Research Article
General Cubic-Quartic Functional Equation

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We obtain the general solution and the generalized Hyers-Ulam stability of the general cubic-quartic functional equation for fixed integers k with k ≠ 0, ±1:

\[ f(x + ky) + f(x - ky) = k^2 f(x + y) + f(x - y) + 2(1 - k^2) f(x) + (k^4 - k^2)/4 (f(2y) - 8f(y)) + f(2x) - 16f(x), \]

where \( \tilde{f}(x) := f(x) + f(-x). \)

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let \((G_1, \cdot)\) be a group and let \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\), such that if a mapping \(h : G_1 \rightarrow G_2\) satisfies the inequality \(d(h(x \cdot y), h(x) \ast h(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \rightarrow G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let \(f : E \rightarrow E'\) be a mapping between Banach spaces such that

\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \]  

for all \(x, y \in E\) and for some \(\delta > 0\). Then there exists a unique additive mapping \(T : E \rightarrow E'\) such that

\[ \|f(x) - T(x)\| \leq \delta \]
for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $T$ is linear. In 1978, Rassias [3] proved the following theorem.

**Theorem 1.1.** Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \tag{1.3}
$$

for all $x, y \in E$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$
\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - \varepsilon^p} \|x\|^p \tag{1.4}
$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$ and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into $E'$ is continuous in real $t$ for each fixed $x \in E$, then $T$ is linear.

In 1990, Rassias during the 27th International Symposium on Functional Equations asked the question whether such a Theorem can also be proved for all real values of $p$ that are greater or equal to one. In 1991, Gajda [4], following the same approach as that of Rassias, provided an affirmative solution to this question for all real values of $p$ that are strictly greater than one. The new concept of stability of the linear mapping that was inspired by Rassias' stability theorem is called Hyers-Ulam-Rassias stability of functional equations.

Jun and Kim [5] introduced the following cubic functional equation:

$$
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.5}
$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function $f$ between real vector spaces $X$ and $Y$ is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and $C$ is symmetric for each fixed one variable and is additive for fixed two variables. The stability of the quartic functional equations was studied by Park and Bae [6], when

$$
f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) + 24f(y) - 6f(x). \tag{1.6}
$$

In fact, they proved that a function $f$ between real vector spaces $X$ and $Y$ is a solution of (1.6) if and only if there exists a unique symmetric multi-additive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$ (see also [7, 8]). It is straightforward to verify that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. (see [9–45]).
In 2008, Gordji et al. [17] provided the solution as well as the stability of a mixed type cubic-quartic functional equation. We only mention here the papers [19, 32, 33] concerning the stability of the mixed type functional equations.

In this paper, we deal with the following general cubic-quartic functional equation:

\[
 f(x + ky) + f(x - ky) = k^2(f(x + y) + f(x - y)) + 2\left(1 - k^2\right)f(x) + \frac{k^4 - k^2}{4} \times (f(2y) - 8f(y)) + \tilde{f}(2x) - 16\tilde{f}(x),
\]

where \(\tilde{f}(x) = f(x) + f(-x).\)

Then it follows easily that the function \(f(x) = ax^4 + bx^3\) satisfies (1.7). We investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7).

### 2. General Solution

In this section, we establish the general solution of functional equation (1.7).

**Theorem 2.1.** Let \(X, Y\) be vector spaces and let \(f : X \to Y\) be a function. Then \(f\) satisfies (1.7) if and only if there exists a unique symmetric multiadditive function \(Q : X \times X \times X \times X \to Y\) and a unique function \(C : X \times X \times X \to Y\) such that \(f(x) = Q(x, x, x, x) + C(x, x, x)\) for all \(x \in X\), where the function \(C\) is symmetric for each fixed one variable and is additive for fixed two variables.

**Proof.** Let \(f\) satisfies (1.7). We decompose \(f\) into the even part and odd part by setting

\[
 f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x))
\]

for all \(x \in X\). By (1.7), we have

\[
 f_e(x + ky) + f_e(x - ky) = \frac{1}{2}[f(x + ky) + f(-x - ky) + f(x - ky) + f(-x + ky)]
\]

\[
 = \frac{1}{2}[f(x + ky) + f(x - ky)] + \frac{1}{2}[f((-x) + (-ky)) + f((-x) - (-ky))]
\]

\[
 = \frac{1}{2}\left[k^2(f(x + y) + f(x - y)) + 2\left(1 - k^2\right)f(x)
\right.
\]

\[
 + \frac{k^4 - k^2}{4}(f(2y) - 8f(y)) + \tilde{f}(2x) - 16\tilde{f}(x)
\]

\[
 + \frac{1}{2}\left[k^2(f(-x - y) + f(-x + y)) + 2\left(1 - k^2\right)f(-x) + \frac{k^4 - k^2}{4}
\right.
\]

\[
 \times (f(-2y) - 8f(-y)) + \tilde{f}(-2x) - 16\tilde{f}(-x)
\]

\[
 \times (f(-2y) - 8f(-y)) + \tilde{f}(-2x) - 16\tilde{f}(-x)
\]
\[
\begin{align*}
&= k^2 \left[ \frac{1}{2} (f(x + y) + f(-(x + y))) \right] + k^2 \left[ \frac{1}{2} (f(x - y) + f(-(x - y))) \right] \\
&+ 2 \left( 1 - k^2 \right) \left[ \frac{1}{2} (f(x) + f(-x)) \right] + \frac{k^4 - k^2}{4} \left[ \frac{1}{2} (f(2y) + f(-2y)) \right] \\
&- \frac{k^4 - k^2}{4} \left[ \frac{1}{2} (8f(y) + 8f(-y)) \right] + \frac{1}{2} \left( \tilde{f}(2x) + \tilde{f}(-2x) \right) \\
&- 16 \left[ \frac{1}{2} \left( \tilde{f}(x) + \tilde{f}(-x) \right) \right] \\
&= k^2 (f_e(x + y) + f_e(x - y)) + 2\left( 1 - k^2 \right) f_e(x) \\
&+ \frac{k^4 - k^2}{4} (f_e(2y) - 8f_e(y)) + \tilde{f}_e(2x) - 16 \tilde{f}_e(x)
\end{align*}
\]

(2.2)

for all \(x, y \in X\). This means that \(f_e\) satisfies (1.7), or

\[
\begin{align*}
&f_e(x + ky) + f_e(x - ky) = k^2 (f_e(x + y) + f_e(x - y)) + 2\left( 1 - k^2 \right) f_e(x) \\
&+ \frac{k^4 - k^2}{4} (f_e(2y) - 8f_e(y)) + \tilde{f}_e(2x) - 16 \tilde{f}_e(x)
\end{align*}
\]

(1.5(e))

for all \(x, y \in X\). Applying the fact that the function \(f_e\) is even for all \(x, y \in X\), (1.5(e)) can be written in the form

\[
\begin{align*}
&f_e(x + ky) + f_e(x - ky) = k^2 (f_e(x + y) + f_e(x - y)) + 2\left( 1 - k^2 \right) f_e(x) \\
&+ \frac{k^4 - k^2}{4} (f_e(2y) - 8f_e(y)) + 2 f_e(2x) - 32 f_e(x)
\end{align*}
\]

(2.3)

for all \(x, y \in X\). Now be setting \(x = y = 0\) in (2.3), we get \(f_e(0) = 0\). Similarly, by setting \(y = 0\) in (2.3), we obtain

\[
f_e(2x) = 16 f_e(x)
\]

(2.4)

for all \(x \in X\). Hence (2.3) can be written as

\[
\begin{align*}
f_e(x + ky) + f_e(x - ky) &= k^2 (f_e(x + y) + f_e(x - y)) + 2\left( 1 - k^2 \right) f_e(x) + 2 \left( k^4 - k^2 \right) f_e(y)
\end{align*}
\]

(2.5)
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for all $x, y \in X$. By substituting $x$ by $x + y$ in (2.5), we have

$$f_e(x + (1 + k)y) + f_e(x + (1 - k)y)$$

$$= k^2(f_e(x + 2y) + f_e(x)) + 2(1 - k^2)f_e(x + y) + 2(k^4 - k^2)f_e(y)$$

(2.6)

for all $x, y \in X$. Substituting $-y$ for $y$ in (2.6), we get by evenness of $f$

$$f_e(x - (1 + k)y) + f_e(x - (1 - k)y)$$

$$= k^2(f_e(x - 2y) + f_e(x)) + 2(1 - k^2)f_e(x - y) + 2(k^4 - k^2)f_e(y)$$

(2.7)

for all $x, y \in X$. Adding (2.6) to (2.7), we obtain

$$f_e(x + (1 + k)y) + f_e(x + (1 - k)y) + f_e(x - (1 + k)y) + f_e(x - (1 - k)y)$$

$$= k^2(f_e(x + 2y) + f_e(x - 2y)) + 2k^2f_e(x) + 2(1 - k^2)(f_e(x + y) + f_e(x - y))$$

$$+ 4(k^4 - k^2)f_e(y)$$

(2.8)

for all $x, y \in X$. By substituting $x$ by $x - ky$ in (2.5), we have

$$f_e(x) + f_e(x - 2ky) = k^2(f_e(x + (1 - k)y) + f_e(x - (k + 1)y)) + 2(1 - k^2)f_e(x - ky)$$

$$+ 2(k^4 - k^2)f_e(y)$$

(2.9)

for all $x, y \in X$. Substituting $-x$ for $x$ in (2.9), we get by evenness of $f_e$

$$f_e(x) + f_e(x + 2ky) = k^2(f_e(x + (k - 1)y) + f_e(x + (k + 1)y)) + 2(1 - k^2)f_e(x + ky)$$

$$+ 2(k^4 - k^2)f_e(y)$$

(2.10)

for all $x, y \in X$. Adding (2.9) to (2.10), we obtain

$$f_e(x + 2ky) + f_e(x - 2ky) = k^2(f_e(x + (1 - k)y) + f_e(x - (k + 1)y) + f_e(x + (k - 1)y)$$

$$+ f_e(x + (k + 1)y)) + 2(1 - k^2)(f_e(x - ky) + f_e(x + ky))$$

$$+ 4(k^4 - k^2)f_e(y) - 2f_e(x)$$

(2.11)
for all \( x, y \in X \). Now, by using (2.5), (2.8), and (2.11), we lead to
\[
f_e(x + 2ky) + f_e(x - 2ky) = k^4(f_e(x + 2y) + f_e(x - 2y)) \\
+ 4k^2 \left(1 - k^2\right)(f_e(x + y) + f_e(x - y)) + 8\left(k^4 - k^2\right)f_e(y) \\
+ \left(6k^4 - 8k^2 + 2\right)f_e(x)
\]
(2.12)
for all \( x, y \in X \). If we replace \( y \) by \( 2y \) in (2.5), we get
\[
f_e(x + 2ky) + f_e(x - 2ky) = k^2(f_e(x + 2y) + f_e(x - 2y)) + 2\left(1 - k^2\right)f_e(x) \\
+ 2\left(k^4 - k^2\right)f_e(2y)
\]
(2.13)
for all \( x, y \in X \). It follows from (2.12) and (2.13) that
\[
k^4(f_e(x + 2y) + f_e(x - 2y)) + 4k^2 \left(1 - k^2\right)(f_e(x + y) + f_e(x - y)) + 8\left(k^4 - k^2\right)f_e(y) \\
+ \left(6k^4 - 8k^2 + 2\right)f_e(x) \\
= k^2(f_e(x + 2y) + f_e(x - 2y)) + 2\left(1 - k^2\right)f_e(x) + 2\left(k^4 - k^2\right)f_e(2y)
\]
(2.14)
for all \( x, y \in X \). So we have
\[
f_e(x + 2y) + f_e(x - 2y) = 4(f_e(x + y) + f_e(x - y)) + 24f_e(y) - 6f_e(x)
\]
(2.15)
for all \( x, y \in X \). This means that \( f_e \) is a quartic function. Thus there exists a unique symmetric multiadditive function \( Q : X \times X \times X \times X \to Y \) such that \( f_e(x) = Q(x, x, x, x) \) for all \( x \in X \). On the other hand, we can show that \( f_o \) satisfies (1.7), or
\[
f_o(x + ky) + f_o(x - ky) = k^2(f_o(x + y) + f_o(x - y)) + 2\left(1 - k^2\right)f_o(x) \\
+ \frac{k^4 - k^2}{4}(f_o(2y) - 8f_o(y)) + \tilde{f}_o(2x) - 16\tilde{f}_o(x)
\]
(1.5(o))
for all \( x, y \in X \). By oddness of \( f_o \) for all \( x, y \in X \), (1.5(o)) can be written as
\[
f_o(x + ky) + f_o(x - ky) = k^2(f_o(x + y) + f_o(x - y)) + 2\left(1 - k^2\right)f_o(x) \\
+ \frac{k^4 - k^2}{4}(f_o(2y) - 8f_o(y))
\]
(2.16)
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for all \( x, y \in X \). Now by setting \( x = y = 0 \) in (3.2), we get \( f_\alpha(0) = 0 \), and by setting \( x = 0 \) in (2.16), we obtain

\[
f_\alpha(2y) = 8f_\alpha(y)
\]  
(2.17)

for all \( y \in X \). Hence (2.16) can be written as

\[
f_\alpha(x + ky) + f_\alpha(x - ky) = k^2(f_\alpha(x + y) + f_\alpha(x - y)) + 2\left(1 - k^2\right)f_\alpha(x)
\]  
(2.18)

for all \( x, y \in X \). Replacing \( x \) by \( x - y \) in (2.18), we obtain

\[
f_\alpha(x + (k - 1)y) + f_\alpha(x - (k + 1)y) = k^2(f_\alpha(x - 2y) + f_\alpha(x)) + 2\left(1 - k^2\right)f_\alpha(x - y)
\]  
(2.19)

for all \( x, y \in X \). Substituting \(-x\) for \( x \) in (2.19), we get by oddness of \( f_\alpha \)

\[
-f_\alpha(x + (1 - k)y) - f_\alpha(x + (k + 1)y) = k^2(-f_\alpha(x + 2y) - f_\alpha(x)) - 2\left(1 - k^2\right)f_\alpha(x + y)
\]  
(2.20)

for all \( x, y \in X \). If we subtract (2.19) from (2.20), we obtain

\[
f_\alpha(x + (k - 1)y) + f_\alpha(x - (k + 1)y) + f_\alpha(x + (1 - k)y) + f_\alpha(x + (k + 1)y) \\
= k^2(f_\alpha(x + 2y) + f_\alpha(x - 2y)) + 2k^2f_\alpha(x) + 2\left(1 - k^2\right)(f_\alpha(x + y) + f_\alpha(x - y))
\]  
(2.21)

for all \( x, y \in X \). By substituting \( x \) by \( x + ky \) in (2.18), we have

\[
f_\alpha(x) + f_\alpha(x + 2ky) = k^2(f_\alpha(x + (k + 1)y) + f_\alpha(x + (k - 1)y)) + 2\left(1 - k^2\right)f_\alpha(x + ky)
\]  
(2.22)

for all \( x, y \in X \). Substituting \(-y\) for \( y \) in (2.22), we get

\[
f_\alpha(x) + f_\alpha(x - 2ky) = k^2(f_\alpha(x - (k + 1)y) + f_\alpha(x - (k - 1)y)) + 2\left(1 - k^2\right)f_\alpha(x - ky)
\]  
(2.23)

for all \( x, y \in X \). Adding (2.22) to (2.23), we obtain

\[
f_\alpha(x + 2ky) + f_\alpha(x - 2ky) = k^2(f_\alpha(x + (k + 1)y) + f_\alpha(x + (k - 1)y) + f_\alpha(x - (k + 1)y) \\
+ f_\alpha(x - (k - 1)y)) + 2\left(1 - k^2\right)(f_\alpha(x + ky) + f_\alpha(x - ky)) - 2f_\alpha(x)
\]  
(2.24)
for all \( x, y \in X \). Now, by using (2.18), (2.21), and (2.24), we lead to

\[
f_o(x + 2ky) + f_o(x - 2ky) = 4k^2(1 - k^2)(f_o(x + y) + f_o(x - y)) + (6k^4 - 8k^2 + 2)f_o(x) + k^4(f_o(x + 2y) + f_o(x - 2y))
\]

(2.25)

for all \( x, y \in X \). If we replace \( y \) by \( 2y \) in (2.18), we get

\[
f_o(x + 2ky) + f_o(x - 2ky) = k^2(f_o(x + 2y) + f_o(x - 2y)) + 2(1 - k^2)f_o(x)
\]

(2.26)

for all \( x, y \in X \). If we compare (2.25) with (2.26), then we conclude that

\[
f_o(x + 2y) + f_o(x - 2y) = 4(f_o(x + y) + f_o(x - y)) - 6f_o(x)
\]

(2.27)

for all \( x, y \in X \). Replacing \( x \) by \( 2x \) in (2.27), we get

\[
f_o(2(x + y)) + f_o(2(x - y)) = 4(f_o(2x + y) + f_o(2x - y)) - 6f_o(2x)
\]

(2.28)

for all \( x, y \in X \). Finally, it follows from (2.17) and (2.28) that

\[
8(f_o(x + y) + f_o(x - y)) = 4(f_o(2x + y) + f_o(2x - y)) - 48f_o(x)
\]

(2.29)

for all \( x, y \in X \). By multiplying both sides of (2.29) by \( 1/4 \), we get

\[
2(f_o(x + y) + f_o(x - y)) = (f_o(2x + y) + f_o(2x - y)) - 12f_o(x)
\]

(2.30)

for all \( x, y \in X \). This means that \( f_o \) is a cubic function and that there exits a unique function \( C : X \times X \times X \rightarrow Y \) such that \( f_o(x) = C(x, x, x) \) for all \( x \in X \) and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all \( x \in X \), we have

\[
f(x) = f_o(x) + f_o(x) = C(x, x, x) + Q(x, x, x, x).
\]

(2.31)

The proof of the converse is trivially.

The following corollary is an alternative result of above Theorem 2.1.

**Corollary 2.2.** Let \( X, Y \) be vector spaces, and let \( f : X \rightarrow Y \) be a function satisfying (1.7). Then the following assertions hold.

(a) If \( f \) is even function, then \( f \) is quartic.

(b) If \( f \) is odd function, then \( f \) is cubic.
3. Stability

We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.7). In the following, let $X$ be a real vector space and let $Y$ be a Banach space. Given $f : X \to Y$, we define the difference operator $D_f : X \times X \to Y$ by

$$
D_f(x, y) = f(x + ky) + f(x - ky) - k^2(f(x + y) + f(x - y)) - 2(1 - k^2)f(x)
$$

(3.1)

$\quad 4 - k^2 - \frac{k^4}{4}(f(2y) - 8f(y)) - \tilde{f}(2x) + 16\tilde{f}(x)$

for all $x, y \in X$.

**Theorem 3.1.** Let $j \in \{-1, 1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$
\sum_{i=(1+j)/2}^{\infty} k^{4ij} \varphi\left(\frac{x}{k^{ij}}, \frac{y}{k^{ij}}\right) < \infty
$$

(3.2)

for all $x, y \in X$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$
\|D_f(x, y)\| \leq \varphi(x, y)
$$

(3.3)

for all $x, y \in X$. Then the limit

$$
Q(x) := \lim_{n \to \infty} k^{4nj} f\left(x\right)
$$

(3.4)

exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfying

$$
\|f(x) - Q(x)\| \leq \frac{1}{k^4} \tilde{\varphi}_e(x)
$$

(3.5)

for all $x \in X$, where

$$
\tilde{\varphi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij} \left[\frac{1}{2} \varphi\left(0, \frac{x}{k^{ij}}\right) + \frac{k^4 - k^2}{16} \varphi\left(\frac{x}{k^{ij}}, 0\right)\right].
$$

(3.6)

**Proof.** Let $j = 1$. It follows from (3.3) and using evenness of $f$ that

$$
\|f(x + ky) + f(x - ky) - k^2(f(x + y) + f(x - y)) - 2(1 - k^2)f(x)
$$

(3.7)

$$
- \frac{k^4 - k^2}{4}(f(2y) - 8f(y)) - 2f(2x) + 32f(x) \leq \varphi(x, y)
$$
for all \( x, y \in X \). Replacing \( x \) and \( y \) by \( 0 \) and \( x \) in (3.7), respectively, we see that

\[
\left\| 2f(kx) + (2k^4 - 4k^2) f(x) + \frac{k^2 - k^4}{4} f(2x) \right\| \leq \varphi(0, x) \tag{3.8}
\]

for all \( x \in X \). If we divide both sides of (3.8) by 2, we get

\[
\left\| f(kx) + \left(k^4 - 2k^2\right) f(x) + \frac{k^2 - k^4}{8} f(2x) \right\| \leq \frac{1}{2} \varphi(0, x) \tag{3.9}
\]

for all \( x \in X \). Putting \( y = 0 \) in (3.7), we obtain

\[
\left\| 2f(2x) - 32f(x) \right\| \leq \varphi(x, 0) \tag{3.10}
\]

for all \( x \in X \). If we multiply both sides of (3.10) by \((k^4 - k^2)/16\), then we have

\[
\left\| \frac{k^4 - k^2}{8} f(2x) - 2 \left(k^4 - k^2\right) f(x) \right\| \leq \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.11}
\]

for all \( x \in X \). It follows from (3.9) and (3.11) that

\[
\left\| f(kx) - k^4 f(x) \right\| \leq \frac{1}{2} \varphi(0, x) + \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.12}
\]

for all \( x \in X \). Let

\[
\psi_e(x) = \frac{1}{2} \varphi(0, x) + \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.13}
\]

for all \( x \in X \). Thus by (3.12), we get

\[
\left\| f(kx) - k^4 f(x) \right\| \leq \psi_e(x) \tag{3.14}
\]

for all \( x \in X \). If we replace \( x \) in (3.14) by \( x/k^{n+1} \) and multiply both sides of (3.14) by \( k^{4n} \), we see that

\[
\left\| k^{4(n+1)} f \left( \frac{x}{k^{n+1}} \right) - k^{4n} f \left( \frac{x}{k^n} \right) \right\| \leq k^{4n} \psi_e \left( \frac{x}{k^{n+1}} \right) \tag{3.15}
\]
for all \( x \in X \) and all nonnegative integers \( n \). So

\[
\left\| k^{4(n+1)} f\left( \frac{x}{k^{n+1}} \right) - k^{4m} f\left( \frac{x}{k^{m}} \right) \right\| \leq \sum_{i=m}^{n} \left\| k^{4(i+1)} f\left( \frac{x}{k^{1+i}} \right) - k^{4i} f\left( \frac{x}{k^{i}} \right) \right\|
\]

(3.16)

for all nonnegative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in X \). By (3.2), we infer that

\[
\sum_{i=m}^{n} k^{4i} q_{e}\left( \frac{x}{k^{1+i}} \right) < \infty, \quad \lim_{n \to \infty} k^{4n} q_{e}\left( \frac{x}{k^{n+1}} \right) = 0
\]

(3.17)

for all \( x \in X \). It follows from (3.16) and (3.17) that the sequence \( \{k^{4n} f(x/k^n)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{k^{4n} f(x/k^n)\} \) converges for all \( x \in X \). So one can define a mapping \( Q : X \to Y \) by (3.4) for all \( x \in X \). Letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.16), we obtain (3.5). It follows from (3.4), (3.15), and (3.17) that

\[
\left\| Q(x) - k^{4} Q\left( \frac{x}{k} \right) \right\| = \lim_{n \to \infty} \left\| k^{4n} f\left( \frac{x}{k^{n}} \right) - k^{4(n+1)} f\left( \frac{x}{k^{(n+1)}} \right) \right\| \leq \lim_{n \to \infty} k^{4n} q_{e}\left( \frac{x}{k^{n+1}} \right) = 0
\]

(3.18)

for all \( x \in X \). So

\[
Q(kx) = k^{4} Q(x)
\]

(3.19)

for all \( x \in X \). On the other hand, it follows from (3.2), (3.3), and (3.4) that

\[
\left\| D_{Q}(x, y) \right\| = \lim_{n \to \infty} k^{4n} \left\| D_{f}\left( \frac{x}{k^{n}}, \frac{y}{k^{n}} \right) \right\| \leq \lim_{n \to \infty} k^{4n} q_{e}\left( \frac{x}{k^{n}}, \frac{y}{k^{n}} \right) = 0
\]

(3.20)

for all \( x, y \in X \). Therefore, by Corollary 2.2, the function \( Q : X \to Y \) is quartic.

To prove the uniqueness of \( Q \), let \( Q' : X \to Y \) be another quartic function satisfying (3.5). Since

\[
\lim_{n \to \infty} k^{4n} \sum_{i=1}^{\infty} k^{4i} q_{e}\left( \frac{x}{k^{n+i}}, \frac{y}{k^{n+i}} \right) = \lim_{n \to \infty} \sum_{i=n+1}^{\infty} k^{4i} q_{e}\left( \frac{x}{k^{i}}, \frac{y}{k^{i}} \right) = 0
\]

(3.21)

for all \( x, y \in X \), hence

\[
\lim_{n \to \infty} k^{4n} q_{e}\left( \frac{x}{k^{n}} \right) = 0
\]

(3.22)
for all $x \in X$. So it follows from (3.5) and (3.22) that

$$
\|Q(x) - Q'(x)\| = \lim_{n \to \infty} k^{4n} \left\| \frac{x}{k^n} - Q' \left( \frac{x}{k^n} \right) \right\| \leq \lim_{n \to \infty} \frac{k^{4n}}{k^4} \psi_c \left( \frac{x}{k^n} \right) = 0
$$

(3.23)

for all $x \in X$. Hence $Q = Q'$. For $j = -1$, the proof of the theorem is similar.

Theorem 3.2. Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$
\tilde{\psi}_c(x, y) = \sum_{i=(1+j)/2}^{\infty} 2^{3ij} \varphi \left( \frac{x}{2^{ij}}, \frac{y}{2^{ij}} \right) < \infty
$$

(3.24)

for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.3). Then the limit

$$
C(x) := \lim_{n \to \infty} 2^{3nj} f \left( \frac{x}{2^{nj}} \right)
$$

(3.25)

exists for all $x \in X$ and $C : X \to Y$ is a unique cubic function satisfying

$$
\left\| f(x) - C(x) \right\| \leq \frac{1}{2(k^4 - k^2)} \tilde{\psi}_c(0, x)
$$

(3.26)

for all $x \in X$.

Proof. Let $j = 1$. It follows from (3.3) and using oddness of $f$ that

$$
\left\| f(x + ky) + f(x - ky) - k^2 (f(x + y) + f(x - y)) - 2(1 - k^2) f(x) \right. \\
- \frac{k^4 - k^2}{4} \left( f(2y) - 8f(y) \right) \right\| \leq \varphi(x, y)
$$

(3.27)

for all $x, y \in X$. Replacing $x$ and $y$ by 0 and $x$ in (3.27), respectively, we see that

$$
\left\| \frac{k^4 - k^2}{4} (f(2x) - 8f(x)) \right\| \leq \varphi(0, x)
$$

(3.28)

for all $x \in X$. If we multiply both sides of (3.28) by $4/(k^4 - k^2)$, we get

$$
\left\| f(2x) - 8f(x) \right\| \leq \frac{4}{k^4 - k^2} \varphi(0, x)
$$

(3.29)
for all \( x \in X \). If we replace \( x \) in (3.29) by \( x/2^{n+1} \) and multiply both sides of (3.29) by \( 2^{3n} \), we see that

\[
\left\| 2^{3(n+1)} f\left( \frac{x}{2^{n+1}} \right) - 2^{3n} f\left( \frac{x}{2^n} \right) \right\| \leq 2^{3n} \frac{4}{k^4 - k^2} q_0 \left( 0, \frac{x}{2^n} \right)
\]

(3.30)

for all \( x \in X \) and all nonnegative integers \( n \). So

\[
\left\| 2^{3(n+1)} f\left( \frac{x}{2^{n+1}} \right) - 2^{3m} f\left( \frac{x}{2^m} \right) \right\| \leq \sum_{i=m}^n \left\| 2^{3(i+1)} f\left( \frac{x}{2^{i+1}} \right) - 2^{3i} f\left( \frac{x}{2^i} \right) \right\|
\]

(3.31)

\[
\leq \frac{4}{k^4 - k^2} \sum_{i=m}^n 2^{3i} q_0 \left( 0, \frac{x}{2^{i+1}} \right)
\]

for all nonnegative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in X \). By (3.24), we infer that

\[
\sum_{i=m}^n 2^{3i} q_0 \left( \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}} \right) < \infty, \quad \lim_{n \to \infty} 2^{3n} q_0 \left( \frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) = 0
\]

(3.32)

for all \( x, y \in X \). It follows from (3.31) and (3.32) that the sequence \( \{2^{3n} f(x/2^n)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^{3n} f(x/2^n)\} \) converges for all \( x \in X \). So one can define a mapping \( C : X \to Y \) by (3.25) for all \( x \in X \). Letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.31), we obtain (3.26). It follows from (3.25), (3.30), and (3.32) that

\[
\left\| C(x) - 2^3 C\left( \frac{x}{2} \right) \right\| = \lim_{n \to \infty} \left\| 2^{3n} f\left( \frac{x}{2^n} \right) - 2^{3(n+1)} f\left( \frac{x}{2^{n+1}} \right) \right\|
\]

(3.33)

\[
\leq \lim_{n \to \infty} \frac{4}{k^4 - k^2} 2^{3n} q_0 \left( 0, \frac{x}{2^{n+1}} \right) = 0
\]

for all \( x \in X \). So

\[
C(2x) = 2^3 C(x)
\]

(3.34)

for all \( x \in X \). On the other hand, it follows from (3.3), (3.24), and (3.25) that

\[
\left\| D_x(x, y) \right\| = \lim_{n \to \infty} \left\| 2^{3n} D_x\left( \frac{x}{2^n}, \frac{y}{2^n} \right) \right\|
\]

(3.35)

\[
\leq \lim_{n \to \infty} 2^{3n} q_0\left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0
\]

for all \( x, y \in X \). Therefore by Corollary 2.2, the function \( C : X \to Y \) is cubic.

To prove the uniqueness of \( C \), let \( C' : X \to Y \) be another cubic function satisfying (3.26). Since

\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} 2^{3i} q_0\left( \frac{x}{2^{n+i}}, \frac{y}{2^{n+i}} \right) = \lim_{n \to \infty} \sum_{i=n+1}^{\infty} 2^{3i} q_0\left( \frac{x}{2^i}, \frac{y}{2^i} \right) = 0
\]

(3.36)
for all \( x, y \in X \), hence

\[
\lim_{n \to \infty} 2^{3n} \bar{q}_c \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0
\]  

(3.37)

for all \( x, y \in X \). So it follows from (3.26) and (3.37) that

\[
\| C(x) - C'(x) \| = \lim_{n \to \infty} 2^{3n} \left\| f \left( \frac{x}{2^n} \right) - C' \left( \frac{x}{2^n} \right) \right\| \leq \lim_{n \to \infty} \frac{1}{2(k^4 - k^2)} 2^{3n} \bar{q}_c \left( 0, \frac{x}{2^n} \right) = 0
\]  

(3.38)

for all \( x \in X \). Hence \( C = C' \).

For \( j = -1 \), the proof of the theorem is similar. \( \square \)

**Theorem 3.3.** Let \( j \in \{1, -1\} \) be fixed. Suppose that a function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.3). If the upper bound \( \phi : X \times X \to [0, \infty) \) is a mapping such that

\[
\sum_{i=\left(1+j\right)/2}^{\infty} \left( \frac{1+j}{2} \right)^{k^4} \varphi \left( \frac{x}{k^{ij}}, \frac{y}{k^{ij}} \right) + \left( \frac{1-j}{2} \right)^{2k^4} \varphi \left( \frac{x}{2^{ij}}, \frac{y}{2^{ij}} \right) \leq \infty,
\]

(3.39)

for all \( x, y \in X \), then there exists a unique quartic function \( Q : X \to Y \) and a unique cubic function \( C : X \to Y \) satisfying

\[
\| f(x) - Q(x) - C(x) \| \leq \frac{1}{2k^4} \left[ \bar{g}_c(x) + \bar{g}_c(-x) \right] + \frac{1}{4(k^4 - k^2)} \left[ \bar{g}_c(0, x) + \bar{g}_c(0, -x) \right]
\]

(3.40)

for all \( x \in X \), where

\[
\bar{g}_c(x) = \sum_{i=\left(1+j\right)/2}^{\infty} k^{4ij} \left[ \frac{1}{2} \varphi \left( 0, \frac{x}{k^{ij}} \right) + \frac{k^4 - k^2}{16} \varphi \left( \frac{x}{k^{ij}}, 0 \right) \right],
\]

(3.41)

Proof. Let \( f_c(x) = (1/2)(f(x) + f(-x)) \) for all \( x \in X \). Then \( f_c(0) = 0 \) and \( f_c \) is even function satisfying \( \| D_{f_c}(x, y) \| \leq (1/2) [\phi(x, y) + \phi(-x, -y)] \) for all \( x, y \in X \). By Theorem 3.1, there exists a unique quartic function \( Q : X \to Y \) satisfying

\[
\| f_c(x) - Q(x) \| \leq \frac{1}{2k^4} \left[ \bar{g}_c(x) + \bar{g}_e(-x) \right]
\]

(3.42)

for all \( x \in X \), where

\[
\bar{g}_c(x) = \sum_{i=\left(1+j\right)/2}^{\infty} k^{4ij} \left[ \frac{1}{2} \varphi \left( 0, \frac{x}{k^{ij}} \right) + \frac{k^4 - k^2}{16} \varphi \left( \frac{x}{k^{ij}}, 0 \right) \right]
\]

(3.43)
for all \( x \in X \). Let now \( f_o(x) = (1/2)(f(x) - f(-x)) \) for all \( x \in X \). Then \( f_o(0) = 0 \) and \( f_o \) is an odd function satisfying \( \|Df_o(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)] \) for all \( x, y \in X \). Hence, in view of Theorem 3.2, there exists a unique cubic function \( C : X \to Y \) such that

\[
\|f_o(x) - Q(x)\| \leq \frac{1}{4(k^4 - k^2)} [\tilde{\psi}_C(0, x) + \tilde{\psi}_C(0, -x)] \tag{3.44}
\]

for all \( x \in X \), where

\[
\tilde{\psi}_C(x, y) = \sum_{i=0}^{\infty} 2^{3n_j} \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \tag{3.45}
\]

for all \( x, y \in X \). On the other hand, we have \( f(x) = f_o(x) + f_o(x) \) for all \( x \in X \). Then by combining (3.42) and (3.44), it follows that

\[
\|f(x) - C(x) - Q(x)\| \leq \|f_o(x) - Q(x)\| + \|f_o(x) - C(x)\|
\]

\[
\leq \frac{1}{2k^4} [\tilde{\psi}_C(x) + \tilde{\psi}_C(-x)] + \frac{1}{4(k^4 - k^2)} [\tilde{\psi}_C(0, x) + \tilde{\psi}_C(0, -x)] \tag{3.46}
\]

for all \( x \in X \).

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.7).

**Corollary 3.4.** Let \( p \in (-\infty, 3) \cup (4, +\infty) \), \( \theta > 0 \). Suppose \( f : X \to Y \) satisfies \( f(0) = 0 \) and inequality

\[
\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.47}
\]

for all \( x, y \in X \). Then there exist a unique quartic function \( Q : X \to Y \) and a unique cubic function \( C : X \to Y \) satisfying

\[
\|f(x) - Q(x) - C(x)\|
\]

\[
\leq \begin{cases} 
\theta\|x\|^p \left( \frac{1}{k^4} \left( \frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left( \frac{1}{k^p - 1} \right) \right), & p > 4, \\
\theta\|x\|^p \left( \frac{1}{k^4} \left( \frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left( \frac{1}{1 - k^p} \right) \right), & p < 3,
\end{cases}
\]

for all \( x \in X \).

**Proof.** In Theorem 3.3, put \( \phi(x, y) = \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \).
Similarly, one can solve Ulam stability problem for functional equation (1.7) when the norm of the Cauchy difference is controlled by the mixed type product-sum function

\[(x, y) \mapsto \theta(\|x\|_X^p \|y\|_Y^p + \|x\|^p + \|y\|^p). \tag{3.49}\]

**Corollary 3.5.** Let \(u, v, p\) be real numbers such that \(u+v, p \in (-\infty, 3) \cup (4, +\infty)\) and \(\theta > 0\). Suppose \(f : X \to Y\) satisfies \(f(0) = 0\) and inequality

\[
\|D_f(x, y)\| \leq \theta(\|x\|_X^p \|y\|_Y^p + \|x\|^p + \|y\|^p) \tag{3.50}
\]

for all \(x, y \in X\). Then there exist a unique quartic function \(Q : X \to Y\) and a unique cubic function \(C : X \to Y\) satisfying

\[
\|f(x) - Q(x) - C(x)\| \leq \begin{cases} 
\theta\|x\|^p \left( \frac{1}{k^4} \left( \frac{1}{2} + \frac{k^4 - k^2}{16} \left( \frac{1}{k^4} - 1 \right) \right) + \frac{1}{2(k^4 - k^2)} \left( \frac{1}{2} - 1 \right) \right), & p > 4, \\
\theta\|x\|^p \left( \frac{1}{k^4} \left( \frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left( \frac{1}{k^4} - 1 \right) \right) + \frac{1}{2(k^4 - k^2)} \left( \frac{1}{2} - 1 \right), & p < 3,
\end{cases} \tag{3.51}
\]

for all \(x \in X\).

Applying Corollary 3.4, one can obtain the stability of the functional equation (1.7) in the following form.

**Corollary 3.6.** Let \(e\) be a positive real number. Suppose \(f : X \to Y\) satisfies \(f(0) = 0\) and \(\|D_f(x, y)\| \leq e\) for all \(x, y \in X\). Then there exists a unique quartic function \(Q : X \to Y\) and a unique cubic function \(C : X \to Y\) satisfying

\[
\|f(x) - Q(x) - C(x)\| \leq e \left( \frac{1}{k^4} \left( \frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left( \frac{k^4}{k^4 - 1} \right) + \frac{1}{2(k^4 - k^2)} \left( \frac{8}{8 - 1} \right) \right) \tag{3.52}
\]

for all \(x \in X\).

**References**


