (2.7)

for all $u \in U(A)$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \cap Z(B)$$  \hspace{1cm} (2.8)
hence, it follows by (2.5) and (2.7) that

\[ 2T(e)T(y) = T(e)T(y) + T(y)T(e) = T(e)f(y) + f(y)T(e) = 2T(e)f(y) \]  

(2.9)

for all \( y \in A \). Since \( T(e) \) is invertible, then \( T(y) = f(y) \) for all \( y \in A \). We have to show that \( f \) is Jordan homomorphism. To this end, let \( x \in A \). By Theorem 4.1.7 of [25], \( x \) is a finite linear combination of unitary elements, that is, \( x = \sum_{j=1}^{n} c_j u_j \ (c_j \in \mathbb{C}, u_j \in U(A)) \), and then it follows from (2.7) that

\[
\begin{align*}
    f(xy + yx) &= T(xy + yx) = T\left( \sum_{j=1}^{n} c_j u_j y + \sum_{j=1}^{n} c_j y u_j \right) \\
    &= \sum_{j=1}^{n} c_j T(u_j y) + T(yu_j) = \sum_{j=1}^{n} c_j (T(u_j)T(y) + T(y)T(u_j)) \\
    &= T\left( \sum_{j=1}^{n} c_j u_j \right) T(y) + T(y)T\left( \sum_{j=1}^{n} c_j u_j \right) = T(x)T(y) + T(y)T(x) \\
    &= f(x)f(y) + f(y)f(x)
\end{align*}
\]

for all \( y \in A \). And this completes the proof of theorem.

\[ \square \]

**Corollary 2.2.** Let \( p \in (0,1), \theta \in [0,\infty) \) be real numbers. Let \( f : A \to B \) be a mapping such that \( f(0) = 0 \) and that

\[
f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u)
\]

(2.11)

for all \( u \in U(A) \), all \( y \in A \), and all \( n = 0, 1, 2, \ldots \). Suppose that

\[
\left\| 2f\left( \frac{\mu x + \mu y}{2} \right) - \mu f(x) - \mu f(y) + f(z^*) - f(z)^* \right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)
\]

(2.12)

for all \( \mu \in \mathbb{T} \) and all \( x, y, z \in A \). If \( \lim_{n} (f(3^n e)/3^n) \in U(B) \cap Z(B) \), then the mapping \( f : A \to B \) is a Jordan \( \ast \)-homomorphism.

**Proof.** Setting \( \phi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p) \) for all \( x, y, z \in A \). Then by Theorem 2.1, we get the desired result.

\[ \square \]

**Theorem 2.3.** Let \( A \) be a \( C^* \)-algebra of real rank zero. Let \( f : A \to B \) be a continuous mapping such that \( f(0) = 0 \) and that

\[
f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u)
\]

(2.13)
for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. Suppose that there exists a function $\phi : (A - \{0\})^2 \times A \rightarrow [0, \infty)$ satisfying (2.2) and $\phi(x, y, z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$. If $\lim_n(f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \rightarrow B$ is a Jordan $*$-homomorphism.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $C$-linear mapping $T : A \rightarrow B$ satisfying (2.3). It follows from (2.13) that

$$T(u y + y u) = \lim_n \frac{f(3^n u y + 3^n y u)}{3^n} = \lim_n \left[ \frac{f(3^n u)}{3^n} f(y) + f(y) \frac{f(3^n u)}{3^n} \right] = T(u) f(y) + f(y) T(u)$$

(2.14)

for all $u \in I_1(A_{sa})$, and all $y \in A$. By additivity of $T$ and (2.14), we obtain that

$$3^n T(u y + y u) = T(u 3^n y) + (3^n y) u = T(u) f(3^n y) + f(3^n y) T(u)$$

(2.15)

for all $u \in I_1(A_{sa})$ and all $y \in A$. Hence,

$$T(u y + y u) = \lim_n \left[ T(u) \frac{f(3^n y)}{3^n} + f(3^n y) \frac{T(u)}{3^n} \right] = T(u) T(y) + T(y) T(u)$$

(2.16)

for all $u \in I_1(A_{sa})$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \cap Z(B).$$

(2.17)

Similar to the proof of Theorem 2.1, it follows from (2.14) and (2.16) that $T = f$ on $A$. So $T$ is continuous. On the other hand, $A$ is real rank zero. One can easily show that $I_1(A_{sa})$ is dense in $A$. Let $v \in I_1(A_{sa}) : \|x\| = 1$. Then there exists a sequence $\{z_n\}$ in $I_1(A_{sa})$ such that $\lim_n z_n = v$. Since $T$ is continuous, it follows from (2.16) that

$$T(v y + y v) = T\left( \lim_n (z_n y + y z_n) \right) = \lim_n T(z_n y + y z_n)$$

$$= \lim_n T(z_n T(y) + \lim T(y) T(z_n)$$

$$T\left( \lim_n z_n \right) T(y) + T(y) \lim z_n$$

(2.18)

$$= T(v) T(y) + T(y) T(v)$$

for all $y \in A$. Now, let $x \in A$. Then we have $x = x_1 + i x_2$, where $x_1 := (x + x^*)/2$ and $x_2 = (x - x^*)/2i$ are self adjoint.
First consider $x_1 = 0$, $x_2 \neq 0$. Since $T$ is $\mathbb{C}$-linear, it follows from (2.18) that

\[
f(xy + yx) = T(xy + yx) = T(i x_2 y + y (ix_2)) = T \left( i \|x_2\| \frac{x_2}{\|x_2\|} y + y \left( i \|x_2\| \frac{x_2}{\|x_2\|} \right) \right) \\
= T \left( i \|x_2\| \frac{x_2}{\|x_2\|} \right) T(y) + T(y) T \left( i \|x_2\| \frac{x_2}{\|x_2\|} \right) \\
= T(i x_2) T(y) + T(y) T(ix_2) = T(x) T(y) + T(y) T(x) \\
= f(x) f(y) + f(y) f(x)
\]

for all $y \in A$.

If $x_2 = 0, x_1 \neq 0$, then by (2.18), we have

\[
f(xy + yx) = T(xy + yx) = T(x_1 y + y(x_1)) = T \left( \|x_1\| \frac{x_1}{\|x_1\|} y + y \left( \|x_1\| \frac{x_1}{\|x_1\|} \right) \right) \\
= T \left( \|x_1\| \frac{x_1}{\|x_1\|} \right) T(y) + T(y) T \left( \|x_1\| \frac{x_1}{\|x_1\|} \right) = T(x_1) T(y) + T(y) T(x_1) \\
= T(x) T(y) + T(y) T(x) = f(x) f(y) + f(y) f(x)
\]

for all $y \in A$.

Finally, consider the case that $x_1 \neq 0, x_2 \neq 0$. Then it follows from (2.18) that

\[
f(xy + yx) = T(xy + yx) = T(x_1 y + i x_2 y + y x_1 + y(ix_2)) \\
= T \left( \|x_1\| \frac{x_1}{\|x_1\|} y + y \left( \|x_1\| \frac{x_1}{\|x_1\|} \right) \right) + T \left( i \|x_2\| \frac{x_2}{\|x_2\|} y + y \left( i \|x_2\| \frac{x_2}{\|x_2\|} \right) \right) \\
= \|x_1\| \left[ T \left( \frac{x_1}{\|x_1\|} \right) T(y) + T(y) T \left( \frac{x_1}{\|x_1\|} \right) \right] + i \|x_2\| \left[ T \left( \frac{x_2}{\|x_2\|} \right) T(y) + T(y) T \left( \frac{x_2}{\|x_1\|} \right) \right] \\
= [T(x_1) + T(ix_2)] T(y) + T(y) [T(x_1) + T(ix_2)] \\
= T(x) T(y) + T(y) T(x) = f(x) f(y) + f(y) f(x)
\]

for all $y \in A$. Hence, $f(xy + yx) = f(x) f(y) + f(y) f(x)$ for all $x, y \in A$, and $f$ is Jordan $*$-homomorphism.

**Corollary 2.4.** Let $A$ be a $C^*$-algebra of rank zero. Let $p \in (0, 1)$, $\theta \in [0, \infty)$ be real numbers. Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that

\[
f(3^n uy + 3^n yu) = f(3^n u) f(y) + f(y) f(3^n u)
\]

(2.22)
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for all \( u \in I_1(A_w) \), all \( y \in A \), and all \( n = 0, 1, 2, \ldots \) Suppose that

\[
\left\| 2f \left( \frac{\mu x + \mu y + \mu z}{3} \right) + f \left( \frac{\mu x - 2\mu y + \mu z}{3} \right) + f \left( \frac{\mu x + \mu y - 2\mu z}{3} \right) - \mu f(x) + f(uv + wv) 
\right\| 
\leq \theta(\|x\|p + \|y\|p + \|z\|p) \tag{2.23}
\]

for all \( \mu \in \mathbb{T} \) and all \( x, y, z \in A \). If \( \lim_n f(3^n e)/3^n \in U(B) \cap Z(B) \), then the mapping \( f : A \to B \) is a Jordan \(*\)-homomorphism.

Proof. Setting \( \phi(x, y, z) := \theta(\|x\|p + \|y\|p + \|z\|p) \) for all \( x, y, z \in A \). Then by Theorem 2.3, we get the desired result. \( \square \)


We investigate the generalized Hyers-Ulam-Aoki-Rassias stability of Jordan \(*\)-homomorphisms on unital C\(^*\)-algebras by using the alternative fixed point.

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (See also [18, 26–43]).

**Theorem 3.1.** Let \( f : A \to B \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \phi : A^5 \to [0, \infty) \) satisfying

\[
\left\| f \left( \frac{\mu x + \mu y + \mu z}{3} \right) + f \left( \frac{\mu x - 2\mu y + \mu z}{3} \right) + f \left( \frac{\mu x + \mu y - 2\mu z}{3} \right) - \mu f(x) + f(uv + wv) 
\right\| 
\leq \phi(x, y, z, u, v, w),
\]

for all \( \mu \in \mathbb{T} \), and all \( x, y, z, u, v, w \in A, w \in U(A) \cup \{0\} \). If there exists an \( L < 1 \) such that \( \phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3) \) for all \( x, y, z, u, v, w \in A \), then there exists a unique Jordan \(*\)-homomorphism \( h : A \to B \) such that

\[
\left\| f(x) - h(x) \right\| \leq \frac{L}{1-L} \phi(x, 0, 0, 0, 0) \tag{3.2}
\]

for all \( x \in A \).

Proof. It follows from \( \phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3) \) that

\[
\lim_{j} 3^{-j} \phi \left( 3^j x, 3^j y, 3^j z, 3^j u, 3^j v, 3^j w \right) = 0 \tag{3.3}
\]

for all \( x, y, z, u, v, w \in A \).

Put \( y = z = w = u = 0 \) and \( \mu = 1 \) in (3.1) to obtain

\[
\left\| 3f \left( \frac{x}{3} \right) - f(x) \right\| \leq \phi(x, 0, 0, 0, 0) \tag{3.4}
\]
for all $x \in A$. Hence,

$$\left\| \frac{1}{3}f(3x) - f(x) \right\| \leq \frac{1}{3}\phi(3x, 0, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0, 0)$$  \hspace{1cm} (3.5)$$

for all $x \in A$.

Consider the set $X := \{ g : A \rightarrow B \}$ and introduce the generalized metric on $X$:

$$d(h, g) := \inf \{ C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0, 0) \ \forall x \in A \}.$$  \hspace{1cm} (3.6)$$

It is easy to show that $(X, d)$ is complete. Now we define the linear mapping $J : X \rightarrow X$ by

$$J(h)(x) = \frac{1}{3}h(3x)$$  \hspace{1cm} (3.7)$$

for all $x \in A$. By Theorem 3.1 of [44],

$$d(J(g), J(h)) \leq Ld(g, h)$$  \hspace{1cm} (3.8)$$

for all $g, h \in X$.

It follows from (3.5) that

$$d(f, J(f)) \leq L.$$  \hspace{1cm} (3.9)$$

Now, from the fixed point alternative [45], $J$ has a unique fixed point in the set $X_1 := \{ h \in X : d(f, h) < \infty \}$. Let $h$ be the fixed point of $J$. $h$ is the unique mapping with

$$h(3x) = 3h(x)$$  \hspace{1cm} (3.10)$$

for all $x \in A$ satisfying there exists $C \in (0, \infty)$ such that

$$\|h(x) - f(x)\| \leq C\phi(x, 0, 0, 0, 0, 0)$$  \hspace{1cm} (3.11)$$

for all $x \in A$. On the other hand, we have $\lim_n d(J^n(f), h) = 0$. It follows that

$$\lim_n \frac{1}{3^n} f(3^n x) = h(x)$$  \hspace{1cm} (3.12)$$

for all $x \in A$. It follows from $d(f, h) \leq (1/(1 - L))d(f, J(f))$, that

$$d(f, h) \leq \frac{L}{1 - L}.$$  \hspace{1cm} (3.13)$$
This implies the inequality (3.2). It follows from (3.1), (3.3), and (3.12) that

\[
\left\| h\left( \frac{x+y+z}{3} \right) + h\left( \frac{x-2y+z}{3} \right) + h\left( \frac{x+y-2z}{3} \right) - h(x) \right\|
\]

\[
= \lim_{n \to \infty} \frac{1}{3^n} \left[ f\left( 3^{n-1} (x+y+z) \right) + f\left( 3^{n-1} (x-2y+z) \right) + f\left( 3^{n-1} (x+y-2z) \right) - f(3^n x) \right]
\]

\[
\leq \lim_{n \to \infty} \frac{1}{3^n} \varphi(3^n x, 3^n y, 3^n z, 0, 0, 0)
\]

\[
= 0
\]

for all \( x, y, z \in A \). So

\[
h\left( \frac{x+y+z}{3} \right) + h\left( \frac{x-2y+z}{3} \right) + h\left( \frac{x+y-2z}{3} \right) = h(x)
\]

(3.15)

for all \( x, y, z \in A \). Put \( w = (x + y + z)/3, t = (x - 2y + z)/3 \) and \( s = (x + y - 2z)/3 \) in above equation, we get \( h(w+t+s) = h(w)+h(t)+h(s) \) for all \( w, t, s \in A \). Hence, \( h \) is Cauchy additive. By putting \( y = z = x, v = w = 0 \) in (3.1), we have

\[
\left\| f(\mu x) - \mu f(x) \right\| \leq \varphi(x, x, x, 0, 0, 0)
\]

(3.16)

for all \( \mu \in T \) and all \( x \in A \). It follows that

\[
\left\| h(\mu x) - \mu h(x) \right\| = \lim_{m \to \infty} \frac{1}{3^m} \left\| f(\mu^m x) - \mu f(3^m x) \right\| \leq \lim_{m \to \infty} \frac{1}{3^m} \varphi(3^m x, 3^m x, 3^m x, 0, 0, 0) = 0
\]

(3.17)

for all \( \mu \in T \), and all \( x \in A \). One can show that the mapping \( h : A \to B \) is \( \mathbb{C} \)-linear. By putting \( x = y = z = u = v = 0 \) in (3.1), it follows that

\[
\left\| h(\omega^*) - (h(\omega))^* \right\| = \lim_{m \to \infty} \left\| \frac{1}{3^m} f((3^m \omega)^*) - \frac{1}{3^m} (f(3^m \omega))^* \right\|
\]

\[
\leq \lim_{m \to \infty} \frac{1}{3^m} \varphi(0, 0, 0, 0, 0, 3^m \omega)
\]

\[
= 0
\]

for all \( \omega \in \mathcal{U}(A) \). By the same reasoning as the proof of Theorem 2.1, we can show that \( h : A \to B \) is \( * \)-preserving.
Since $h$ is $\mathbb{C}$-linear, by putting $x = y = z = w = 0$ in (3.1), it follows that
\[
\|h(\mu x + \mu y + \mu z)\| = \lim_{m} \left\| \frac{1}{3^m} f \left( \frac{2}{3^m} \mu x + \mu y + \mu z \right) + \frac{1}{3^m} f \left( \frac{3^m}{3^2} \mu x + \mu y + \mu z \right) - \mu f(x) - f(\mu x + \mu y + \mu z) \right\|
\]
\[
\leq \lim_{m} \frac{1}{3^m} \phi(0, 0, 3^m, 3^m, 0) \leq \lim_{m} \frac{1}{3^m} \phi(0, 0, 3^m, 3^m, 0)
\]
\[
= 0
\]
for all $x, y, z, u, v \in A$. Thus $h : A \to B$ is Jordan $*$-homomorphism satisfying (3.2), as desired.

We prove the following Hyers-Ulam-Aoki-Rassias stability problem for Jordan $*$-homomorphisms on unital $C^*$-algebras.

**Corollary 3.2.** Let $p \in (0, 1), \theta \in [0, \infty)$ be real numbers. Suppose $f : A \to A$ satisfies
\[
\left\| f \left( \frac{\mu x + \mu y + \mu z}{3} \right) + f \left( \frac{\mu x - 2\mu y + \mu z}{3} \right) + f \left( \frac{\mu x + \mu y - 2\mu z}{3} \right) - \mu f(x) + f(\mu x + \mu y + \mu z) \right\|
\]
\[
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p),
\]
for all $\mu \in \mathbb{T}$ and all $x, y, z, u, v, w \in A, w \in U(A) \cup \{0\}$. Then there exists a unique Jordan $*$-homomorphism $h : A \to B$ such that such that
\[
\|f(x) - h(x)\| \leq \frac{3^p \theta}{3 - 3^p} \|x\|^p
\]
for all $x \in A$.

**Proof.** Setting $\phi(x, y, z, u, v, w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ all $x, y, z, u, v, w \in A$. Then by $L = 3^{p-1}$ in Theorem 3.2, one can prove the result.

**References**


