Research Article

The Lagrangian Stability for a Class of Second-Order Quasi-Periodic Reversible Systems

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1. Introduction

The Kolmogorov-Arnold-Moser (KAM) theory was developed for conservative (Hamiltonian) dynamical systems that are nearly integrable. Integrable systems in their phase space contain lots of invariant tori, and KAM theory establishes persistence of such tori, which carry quasi-periodic motions, whereas parallel results exist for other classes of dynamical systems as well. In particular, the reversible KAM theory (starting with Moser’s paper [1]) is to a great extent parallel to the Hamiltonian ones, see [2–6] and references therein. In the case of reversible diffeomorphisms, however, some special effects are exhibited [7], and the weakly reversible KAM theory has been developed in [8, 9].

In this paper, we will consider the \( P \)-Laplace equations with quasi-periodic reversible structure. Firstly, we give some concepts of reversible system, a mechanical system of \( s \) particles with the interaction forces which are independent of velocities or even functions in velocities; such a system ruled by the Newton equation, \( d^2r/dt^2 = F(r, v), \ r \in R^{3s}, \ v = dr/dt, \ F(r, -v) \equiv F(r, v), \) is time reversible, that is, reversing all the velocities \( v \) reverses all the trajectories in the configuration space \( R^{3s} \). More generally, an autonomous differential equation \( du/dt = V(u) \) and the corresponding vector field \( V \) are said to be reversible if there...
exists a phase space involution $G$ (a mapping which $G^2 = Id$) that reverses the direction of
time: $TG \circ V = -V \circ G$, where $TG$ is the differential of $G$; that is, $G$ transforms the field $V$ into
the opposite field $-V$. This $u(t)$, in addition to $G(u(-t))$, is a solution of the equation.

In the above example, $G: (r,v) \mapsto (r,-v)$. A system
\begin{align}
\frac{dx}{dt} &= f(x,t), \\
\frac{dy}{dt} &= g(x,t), \\
&x \in \mathbb{R}^n, y \in \mathbb{R}^l,
\end{align}
(1.1)
is reversible with respect to the involution
\begin{equation}
G: (x,y) \mapsto (-x,y)
\end{equation}
(1.2)
if and only if $f$ is even in $x$ and $g$ is odd in $x$.

We can refer to [2, 8–11] for more detailed concepts of reversibility. Side by side with
reversible vector fields, there are reversible diffeomorphisms. A mapping $A$ is said to be
reversible if there exists an involution $G$ that conjugates $A$ with its inverse $A^{-1}$, that is, $AGA = G$.
The flow map of a reversible vector field for each fixed time is reversible with respect to
the same involution, and vice versa.

In [12] Liu considered the following quasi-periodic mappings:
\begin{equation}
A: (x,y) \mapsto (x + \omega + y + f(x,y), y + g(x,y)),
\end{equation}
(1.3)
where $f$ and $g$ are quasi-periodic in $x$ with frequencies $\mu_1, \ldots, \mu_m$ and real analytic in $x$ and
$y$, the variable $y$ ranges in a neighborhood of the origin of the real line $\mathbb{R}$, and $\omega$ is a positive
constant. He supposes that the mapping $A$ is reversible with the involution $R : (x,y) \mapsto
(-x,y)$, that is, $RAR = A^{-1}$. Such a map is often met when the vector field is quasi-periodic
in time and reversible with respect to the involution $R$. In fact, the phase flow induces such a
map on a cross-section transversal to the vector field.

The invariant curve theorem of reversible systems was first obtained by Moser [1] who
then developed it [13] (for continuous systems), it was also developed by Sevryuk [9] (for
both continuous and discrete system). In [1], the author also studied the existence of invariant
tori of a reversible system depending quasi-periodically on time. In [12] Liu obtained an
invariant curve theorem for reversible quasi-periodic mappings, as application, he studies
the existence of quasi-periodic solutions and the boundedness of solutions for a pendulum-
type equation
\begin{equation}
x'' + f(x,t)x' + g(x,t) = 0
\end{equation}
(1.4)
and an asymmetric oscillator depending quasi-periodically on time. By some theorems in
[12], in this paper, we consider two-order differential equations
\begin{equation}
\left(\Phi_p(x')\right)' + f(x,t)\Phi_p(x') + g(x,t) = 0,
\end{equation}
(1.5)
where $\Phi_p(s) = |s|^{(p-2)}s, \ p > 0, \ \omega > 0$; if $p = 2$, it becomes (1.4).

Our main result is the following theorem.
2. Main Result

We first give some definitions [12].

**Definition 1.** A function \( f : \mathbb{R} \to \mathbb{R} \) is called real analytic quasi-periodic with frequencies \( \mu_1, \ldots, \mu_m \) if it can be represented as a Fourier series of the type

\[
 f(t) = \sum_k f_k e^{(k,\mu)t},
\]

where \( k = (k_1, \ldots, k_m), \mu = (\mu_1, \ldots, \mu_m), \langle k, \mu \rangle = \sum k_i \mu_i \neq 0; \) if \( k \neq 0 \), the coefficients \( f_k \) decay exponentially with \( |k| = |k_1| + \cdots + |k_m| \).

**Definition 2.** The vector \( \mu = (\mu_1, \ldots, \mu_m) \) satisfies the Diophantine condition if:

\[
 |\langle k, \mu \rangle| \geq \frac{c}{|k|^\sigma}, \quad c, \sigma > 0
\]

for all integer vector \( k \neq 0 \).

For our study of (1.5), where \( f(x, t) \) and \( g(x, t) \) are real analytic in \( x \) and \( t \), \( 2a\pi_{\omega} \) periodic in \( x \), and quasi-periodic in \( t \) with frequencies \( (\omega_1, \ldots, \omega_m) \), where the number \( \pi_{\omega} \) is defined by

\[
 \pi_{\omega} = 2 \int_0^{(p-1)/p} \frac{ds}{[1 - s^p/(p-1)]^{1/p}}.
\]

Moreover, we assume

\[
 f(-x, -t) = f(x, t), \quad g(-x, -t) = g(x, t).
\]

**Theorem 2.1.** Suppose that \( (\omega_1, \ldots, \omega_m) \) satisfy the Diophantine condition

\[
 |\langle k, \mu \rangle| \geq \frac{c_0}{|k|^\sigma_0}, \quad \text{for } k \in \mathbb{Z}^m \setminus \{0\},
\]

where \( c_0, \sigma_0 \) are positive constants. Then there are infinitely many quasi-periodic solutions with large amplitude, and the solutions of (1.5) satisfy

\[
 \sup_{t \in \mathbb{R}} |x'(t)| < +\infty.
\]

3. Coordination Transformation

In this section we first make a coordination transformation then study the boundedness of all solutions of the new system.
Equation (1.5) is equivalent to the planar system:

\begin{align}
    x' &= \Phi_q(y), \\
    y' &= -f(x, t)y - g(x, t).
\end{align} \tag{3.1}

It is easy to verify that planar system (3.1) is reversible with respect to the involution $R : (x, y) \mapsto (-x, y)$.

Since we are concerned with the boundedness of solutions and the existence of quasi-periodic solutions with large amplitude, we may assume $|y| \geq 1$. Instead of considering planar system (3.1), we are concerned with the following system:

\begin{align}
    \frac{dt}{dx} &= \frac{1}{\Phi_q(y)}, \\
    \frac{dy}{dx} &= \frac{f(x, t)y}{\Phi_q(y)} - \frac{g(x, t)}{\Phi_q(y)}.
\end{align} \tag{3.2}

We will prove that if $(t(x), y(x))$ is a solution of system (3.2), then $|y|$ is bounded.

4. Poincaré Map

In this section, we first introduce new action variable then give an expression for the Poincaré map of the new system.

Introduce a new action variable $v$ and a small parameter $\varepsilon$ as follows:

\begin{align}
    y &= \frac{1}{\Phi_p(\varepsilon)v}, \quad v \in \left[\frac{1}{\gamma}, \gamma\right], \quad \gamma > 1. \tag{4.1}
\end{align}

So system (3.2) is changed into the following form:

\begin{align}
    \frac{dt}{dx} &= \varepsilon \Phi_q(v), \\
    \frac{dv}{dx} &= \varepsilon|v|^p f(x, t) + \varepsilon|v|^{p+1} g(x, t). \tag{4.2}
\end{align}

It is easy to verify that system (4.2) is reversible with respect to the involution $(t, v) \mapsto (-t, v)$. We make the ansatz that the solution $(t(x, v_0, t_0, \varepsilon), \rho(x, v_0, t_0, \varepsilon))$ has the following form:

\begin{align}
    t &= t_0 + \varepsilon T(x, v_0, t_0; \varepsilon), \quad v = V_0 + \varepsilon V(x, v_0, t_0; \varepsilon). \tag{4.3}
\end{align}
Abstract and Applied Analysis

The functions $T$ and $V$ satisfy

$$
\frac{dT}{dx} = \Phi_\varrho(V_0 + \varepsilon V) = \Phi_\varrho(V_0) + O(\varepsilon),
$$

$$
\frac{dV}{dx} = f(x, t_0 + \varepsilon T)|V_0 + \varepsilon V|^q + \Phi_p(\varepsilon)|V_0 + \varepsilon V|^q g(x, t_0 + \varepsilon T)
= f(x, t_0)V_0^q + O(\varepsilon).

(4.4)

Denote by $P$ the Poincaré map of (4.2); then, from the above equations, it is easy to see that

$$
P(t_0, V_0) = \left( t_0 + 2\alpha x_0 \varepsilon \Phi_\varrho(V_0) + O\left( \varepsilon^2 \right), V_0 + \varepsilon V_0^q \int_0^{2\alpha \pi} f(x, t_0) \, dx + O\left( \varepsilon^2 \right) \right).

(4.5)

5. The Proof of Theorem 2.1

In this part, we will prove the map $P$ has an invariant curve, and then boundedness of solutions of (1.5) follows from the standard arguments [14–18]. In the following, we will apply the invariant curves of quasi-periodic reversible mapping theorem to prove our conclusion.

Now we state Liu’s result [12].

Consider the map

$$
M_\delta : (x, y) \mapsto (x + \alpha + \delta L(x, y) + \delta f(x, y, \delta), y + \delta M(x, y) + \delta g(x, y, \delta)),

(5.1)

where $L, M, f, g$ are quasi-periodic in $x$ with the frequencies $\mu_1, \ldots, \mu_m$, $f(x, y, 0) = g(x, y, 0) = 0$.

We also assume that these functions are real analytic in a complex neighborhood of the domain $R \times [a_1, b_1]$. The functions $L$ and $M$ can be represented in the form

$$
L(x, y) := \overline{L}(x, y) + \overline{L}(x, y) = \sum_{k \in K} L_k(y) e^{(i(k, \mu))x} + \sum_{k \not\in K} L_k(y) e^{(i(k, \mu))x},
$$

$$
M(x, y) := \overline{M}(x, y) + \overline{M}(x, y) = \sum_{k \in K} M_k(y) e^{(i(k, \mu))x} + \sum_{k \not\in K} M_k(y) e^{(i(k, \mu))x}.

(5.2)

Note that

$$
e^{i(k, \mu)x} - 1 \neq 0, \text{ for } k \in K \setminus K,
$$

$$
\overline{L}(x + \alpha) \equiv \overline{L}(x), \quad \overline{M}(x + \alpha) \equiv \overline{M}(x).

(5.3)

Lemma 5.1 ([12, Theorem 4]). Suppose the function $L$ satisfies

$$
\overline{L}(x, y) > 0, \quad \frac{\partial \overline{L}}{\partial y} > 0,

(5.4)
and there is a real analytic function $I(x, y) = I(x + a, y)$ satisfying
\[ \frac{\partial I}{\partial y} > 0, \]
\[ \bar{L}(x, y) \frac{\partial I}{\partial x}(x, y) + \bar{M}(x, y) \frac{\partial I}{\partial y}(x, y) \equiv 0. \tag{5.5} \]

Moreover, suppose that there are two numbers $\tilde{a}$ and $\tilde{b}$ such that $a_1 < \tilde{a} < \tilde{b} < b_1$ and
\[ I_M(a_1) < I_m(\tilde{a}) \leq I_M(\tilde{a}) < I_M(\tilde{b}) \leq I_m(b_1), \tag{5.6} \]
where
\[ I_M(y) = \max_{x \in \mathbb{R}} I(x, y), \quad I_m(y) = \min_{x \in \mathbb{R}} I(x, y). \tag{5.7} \]
Then there exist $\epsilon > 0$ and $\Delta > 0$ such that if $\delta < \Delta$ and
\[ \|f(\cdot, \cdot, \delta)\| + \|g(\cdot, \cdot, \delta)\| < \epsilon, \tag{5.8} \]
the mapping $M_\delta$ has an invariant curve which is of the form $y = \phi(x)$, and $\phi$ is quasi-periodic in $x$ with frequencies $\mu_1, \ldots, \mu_m$. The constants $\epsilon$ and $\Delta$ depend on $a$, $\tilde{a}$, $\tilde{b}$, $b$, $L$, $M$, and $I$. In particular, $\epsilon$ is independent of $\delta$. If $a = 0$, the conclusion also holds; in this case, $K = \mathbb{Z}^m$, $\bar{L} = \bar{M} = 0$, $\bar{L} = L$, and $M = M$.

For the Poincaré map $P(t_0, V_0)$ of (4.2), let
\[ I(t_0, V_0) = V_0 e^{-(1/2a_x)} \int_0^\infty \int_0^{2a_x} f(x, t) dx dt. \tag{5.9} \]
Since $f(-t, -x) = -f(t, x)$, we know that $I(t_0, V_0)$ is even and quasi-periodic in $t_0$ with frequencies $\mu_1, \ldots, \mu_m$. And it is easy to see that
\[ \frac{\partial I}{\partial V_0} = e^{-(1/2a_x)} \int_0^{2a_x} \int_0^{2a_x} f(x, t) dx dt > 0, \]
\[ \bar{L}(V_0, t_0) \frac{\partial I}{\partial t_0}(V_0, t_0) + \bar{M}(V_0, t_0) \frac{\partial I}{\partial V_0}(V_0, t_0) \]
\[ = 2a_x \Phi_y(v_0) \cdot V_0 \left( -\frac{1}{2a_x} \right) \int_0^{2a_x} f(x, t_0) dx \cdot e^{-(1/2a_x)} \int_0^{2a_x} \int_0^{2a_x} f(x, t) dx dt \]
\[ + V_0^2 \int_0^{2a_x} f(x, t_0) dx \cdot e^{-(1/2a_x)} \int_0^{2a_x} \int_0^{2a_x} f(x, t) dx dt. \]
Moreover, if we define \( \alpha \) and \( \beta \) by

\[
I(t_0, V_0) = V_0 e^{-\frac{1}{2a \pi p}} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt,
\]

\[
\alpha = \min_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right), \quad \beta = \max_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right),
\]

then \( \beta \gg \alpha > 0 \). Now we choose the constants \( \gamma, \gamma_1, \gamma_2 \) as

\[
\gamma = \left( \frac{2\beta}{\alpha} \right)^3 > 1, \quad \gamma_1 = \left( \frac{2\beta}{\alpha} \right)^{-1}, \quad \gamma_2 = \left( \frac{2\beta}{\alpha} \right).
\]

Then

\[
\begin{align*}
I_M \left( \frac{1}{\gamma} \right) & = \frac{1}{\gamma} \max_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = \frac{a^3}{8\beta^2}, \\
I_m(\gamma) & = \gamma \min_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = \frac{8\beta^3}{a^2}, \\
I_m(\gamma_1) & = \gamma_1 \min_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = \frac{a^2}{2\beta}, \\
I_M(\gamma_1) & = \gamma_1 \max_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = \frac{a}{2}, \\
I_m(\gamma_2) & = \gamma_2 \min_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = 2\beta, \\
I_M(\gamma_2) & = \gamma_2 \max_{t_0 \in \mathbb{R}} \exp \left( -\frac{1}{2a \pi p} \int_0^{t_0} \int_0^{2a \pi p} f(x, t)dxdt \right) = -\frac{2\beta^2}{a}.
\end{align*}
\]

We have already demonstrated that the map \( P \) satisfies all the conditions in Lemma 5.1; hence, \( P \) has an invariant curve; thus, all solutions of (1.5) are bounded.

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