Research Article

Tight Representations of 0-E-Unitary Inverse Semigroups

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We study the tight representation of a semilattice in $\{0,1\}$ by some examples. Then we introduce the concept of the complex tight representation of an inverse semigroup $S$ by the concept of the tight representation of the semilattice of idempotents $E$ of $S$ in $\{0,1\}$. Specifically we describe the tight representation of a $0$-$E$-unitary inverse semigroup and prove that if $\sigma$ is a tight semilattice representation of the $0$-$E$-unitary inverse semigroup $S$ in $\{0,1\}$, then $\sigma$ is a complex tight representation.

1. Introduction

A semigroup is a set equipped with an associative binary operation. A monoid is a semigroup with an identity. A semigroup $S$ is said to be an inverse semigroup, provided there exists, for each $s$ in $S$, a unique element $s^*$ in $S$ such that

$$s = ss^*s, \quad s^* = s^*ss^* \quad (1.1)$$

Good references for inverse semigroups are [1–3].

For a given set $X$, let $I(X)$ be the set of all bijective functions $f : A \to B$, where $A$ and $B$ are subsets of $X$. The multiplication on $I(X)$ is by composition of functions, defined on the largest possible domain. More precisely, for $f, g \in I(X)$, let $fg$ be the function with $\text{dom}(fg) = g^{-1}(\text{ran}(g) \cap \text{dom}(f))$, and $f(g(x)) = f(g(x))$. The involution on $I(X)$ sends a function to its inverse. $I(X)$ is called the inverse semigroup of partial bijections on $X$.

By the Wagner-Preston representation theorem, (see [1, 1.5.1]) every inverse semigroup is an inverse semigroup of partial bijection.
Let $S$ be an inverse semigroup. An idempotent is an element $e \in S$ such that $e^2 = e$. The set of idempotents of $S$ is usually denoted by $E(S)$, or just $E$. A partial bijection is idempotent if and only if it is the identity function on its domain.

The natural partial order $\leq$ on $S$ is defined by

$$s \leq t \text{ iff } s = te \text{ for some idempotent } e.$$ (1.2)

The natural partial order induces a semilattice structure on the set $E(S)$ of idempotents by

$$e \leq f \text{ iff } e = ef.$$ (1.3)

So, one often refers to $E(S)$ as the semilattices of idempotents of $S$. For $f, g$ in $I(X)$, $f \leq g$ if and only if $g$ restricted to $\text{dom}(f)$ is $f$.

Let $B_n = \{(i,j) : 1 \leq i, j \leq n\} \cup \{0\}$. Define a multiplication on $B_n$ by

$$(i,j)(k,l) = \begin{cases} (i,l), & i = j, \\ 0, & \text{otherwise}, \end{cases}$$ (1.4)

and $(i,j)0 = 0(i,j) = 0$. Define the involution on $B_n$ by $(i,j)^* = (j,i)$. The inverse semigroup $B_n$ in called a Brandt semigroup.

## 2. Tight Representations of Semilattices

In this section we define the tight representation of a semilattice $E$ on $\{0,1\}$ and introduce two characteristic functions on $E$ that are tight representations. One can see more about representations and semilattices in [4–7].

**Definition 2.1.** Let $E$ be a partially ordered set. A subset $F \subseteq E$ is said to be connected if, for every $f_1$ and $f_2$ in $F$, there exists an element $f$ in $F$ such that

$$f \leq f_1, \quad f \leq f_2.$$ (2.1)

A component of $E$ is a maximal connected subset of $E$. For a partially ordered set $E$ with the minimum element $0$, we denote by $E_{\text{min}}$ the set of all minimal elements of $E^* = E \setminus \{0\}$.

**Definition 2.2.** Given a partially ordered set $E$ with smallest element $0$, we say that two elements $s$ and $t$ in $E$ are disjoint, in symbols $s \perp t$, if there is no nonzero $u \in E$ such that $u \leq s, t$. Otherwise we say that $s$ and $t$ intersect, in symbols $s \cap t \neq \emptyset$.

For any subset $U$ of $E$, we say that a subset $V \subseteq U$ is a cover for $U$ if, for every nonzero $u \in U$, there exists $v \in V$ such that $u \cap v \neq \emptyset$.

A semilattice is a partially ordered set $E$ such that for every $s, t \in E$, the set $\{u \in E : u \leq s, t\}$ contains a maximum element.

From now on we will fix a semilattice $E$. 
Definition 2.3. For a finite subset \( F \subseteq E \), define \([0, F]\) to be the subset of \( E \) given by
\[
[0, F] = \{ e \in E : e \leq f, \, \forall f \in F \},
\]
and denote by \( F^\perp \) the subset of \( E \) given by
\[
F^\perp = \{ e \in E : e \perp f, \, \forall f \in F \}.
\]

It is obvious that \( 0 \in [0, F] \) and if \( F \) is not contained in a component of \( E^* \), then \([0, F] = \{0\} \).

If \( F \) and \( G \) are finite subsets of \( E \), we denote by \( E^\perp \) the subset \([0, F] \cap G^\perp \) of \( E \).

Notice that if \( F = G = \emptyset \), then \( E^\perp = E \), if \( F = \emptyset \), \( E^\perp = G^\perp \) and if \( G = \emptyset \), \( E^\perp = [0, F] \).

If \( e \leq f \), then \( E^{[e, \{f\}]} = \{0\} \) and \( E^{[e, \{f\}]} = \emptyset \). However \( E^{[f], \{e\}} \) is not necessarily zero. Note that if \( e \) and \( f \) belong to different components of \( E^* \), then \( E^{[f], \{e\}} = (0, e] \). For elements \( e \) and \( f \) in \( E \) such that \( e \leq f \), \( e \) is said to be dense in \( f \) if \( E^{[f], \{e\}} = \{0\} \).

Definition 2.4. A map \( \sigma : E \to \{0, 1\} \) is said to be a representation of \( E \) in \( \{0, 1\} \), if \( \sigma(0) = 0 \) and \( \sigma(x \land y) = \sigma(x) \cdot \sigma(y) \), for all \( x, y \in E \). We say that \( \sigma \) is tight if for all finite subsets \( F, G \subseteq E \), and for all finite cover \( H \) for \( E^\perp \), one has that
\[
\text{sgn} \left( \sum_{h \in H} \sigma(h) \right) = \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)).
\]

Proposition 2.5. Let \( e \) and \( f \) be in \( E \) with \( e \) being dense in \( f \). Then \( \sigma(e) = \sigma(f) \) for every tight representation \( \sigma \) of \( E \) in \( \{0, 1\} \).

Proof. Suppose that \( \sigma \) is a tight representation of \( E \) in \( \{0, 1\} \) and choose \( e, f \) in \( E \) such that \( E^{[f], \{e\}} = \{0\} \). Then \( \emptyset \) is a cover for \( E^{[f], \{e\}} \). So by the definition of tight representation we have \( \sigma(f)(1 - \sigma(e)) = 0 \). Therefore \( \sigma(f) \leq \sigma(e) \). On the other hand, since \( e \leq f \), then \( \sigma(e) \leq \sigma(f) \).

Theorem 2.6. Let \( E \) be a semilattice with minimum element 0. If \( e \in E_{\text{min}} \), then \( E_{[e, \infty]} \) is a tight representation of \( E \) in \( \{0, 1\} \).

Proof. Set \( \sigma = E_{[e, \infty]} \). If \( x, y \in E \) are such that \( x \leq y \), then \( \sigma(x) \leq \sigma(y) \). On the other hand if \( x \) and \( y \) are disjoint, then \( \sigma(x) \) and \( \sigma(y) \) are disjoint too. So \( \sigma(x) \leq 1 - \sigma(y) \). More generally, if \( F \) and \( G \) are finite subsets of \( E \), and \( h \in E \) is such that \( h \leq f \) for every \( f \in F \), and \( h \perp g \), for every \( g \in G \), then
\[
\sigma(h) \leq \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)).
\]

Conversely, let \( F, G \) be finite subsets of \( E \), and let \( H \) be a cover for \( E^\perp \). To prove the inequality
\[
\text{sgn} \left( \sum_{h \in H} \sigma(h) \right) \geq \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)),
\]
we see that if the right-hand side is 0, then the inequality holds obviously. So suppose that the right-hand side is 1. Then we show that the left-hand side is 1 too. Since $\sigma = \chi_{[e,\infty)}$, we have $F \subseteq [e,\infty)$ and $G \cap [0,\infty) = \emptyset$. Also $e \in E^c G$. Then there exists $h \in H$ such that $h \cap e \neq \emptyset$. This means that there exists a nonzero $t \in E$ such that $t \leq h, e$. Since $e \in E_{\min}$, then $e \leq h$ and so $h \in [0,\infty)$ and $\sigma(h) = 1$. Therefore the left-hand side is 1 too.

By the definition of $E_{\min}$, one can show that every element of $E_{\min}$ is the minimum element of some component of $E^\ast$. But it may happen that some component of $E^\ast$ does not have a minimum element. So the following theorem holds.

**Theorem 2.7.** If $F$ is a component of $E^\ast$, then $\chi_F$ is a tight representation of $E$ in $[0,1]$.

### 3. Complex Tight Representations of 0-$E$-Unitary Inverse Semigroups

The class of $E$-unitary inverse semigroups is one of the most important in inverse semigroup theory. When an inverse semigroup contains a zero, then every element of $E$ must be idempotent. Thus motivated by Szendrei [8], we define the class of 0-$E$-unitary inverse semigroups (although she called them $E^\ast$-unitary). The term 0-$E$-unitary appears to be due to Meakin and Sapir [9]. More references for 0-$E$-unitary inverse semigroups are [10–12].

Throughout this section we define complex tight representations of inverse semigroups and prove that every semilattice tight representation on a 0-$E$-unitary inverse semigroup is a complex tight representation.

**Definition 3.1.** An inverse semigroup $S$ with semilattice of idempotent $E$ is $E$-unitary if, for every $e \in E$, $e \leq s$ for some $s \in S$ implies that $s$ is idempotent.

**Proposition 3.2** (see [1]). Let $S$ be an inverse semigroup. For $s, t \in S$, the following are equivalent:

- (i) $s \leq t$,
- (ii) there exists $f \in E$ such that $s = ft$,
- (iii) $s = ts^*s$,
- (iv) $s = ss^*t$,
- (v) $s^* \leq t^*$.

**Proposition 3.3.** Let $S$ be an inverse semigroup and $e$ is an idempotent in $E$. If $s \in S$ such that $s \leq e$, then $s$ is also an idempotent.

**Proof.** If $s \leq e$, then by the previous proposition there exists an idempotent $f \in E$ such that $s = ef$. Since the semilattice of idempotents is closed under multiplication, we have $s \in E$. \square

**Definition 3.4.** An inverse semigroup $S$ is said to be a 0-$E$-unitary if, for every nonzero idempotent $e$, $e \leq s$ for some $s \in S$ implies $s$ is idempotent. The components of $E^\ast$ are in the form $[s,\infty)$ or $(s,\infty)$ for some nonzero element $s \in S$. By Proposition 3.3, if $F$ is any component of $S^* = S \setminus \{0\}$, then $F \subseteq E$ or $F \cap E = \emptyset$.

**Lemma 3.5** (see [4]). If $S$ is a 0-$E$-unitary inverse semigroup and $s, t \in S$ are such that $s^*s = t^*t$ and $se = te$ for some nonzero idempotent $e \leq s^*s$, then $s = t$. 

Proposition 3.6. If $S$ is a 0-$E$-unitary inverse semigroup with zero, then $S$ is a semilattice with respect to natural order.

Proof. Let $s, t \in S$. If there is no nonzero $u \in S$ such that $u \leq s, t$, then $st = 0$. So $0$ is the infimum of $s, t$. Now suppose that there exists a nonzero element $u$ such that $u \leq s, t$. By [1], $u \leq s, t$. Then $u^*u \leq f$. Setting $s_1 = s, t_1 = t$, we have

$$s_1^*s_1 = f s^*s f = f = ft^* tf = t_1^*t_1.$$  \hspace{1cm} (3.1)$$

Since

$$s_1u^*u = sf u^*u = su^*u = u = tu^*u = tf u^*u = t_1u^*u,$$  \hspace{1cm} (3.2)$$

by Lemma 3.5 we have $s_1 = t_1$. So

$$st^*t = ss^*st^*t = sf = s_1 = t_1 = tf = ts^*s.$$  \hspace{1cm} (3.3)$$

Since $0 \neq u_1 \leq s_1, t_1$ we may apply the above argument to $s_1, u_1, t_1$ in order to prove that $s^*tt^* = t^*ss^*$, which implies that $tt^*s = ss^*t$. The fact that $u \leq s, t$ implies that $su^*u = u = tu^*u$. So

$$t^*su^*u = t^*tu^*u = u^*u.$$  \hspace{1cm} (3.4)$$

Since $S$ is 0-$E$-unitary, $t^*s$ is an idempotent. Also we can prove similarly that $ts^*$ is an idempotent. Thus $st^*t = ts^*t = tt^*s$. Therefore

$$st^*t = ts^*t = tt^*s = ss^*t.$$  \hspace{1cm} (3.5)$$

We claim that $st^*t$ is the infimum of $s, t$. It is obvious that $st^*t \leq s, t$. Since

$$u = su^*u = sf u^*u = ss^*st^*tu^*u = st^*tu^*u,$$  \hspace{1cm} (3.6)$$

then $u \leq st^*t$.

Note that if $\sigma$ is a representation of an inverse semigroup $S$ in the complex plane (as a Hilbert space), then $\sigma(e) = 0$ or 1, for every idempotent element $e \in E(S)$. Such representations are called complex representations.

Now we will fix an inverse semigroup $S$ with 0.

Definition 3.7. A complex representation $\sigma$ of $S$ on the complex plane is said to be tight if the restriction of $\sigma$ to $E(S)$ is a tight representation of $E(S)$ in $[0, 1]$. From the definition one can show that if $s_0$ is a minimum element of $S^* = S \setminus \{0\}$, then $\chi_{[s_0, \infty)}$ is a complex tight representation on $S$. Also if $T$ is a component of $S^*$, then $\chi_T$ is a complex tight representation on $S$. 

Since every 0-E-unitary inverse semigroup is a semilattice with zero, a representation of $S$ in $\{0,1\}$ is both a representation of the semilattice $S$ in $\{0,1\}$ and a complex representation of the inverse semigroup $S$.

**Theorem 3.8.** Let $S$ be a 0-E-unitary inverse semigroup and let $\sigma$ be a representation of $S$ in $\{0,1\}$. If $\sigma$ is tight as a semilattice representation, then it is tight as a complex representation.

**Proof.** Suppose that $\sigma$ is a semilattice tight representation of $S$ in $\{0,1\}$. Let $F$ and $G$ be finite subsets of $E$ and $H$ a cover for $E^F, G$. Since $E \subseteq S$, then $E^F, G \subseteq S^F, G$. Since $H$ is a cover of $E^F, G$, then there is a cover $K$ of $S^F, G$ such that $H \subseteq K$. Therefore

$$\sum_{h \in H} \sigma(h) \leq \sum_{k \in K} \sigma(k),$$

and hence

$$\text{sgn} \left( \sum_{k \in K} \sigma(k) \right) \geq \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)).$$

(3.8)

Then $\sigma|_E$ is a tight representation of $E$ in $\{0,1\}$ and therefore $\sigma$ is a complex tight representation of $S$ in $\{0,1\}$.

**References**
