Research Article

Weighted Asymptotically Periodic Solutions of Linear Volterra Difference Equations

Josef Diblík,1, 2 Miroslava Růžičková,3 Ewa Schmeidel,4 and Małgorzata Zbaszyńska4

1 Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 66237 Brno, Czech Republic
2 Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 61600 Brno, Czech Republic
3 Department of Mathematics, University of Žilina, 01026 Žilina, Slovakia
4 Faculty of Electrical Engineering, Institute of Mathematics, Poznań University of Technology, 60965 Poznań, Poland

Correspondence should be addressed to Ewa Schmeidel, ewa.schmeidel@put.poznan.pl

Received 16 January 2011; Accepted 17 March 2011

Academic Editor: Elena Braverman

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A linear Volterra difference equation of the form

\[ x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \]

where \( x : \mathbb{N}_0 \to \mathbb{R} \), \( a : \mathbb{N}_0 \to \mathbb{R} \), \( K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R} \), and \( b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\} \) is \( \omega \)-periodic, is considered. Sufficient conditions for the existence of weighted asymptotically periodic solutions of this equation are obtained. Unlike previous investigations, no restriction on \( \prod_{j=0}^{\omega-1} b(j) \) is assumed. The results generalize some of the recent results.

1. Introduction

In the paper, we study a linear Volterra difference equation

\[ x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \quad (1.1) \]

where \( n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), \( a : \mathbb{N}_0 \to \mathbb{R} \), \( K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R} \), and \( b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\} \) is \( \omega \)-periodic, \( \omega \in \mathbb{N} := \{1, 2, \ldots\} \). We will also adopt the customary notations

\[ \sum_{i=k+s}^{k} \Theta(i) = 0, \quad \prod_{i=k+s}^{k} \Theta(i) = 1, \quad (1.2) \]
where \( k \) is an integer, \( s \) is a positive integer, and “\( O \)” denotes the function considered independently of whether it is defined for the arguments indicated or not.

In [1], the authors considered (1.1) under the assumption

\[
\prod_{j=0}^{\omega-1} b(j) = 1,
\]

and gave sufficient conditions for the existence of asymptotically \( \omega \)-periodic solutions of (1.1) where the notion for an asymptotically \( \omega \)-periodic function has been given by the following definition.

**Definition 1.1.** Let \( \omega \) be a positive integer. The sequence \( y : \mathbb{N}_0 \to \mathbb{R} \) is called \( \omega \)-periodic if \( y(n + \omega) = y(n) \) for all \( n \in \mathbb{N}_0 \). The sequence \( y \) is called asymptotically \( \omega \)-periodic if there exist two sequences \( u, v : \mathbb{N}_0 \to \mathbb{R} \) such that \( u \) is \( \omega \)-periodic, \( \lim_{n \to \infty} v(n) = 0 \), and

\[
y(n) = u(n) + v(n)
\]

for all \( n \in \mathbb{N}_0 \).

In this paper, in general, we do not assume that (1.3) holds. Then, we are able to derive sufficient conditions for the existence of a weighted asymptotically \( \omega \)-periodic solution of (1.1). We give a definition of a weighted asymptotically \( \omega \)-periodic function.

**Definition 1.2.** Let \( \omega \) be a positive integer. The sequence \( y : \mathbb{N}_0 \to \mathbb{R} \) is called weighted asymptotically \( \omega \)-periodic if there exist two sequences \( u, v : \mathbb{N}_0 \to \mathbb{R} \) such that \( u \) is \( \omega \)-periodic and \( \lim_{n \to \infty} v(n) = 0 \), and, moreover, if there exists a sequence \( w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\} \) such that

\[
\frac{y(n)}{w(n)} = u(n) + v(n),
\]

for all \( n \in \mathbb{N}_0 \).

Apart from this, when we assume

\[
\prod_{k=0}^{\omega-1} b(k) = -1,
\]

then, as a consequence of our main result (Theorem 2.2), the existence of an asymptotically \( 2\omega \)-periodic solution of (1.1) is obtained.

For the reader’s convenience, we note that the background for discrete Volterra equations can be found, for example, in the well-known monograph by Agarwal [2], as well as by Elaydi [3] or Kocić and Ladas [4]. Volterra difference equations were studied by many others, for example, by Appleby et al. [5], by Elaydi and Murakami [6], by Győri and Horváth [7], by Győri and Reynolds [8], and by Song and Baker [9]. For some results on periodic solutions of difference equations, see, for example, [2–4, 10–13] and the related references therein.
2. Weighted Asymptotically Periodic Solutions

In this section, sufficient conditions for the existence of weighted asymptotically \( \omega \)-periodic solutions of (1.1) will be derived. The following version of Schauder’s fixed point theorem given in [14] will serve as a tool used in the proof.

Lemma 2.1. Let \( \Omega \) be a Banach space and \( S \) its nonempty, closed, and convex subset and let \( T \) be a continuous mapping such that \( T(S) \) is contained in \( S \) and the closure \( \overline{T(S)} \) is compact. Then, \( T \) has a fixed point in \( S \).

We set

\[
\beta(n) := \prod_{j=0}^{n-1} b(j), \quad n \in \mathbb{N}_0, \quad (2.1)
\]

\[
\mathcal{B} := \beta(\omega). \quad (2.2)
\]

Moreover, we define

\[
n^* := n - 1 - \omega \left\lfloor \frac{n-1}{\omega} \right\rfloor, \quad (2.3)
\]

where \( \lfloor \cdot \rfloor \) is the floor function (the greatest-integer function) and \( n^* \) is the “remainder” of dividing \( n-1 \) by \( \omega \). Obviously, \( \{\beta(n^*)\}, n \in \mathbb{N} \) is an \( \omega \)-periodic sequence.

Now, we derive sufficient conditions for the existence of a weighted asymptotically \( \omega \)-periodic solution of (1.1).

Theorem 2.2 (Main result). Let \( \omega \) be a positive integer, \( b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\} \) be \( \omega \)-periodic, \( a : \mathbb{N}_0 \to \mathbb{R} \), and \( K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R} \). Assume that

\[
\sum_{i=0}^{\infty} \left| a(i) \right| \beta(i+1) < \infty,
\]

\[
\sum_{j=0}^{\infty} \sum_{i} \left| K(j,i) \beta(i) \right| \frac{1}{\beta(j+1)} < 1, \quad (2.4)
\]

and that at least one of the real numbers in the left-hand sides of inequalities (2.4) is positive.

Then, for any nonzero constant \( c \), there exists a weighted asymptotically \( \omega \)-periodic solution \( x : \mathbb{N}_0 \to \mathbb{R} \) of (1.1) with \( u, v : \mathbb{N}_0 \to \mathbb{R} \) and \( w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\} \) in representation (1.5) such that

\[
w(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor}, \quad u(n) := c\beta(n^* + 1), \quad \lim_{n \to \infty} v(n) = 0, \quad (2.5)
\]

that is,

\[
\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^* + 1) + v(n), \quad n \in \mathbb{N}_0. \quad (2.6)
\]
Proof. We will use a notation

\[ M := \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| K(j, i) \beta(i) \right| / \beta(j + 1), \]  

(2.7)

whenever this is useful.

Case 1. First assume \( c > 0 \). We will define an auxiliary sequence of positive numbers \( \{a(n)\} \), \( n \in \mathbb{N}_0 \). We set

\[ a(0) := \frac{\sum_{i=0}^{\infty} \left| a(i) \right| / \beta(i + 1)}{1 - \sum_{j=0}^{\infty} \left| \sum_{i=0}^{j} \left( K(j, i) \beta(i) \right) / \beta(j + 1) \right|}, \]

(2.8)

where the expression on the right-hand side is well defined due to (2.4). Moreover, we define

\[ a(n) := \sum_{i=n}^{\infty} \left| a(i) / \beta(i + 1) \right| + (c + a(0)) \sum_{j=n}^{\infty} \sum_{i=0}^{j} \left| K(j, i) \beta(i) \right| / \beta(j + 1), \]

(2.9)

for \( n \geq 1 \). It is easy to see that

\[ \lim_{n \to \infty} a(n) = 0. \]

(2.10)

We show, moreover, that

\[ a(n) \leq a(0), \]

(2.11)

for any \( n \in \mathbb{N} \). Let us first remark that

\[ a(0) = \sum_{i=0}^{\infty} \left| a(i) / \beta(i + 1) \right| + (c + a(0)) \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| K(j, i) \beta(i) \right| / \beta(j + 1). \]

(2.12)

Then, due to the convergence of both series (see (2.4)), the inequality

\[ a(0) = \sum_{i=0}^{\infty} \left| a(i) / \beta(i + 1) \right| + (c + a(0)) \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| K(j, i) \beta(i) \right| / \beta(j + 1) \]

\[ \geq \sum_{i=0}^{\infty} \left| a(i) / \beta(i + 1) \right| + (c + a(0)) \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| K(j, i) \beta(i) \right| / \beta(j + 1) = a(n) \]

(2.13)

obviously holds for every \( n \in \mathbb{N} \) and (2.11) is proved.
Let $B$ be the Banach space of all real bounded sequences $z : \mathbb{N}_0 \to \mathbb{R}$ equipped with the usual supremum norm $\|z\| = \sup_{n \in \mathbb{N}_0} |z(n)|$ for $z \in B$. We define a subset $S \subset B$ as

$$S := \{ z \in B : c - a(0) \leq z(n) \leq c + a(0), \ n \in \mathbb{N}_0 \}. \quad (2.14)$$

It is not difficult to prove that $S$ is a nonempty, bounded, convex, and closed subset of $B$.

Let us define a mapping $T : S \to B$ as follows:

$$(Tz)(n) = c - \sum_{i=n}^{\infty} \frac{a(i)}{\beta(i+1)} - \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{K(j,i)\beta(i)}{\beta(j+1)} z(i), \quad (2.15)$$

for any $n \in \mathbb{N}_0$.

We will prove that the mapping $T$ has a fixed point in $S$.

We first show that $T(S) \subset S$. Indeed, if $z \in S$, then $|z(n) - c| \leq a(0)$ for $n \in \mathbb{N}_0$ and, by (2.11) and (2.15), we have

$$|(Tz)(n) - c| \leq \sum_{i=n}^{\infty} \frac{a(i)}{\beta(i+1)} + (c + a(0)) \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{K(j,i)\beta(i)}{\beta(j+1)} = a(n) \leq a(0). \quad (2.16)$$

Next, we prove that $T$ is continuous. Let $z^{(p)}$ be a sequence in $S$ such that $z^{(p)} \to z$ as $p \to \infty$. Because $S$ is closed, $z \in S$. Now, utilizing (2.15), we get

$$\left| (Tz^{(p)})(n) - (Tz)(n) \right| = \left| \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{K(j,i)\beta(i)}{\beta(j+1)} (z^{(p)}(i) - z(i)) \right|$$

$$\leq M \sup_{i \geq 0} \left| z^{(p)}(i) - z(i) \right| = M \left\| z^{(p)} - z \right\|, \quad n \in \mathbb{N}_0. \quad (2.17)$$

Therefore,

$$\left\| Tz^{(p)} - Tz \right\| \leq M \left\| z^{(p)} - z \right\|, \quad (2.18)$$

This means that $T$ is continuous.

Now, we show that $\overline{T(S)}$ is compact. As is generally known, it is enough to verify that every $\epsilon$-open covering of $\overline{T(S)}$ contains a finite $\epsilon$-subcover of $\overline{T(S)}$, that is, finitely many of these open sets already cover $\overline{T(S)}$ ([15], page 756 (12)). Thus, to prove that $\overline{T(S)}$ is compact, we take an arbitrary $\epsilon > 0$ and assume that an open $\epsilon$-cover $C_\epsilon$ of $\overline{T(S)}$ is given. Then, from (2.10), we conclude that there exists an $n_\epsilon \in \mathbb{N}$ such that $a(n) < \epsilon/4$ for $n \geq n_\epsilon$. 

Suppose that $x^1_T \in \overline{T(S)}$ is one of the elements generating the $\varepsilon$-cover $C_\varepsilon$ of $\overline{T(S)}$. Then (as follows from (2.16)), for an arbitrary $x_T \in \overline{T(S)}$,

$$\left| x^1_T(n) - x_T(n) \right| < \varepsilon \quad (2.19)$$

if $n \geq n_\varepsilon$. In other words, the $\varepsilon$-neighborhood of $x^1_T - c^*$:

$$\| x^1_T - c^* \| < \varepsilon, \quad (2.20)$$

where $c^* = \{c, c, \ldots\} \in S$ covers the set $\overline{T(S)}$ on an infinite interval $n \geq n_\varepsilon$. It remains to cover the rest of $\overline{T(S)}$ on a finite interval for $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$ by a finite number of $\varepsilon$-neighborhoods of elements generating $\varepsilon$-cover $C_\varepsilon$. Supposing that $x^1_T$ itself is not able to generate such cover, we fix $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$ and split the interval

$$[c - \alpha(n), c + \alpha(n)] \quad (2.21)$$

into a finite number $h(\varepsilon, n)$ of closed subintervals

$$I_1(n), I_2(n), \ldots, I_{h(\varepsilon, n)}(n) \quad (2.22)$$

each with a length not greater than $\varepsilon/2$ such that

$$\bigcup_{i=1}^{h(\varepsilon, n)} I_i(n) = [c - \alpha(n), c + \alpha(n)], \quad (2.23)$$

$$\text{int } I_i(n) \cap \text{int } I_j(n) = \emptyset, \quad i, j = 1, 2, \ldots, h(\varepsilon, n), \quad i \neq j.$$ 

Finally, the set

$$\bigcup_{n=0}^{n_\varepsilon-1} [c - \alpha(n), c + \alpha(n)] \quad (2.24)$$

equals

$$\bigcup_{n=0}^{n_\varepsilon-1} \bigcup_{i=1}^{h(\varepsilon, n)} I_i(n) \quad (2.25)$$

and can be divided into a finite number

$$M_\varepsilon := \sum_{n=0}^{n_\varepsilon-1} h(\varepsilon, n) \quad (2.26)$$
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of different subintervals (2.22). This means that, at most, \( M_\varepsilon \) of elements generating the cover \( C_\varepsilon \) are sufficient to generate a finite \( \varepsilon \)-subcover of \( \overline{T(S)} \) for \( n \in \{0, 1, \ldots, n_\varepsilon - 1\} \). We remark that each of such elements simultaneously plays the same role as \( x^1(n) \) for \( n \geq n_\varepsilon \). Since \( \varepsilon > 0 \) can be chosen as arbitrarily small, \( \overline{T(S)} \) is compact.

By Schauder’s fixed point theorem, there exists a \( z \in S \) such that \( z(n) = (Tz)(n) \) for \( n \in \mathbb{N}_0 \). Thus,

\[
z(n) = c - \lim_{n \to \infty} \frac{a(i)}{p(i + 1)} - \sum_{j=0}^{\infty} \frac{\beta(i)}{\beta(j + 1)} K(j, i) z(i),
\]

(2.27)

for any \( n \in \mathbb{N}_0 \).

Due to (2.10) and (2.16), for fixed point \( z \in S \) of \( T \), we have

\[
\lim_{n \to \infty} |z(n) - c| = \lim_{n \to \infty} |(Tz)(n) - c| \leq \lim_{n \to \infty} \alpha(n) = 0,
\]

(2.28)

or, equivalently,

\[
\lim_{n \to \infty} z(n) = c.
\]

(2.29)

Finally, we will show that there exists a connection between the fixed point \( z \in S \) and the existence of a solution of (1.1) which divided by \( B^{[n-1]/\varepsilon] \) provides an asymptotically \( \omega \)-periodic sequence. Considering (2.27) for \( z(n+1) \) and \( z(n) \), we get

\[
\Delta z(n) = \frac{a(n)}{\beta(n + 1)} + \sum_{i=0}^{n} \frac{\beta(i)}{\beta(n + 1)} K(n, i) z(i),
\]

(2.30)

where \( n \in \mathbb{N}_0 \). Hence, we have

\[
z(n + 1) - z(n) = \frac{a(n)}{\beta(n + 1)} + \frac{1}{\beta(n + 1)} \sum_{i=0}^{n} \beta(i) K(n, i) z(i), \quad n \in \mathbb{N}_0.
\]

(2.31)

Putting

\[
z(n) = \frac{x(n)}{\beta(n)}, \quad n \in \mathbb{N}_0
\]

(2.32)

in (2.31), we get (1.1) since

\[
\frac{x(n + 1)}{\beta(n + 1)} - \frac{x(n)}{\beta(n)} = \frac{a(n)}{\beta(n + 1)} + \frac{1}{\beta(n + 1)} \sum_{i=0}^{n} K(n, i) x(i), \quad n \in \mathbb{N}_0
\]

(2.33)
yields
\[ x(n + 1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n, i)x(i), \quad n \in \mathbb{N}_0. \]  
(2.34)

Consequently, \( x \) defined by (2.32) is a solution of (1.1). From (2.29) and (2.32), we obtain
\[ \frac{x(n)}{\beta(n)} = z(n) = c + o(1), \]  
(2.35)

for \( n \to \infty \) (where \( o(1) \) is the Landau order symbol). Hence,
\[ x(n) = \beta(n)(c + o(1)), \quad n \to \infty. \]  
(2.36)

It is easy to show that the function \( \beta \) defined by (2.1) can be expressed in the form
\[ \beta(n) = \prod_{j=0}^{n-1} b(j) = B^{[(n-1)/\omega]} \cdot \beta(n^* + 1), \]  
(2.37)

for \( n \in \mathbb{N}_0 \). Then, as follows from (2.36),
\[ x(n) = B^{[(n-1)/\omega]} \cdot \beta(n^* + 1)(c + o(1)), \quad n \to \infty, \]  
(2.38)

or
\[ \frac{x(n)}{B^{[(n-1)/\omega]}} = c\beta(n^* + 1) + \beta(n^* + 1)o(1), \quad n \to \infty. \]  
(2.39)

The proof is completed since the sequence \( \{\beta(n^* + 1)\} \) is \( \omega \)-periodic, hence bounded and, due to the properties of Landau order symbols, we have
\[ \beta(n^* + 1)o(1) = o(1), \quad n \to \infty, \]  
(2.40)

and it is easy to see that the choice
\[ u(n) := c\beta(n^* + 1), \quad w(n) := B^{[(n-1)/\omega]}, \quad n \in \mathbb{N}_0, \]  
(2.41)

and an appropriate function \( v : \mathbb{N}_0 \to \mathbb{R} \) such that
\[ \lim_{n \to \infty} v(n) = 0 \]  
(2.42)

finishes this part of the proof. Although for \( n = 0 \), there is no correspondence between formula (2.36) and the definitions of functions \( u \) and \( w \), we assume that function \( v \) makes up for this.
Case 2. If $c < 0$, we can proceed as follows. It is easy to see that arbitrary solution $y = y(n)$ of the equation

$$y(n + 1) = -a(n) + b(n)y(n) + \sum_{i=0}^{n} K(n, i)y(i)$$ \hspace{1cm} (2.43)

defines a solution $x = x(n)$ of (1.1) since a substitution $y(n) = -x(n)$ in (2.43) turns (2.43) into (1.1). If the assumptions of Theorem 2.2 hold for (1.1), then, obviously, Theorem 2.2 holds for (2.43) as well. So, for an arbitrary $c > 0$, (2.43) has a solution that can be represented by formula (2.6), that is,

$$\frac{y(n)}{B[n-1/ω]} = c\beta(n^* + 1) + v(n), \quad n \in \mathbb{N}_0.$$ \hspace{1cm} (2.44)

Or, in other words, (1.1) has a solution that can be represented by formula (2.44) as

$$\frac{x(n)}{B[n-1/ω]} = c_0\beta(n^* + 1) + v^*(n), \quad n \in \mathbb{N}_0,$$ \hspace{1cm} (2.45)

with $c_0 = -c$ and $v^*(n) = -v(n)$. In (2.45), $c_0 < 0$ and the function $v^*(n)$ has the same properties as the function $v(n)$. Therefore, formula (2.6) is valid for an arbitrary negative $c$ as well.

Now, we give an example which illustrates the case where there exists a solution to equation of the type (1.1) which is weighted asymptotically periodic, but is not asymptotically periodic.

Example 2.3. We consider (1.1) with

$$a(n) = (-1)^{n+1} \left(1 - \frac{1}{3^{n+1}}\right),$$

$$b(n) = 3(-1)^{n},$$ \hspace{1cm} (2.46)

$$K(n, i) = (-1)^{n+i(i-1)/2} \frac{1}{3^i},$$

that is, the equation

$$x(n + 1) = (-1)^{n+1} \left(1 - \frac{1}{3^{n+1}}\right) + 3(-1)^{n}x(n) + \sum_{i=0}^{n} (-1)^{n+i(i-1)/2} \frac{1}{3^i}x(i).$$ \hspace{1cm} (2.47)
The sequence $b(n)$ is 2-periodic and

\[ \beta(n) = \prod_{j=0}^{n-1} b(j) = (-1)^{n(n+1)/2} 3^n, \]

\[ \mathcal{B} = \beta(\omega) = \beta(2) = -9, \]

\[ \beta(n^* + 1) = -3 + 6(-1)^{n+1}, \]

\[ \frac{a(n)}{\beta(n + 1)} = (-1)^{(-1^{n+2})/2} \left( \frac{1}{3^{n+1}} - \frac{1}{3^{2(n+1)}} \right), \]

\[ \sum_{i=0}^{\infty} \left| \frac{a(i)}{\beta(i + 1)} \right| < \infty, \quad (2.48) \]

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{i} K(j, i) \beta(i) \beta(j + 1) \]

\[ = \frac{1}{3} \left( \sum_{i=0}^{\infty} \frac{1}{3^i} \right) \left( \sum_{i=0}^{\infty} \frac{1}{3^i} \right) = \frac{1}{3} \cdot \frac{1}{1 - 1/3} \cdot \frac{1}{1 - 1/3} \]

\[ = \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{3}{4} < 1. \]

By virtue of Theorem 2.2, for any nonzero constant $c$, there exists a solution $x : \mathbb{N}_0 \to \mathbb{R}$ of (1.1) which is weighed asymptotically 2-periodic. Let, for example, $c = 2/3$. Then,

\[ w(n) = (-9)^{(n-1)/2}, \]

\[ u(n) = c\beta(n^* + 1) = \frac{2}{3} \left( -3 + 6(-1)^{n+1} \right) = -2 + 4(-1)^{n+1}, \quad (2.49) \]

and the sequence $x(n)$ given by

\[ \frac{x(n)}{(-9)^{(n-1)/2}} = -2 + 4(-1)^{n+1} + v(n), \quad n \in \mathbb{N}_0, \quad (2.50) \]

or, equivalently,

\[ x(n) = (-9)^{(n-1)/2} \left( -2 + 4(-1)^{n+1} \right) + v(n), \quad n \in \mathbb{N}_0 \quad (2.51) \]

is such a solution. We remark that such solution is not asymptotically 2-periodic in the meaning of Definition 1.1.
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It is easy to verify that the sequence $x^*(n)$ obtained from (2.51) if $v(n) = 0, n \in \mathbb{N}_0$, that is,

$$x^*(n) = (-9)^{\lfloor (n-1)/2 \rfloor} \left(-2 + 4(-1)^{n+1}\right) = \frac{2}{3} \cdot (-1)^{n(n-1)/2} \cdot 3^n, \quad n \in \mathbb{N}_0$$  (2.52)

is a true solution of (2.47).

3. Concluding Remarks and Open Problems

It is easy to prove the following corollary.

**Corollary 3.1.** Let Theorem 2.2 be valid. If, moreover, $|B| < 1$, then every solution $x = x(n)$ of (1.1) described by formula (2.6) satisfies

$$\lim_{n \to \infty} x(n) = 0.\quad (3.1)$$

If $|B| > 1$, then, for every solution $x = x(n)$ of (1.1) described by formula (2.6), one has

$$\liminf_{n \to \infty} x(n) = -\infty \quad (3.2)$$

or/and

$$\limsup_{n \to \infty} x(n) = \infty. \quad (3.3)$$

Finally, if $B > 1$, then, for every solution $x = x(n)$ of (1.1) described by formula (2.6), one has

$$\lim_{n \to \infty} x(n) = \infty, \quad (3.4)$$

and if $B < -1$, then, for every solution $x = x(n)$ of (1.1) described by formula (2.6), one has

$$\lim_{n \to \infty} x(n) = -\infty. \quad (3.5)$$

Now, let us discuss the case when (1.6) holds, that is, when

$$B = \prod_{j=0}^{\omega-1} b(j) = -1. \quad (3.6)$$

**Corollary 3.2.** Let Theorem 2.2 be valid. Assume that $B = -1$. Then, for any nonzero constant $c$, there exists an asymptotically $2\omega$-periodic solution $x = x(n), n \in \mathbb{N}_0$ of (1.1) such that

$$x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + z(n), \quad n \in \mathbb{N}_0, \quad (3.7)$$
\[ u(n) := c\beta(n^* + 1), \quad \lim_{n \to \infty} z(n) = 0. \] (3.8)

**Proof.** Putting \( B = -1 \) in Theorem 2.2, we get
\[ x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + (-1)^{\lfloor (n-1)/\omega \rfloor} v(n), \] (3.9)

with
\[ u(n) := c\beta(n^* + 1), \quad \lim_{n \to \infty} v(n) = 0. \] (3.10)

Due to the definition of \( n^* \), we see that the sequence
\[ \{ \beta(n^* + 1) \} = \{ \beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), \ldots \}, \] (3.11)
is an \( \omega \)-periodic sequence. Since
\[ \left\{ \left\lfloor \frac{n-1}{\omega} \right\rfloor \right\} = \left\{ -1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots \right\}, \] (3.12)
for \( n \in \mathbb{N}_0 \), we have
\[ \left\{ (-1)^{\lfloor (n-1)/\omega \rfloor} \right\} = \left\{ -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots \right\}. \] (3.13)

Therefore, the sequence
\[ \left\{ (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) \right\} = c\{ -\beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), -\beta(1), -\beta(2), \ldots, -\beta(\omega), \ldots \} \] (3.14)
is a \( 2\omega \)-periodic sequence. Set
\[ z(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} v(n). \] (3.15)

Then,
\[ \lim_{n \to \infty} z(n) = 0. \] (3.16)

The proof is completed.
Remark 3.3. From the proof, we see that Theorem 2.2 remains valid even in the case of $c = 0$. Then, there exists an “asymptotically weighted $\omega$-periodic solution” $x = x(n)$ of (1.1) as well. The formula (2.6) reduces to

$$x(n) = B^{[(n-1)/\omega]}v(n) = o(1), \quad n \in \mathbb{N}_0, \quad (3.17)$$

since $u(n) = 0$. In the light of Definition 1.2, we can treat this case as follows. We set (as a singular case) $u \equiv 0$ with an arbitrary (possibly other than “$\omega$”) period and with $v = o(1)$, $n \to \infty$.

Remark 3.4. The assumptions of Theorem 2.2 [1] are substantially different from those of the present Theorem 2.2. However, it is easy to see that Theorem 2.2 [1] is a particular case of the present Theorem 2.2 if (1.3) holds, that is, if $B = 1$. Therefore, our results can be viewed as a generalization of some results in [1].

In connection with the above investigations, some open problems arise.

Open Problem 1. The results of [1] are extended to systems of linear Volterra discrete equations in [16, 17]. It is an open question if the results presented can be extended to systems of linear Volterra discrete equations.

Open Problem 2. Unlike the result of Theorem 2.2 [1] where a parameter $c$ can be arbitrary, the assumptions of the results in [16, 17] are more restrictive since the related parameters should satisfy certain inequalities as well. Different results on the existence of asymptotically periodic solutions were recently proved in [8]. Using an example, it is shown that the results in [8] can be less restrictive. Therefore, an additional open problem arises if the results in [16, 17] can be improved in such a way that the related parameters can be arbitrary and if the expected extension of the results suggested in Open Problem 1 can be given in such a way that the related parameters can be arbitrary as well.

Acknowledgments

The first author has been supported by the Grant P201/10/1032 of the Czech Grant Agency (Prague), by the Council of Czech Government MSM 00216 30519, and by the project FEKT/FSI-S-11-1-1159. The second author has been supported by the Grant VEGA 1/0090/09 of the Grant Agency of Slovak Republic and by the Grant APVV-0700-07 of the Slovak Research and Development Agency.

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