Research Article

Some Identities on Bernstein Polynomials Associated with $q$-Euler Polynomials

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We investigate some interesting properties of the $q$-Euler polynomials. The purpose of this paper is to give some relationships between Bernstein and $q$-Euler polynomials, which are derived by the $p$-adic integral representation of the Bernstein polynomials associated with $q$-Euler polynomials.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the field of $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively (see [1–15]). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The normalized $p$-adic absolute value is defined by $|p|_p = 1/p$. As an indeterminate, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x,$$

(1.1)
(see [7–10]). For \( n \in \mathbb{N} \), we can derive the following integral equation from (1.1):

\[
I_{-1}(f_n) = (-1)^n \int_{\mathbb{R}} f(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l),
\]

where \( f_n(x) = f(x + n) \) (see [7–11]). As well-known definition, the Euler polynomials are given by the generating function as follows:

\[
\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},
\]

(see [3, 5, 7–15]), with usual convention about replacing \( E^n(x) \) by \( E_n(x) \). In the special case \( x = 0 \), \( E_n(0) = E_n \) are called the \( n \)th Euler numbers. From (1.3), we can derive the following recurrence formula for Euler numbers:

\[
E_0 = 1, \quad (E + 1)^n + E_n = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}
\]

(see [12]), with usual convention about replacing \( E^n \) by \( E_n \). By the definitions of Euler numbers and polynomials, we get

\[
E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l,
\]

(see [3, 5, 7–15]). Let \( C[0,1] \) denote the set of continuous functions on \([0,1]\). For \( f \in C[0,1] \), Bernstein introduced the following well-known linear positive operator in the field of real numbers \( \mathbb{R} \):

\[
\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{k,n}(x),
\]

where \( \binom{n}{k} = n(n-1) \cdots (n-k+1)/k! = n!/k!(n-k)! \) (see [1, 2, 7, 11, 12, 14]). Here, \( \mathbb{B}_n(f \mid x) \) is called the Bernstein operator of order \( n \) for \( f \). For \( k, n \in \mathbb{Z}_+ \), the Bernstein polynomials of degree \( n \) are defined by

\[
B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad \text{for } x \in [0,1].
\]

In this paper, we study the properties of \( q \)-Euler numbers and polynomials. From these properties, we investigate some identities on the \( q \)-Euler numbers and polynomials. Finally, we give some relationships between Bernstein and \( q \)-Euler polynomials, which are derived by the \( p \)-adic integral representation of the Bernstein polynomials associated with \( q \)-Euler polynomials.
\section{q-Euler Numbers and Polynomials}

In this section, we assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \). Let \( f(x) = q^x e^{xt} \). From (1.1) and (1.2), we have

\[
\int_{Z_p} f(x) d\mu_{-1}(x) = \frac{2}{qe^t + 1}. \tag{2.1}
\]

Now, we define the q-Euler numbers as follows:

\[
\frac{2}{qe^t + 1} = e^{E(t)} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{2.2}
\]

with the usual convention about replacing \( E_{n,q} \) by \( E_{n,q} \).

By (2.2), we easily get

\[
E_{0,q} = \frac{2}{q + 1}, \quad q(E_q + 1)^n + E_{n,q} = \begin{cases} 
2 & \text{if } n = 0, \\
0 & \text{if } n > 0,
\end{cases} \tag{2.3}
\]

with usual convention about replacing \( E_{n,q} \) by \( E_{n,q} \).

We note that

\[
\frac{2}{qe^t + 1} = \frac{2}{e^t + q^{-1}} \cdot \frac{2}{1 + q} = \frac{2}{1 + q} \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}, \tag{2.4}
\]

where \( H_n(-q^{-1}) \) is the \( n \)th Frobenius-Euler numbers.

From (2.1), (2.2), and (2.4), we have

\[
\int_{Z_p} q^x e^{xt} d\mu_{-1}(x) = E_{n,q} = \frac{2}{1 + q} H_n(-q^{-1}), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.5}
\]

Now, we consider the q-Euler polynomials as follows:

\[
\frac{2}{qe^t + 1} e^{xt} = e^{E_{n}(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \tag{2.6}
\]

with the usual convention \( E_{n,q}(x) \) by \( E_{n,q}(x) \).

From (1.2), (2.1), and (2.6), we get

\[
\int_{Z_p} q^x e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.7}
\]
By comparing the coefficients on the both sides of (2.6) and (2.7), we get the following Witt’s formula for the \(q\)-Euler polynomials as follows:

\[
\int_{z_p} q^n (x + y)^n \, d\mu_\eta(y) = E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}. \tag{2.8}
\]

From (2.6) and (2.8), we can derive the following equation:

\[
\frac{2q}{qe^x + 1} e^{(1-x)t} = \frac{2}{1 + q^{-1} e^{-l}} e^{-xt} = \sum_{n=0}^{\infty} E_{n,q^{-1}}(x)(-1)^n t^n. \tag{2.9}
\]

By (2.6) and (2.9), we obtain the following reflection symmetric property for the \(q\)-Euler polynomials.

**Theorem 2.1.** For \(n \in \mathbb{Z}_+\), one has

\[
(-1)^n E_{n,q^{-1}}(x) = q E_{n,q}(1 - x). \tag{2.10}
\]

From (2.5), (2.6), (2.7), and (2.8), we can derive the following equation: for \(n \in \mathbb{N}\),

\[
E_{n,q}(2) = (E_q + 1 + 1)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}(1)
\]

\[
= E_{0,q} + \frac{1}{q} \sum_{l=1}^{n} \binom{n}{l} q E_{l,q}(1) = \frac{2}{1 + q} - \frac{1}{q} \sum_{l=1}^{n} \binom{n}{l} E_{l,q}
\]

\[
= \frac{2}{1 + q} + \frac{2}{q(1 + q)} - \frac{1}{q} \sum_{l=0}^{n} \binom{n}{l} E_{l,q}
\]

\[
= \frac{2}{q} - \frac{1}{q^2} E_{n,q}(1) = \frac{2}{q} + 1 q E_{n,q},
\tag{2.11}
\]

by using recurrence formula (2.3). Therefore, we obtain the following theorem.

**Theorem 2.2.** For \(n \in \mathbb{N}\), one has

\[
q E_{n,q}(2) = 2 + \frac{1}{q} E_{n,q}. \tag{2.12}
\]
By using (2.5) and (2.8), we get

\[
\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} q^{-x}(x-1)^n d\mu_{-1}(x)
\]

\[
= (-1)^n E_{n,q^{-1}}(-1) = q \int_{\mathbb{Z}_p} (x+2)^n d\mu_{-1}(x) = q \left( \frac{2}{q} + \frac{1}{q^2} E_{n,q} \right)
\]

(2.13)

\[
= 2 + \frac{1}{q} E_{n,q} = 2 + \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x), \quad \text{for } n > 0.
\]

Therefore, we obtain the following theorem.

**Theorem 2.3.** For \( n \in \mathbb{N} \), one has

\[
\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = 2 + \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x). \quad (2.14)
\]

By using Theorem 2.3, we will study for the \( p \)-adic integral representation on \( \mathbb{Z}_p \) of the Bernstein polynomials associated with \( q \)-Euler polynomials in Section 3.

### 3. Bernstein Polynomials Associated with \( q \)-Euler Numbers and Polynomials

Now, we take the \( p \)-adic integral on \( \mathbb{Z}_p \) for the Bernstein polynomials in (1.7) as follows:

\[
\int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} q^x d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \int_{\mathbb{Z}_p} x^{n-j} q^x d\mu_{-1}(x)
\]

(3.1)

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} E_{n-j,q}
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j,q}, \quad \text{where } n, k \in \mathbb{Z}_+.
\]

By the definition of Bernstein polynomials, we see that

\[
B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{where } n, k \in \mathbb{Z}_+. \quad (3.2)
\]
Let \( n, k \in \mathbb{Z}_+ \) with \( n > k \). Then, by (3.2), we get

\[
\int_{\mathbb{Z}_p} q^r B_{k,n}(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} q^r B_{n-k,n}(1-x) d\mu_{-1}(x)
\]

\[
= \binom{n}{n-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \int_{\mathbb{Z}_p} (1-x)^{n-j} q^r d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left( 2 + q \int_{\mathbb{Z}_p} x^{n-j} q^r d\mu_{-1}(x) \right)
\]

\[
= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (2 + qE_{n-j,q}^{-1})
\]

\[
= \begin{cases} 
2 + qE_{n,q}^{-1} & \text{if } k = 0, \\
\binom{n}{k} q \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j,q}^{-1} & \text{if } k > 0.
\end{cases}
\]

(3.3)

Thus, we obtain the following theorem.

**Theorem 3.1.** For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), one has

\[
\int_{\mathbb{Z}_p} q^{1-x} B_{k,n}(x) d\mu_{-1}(x) = \begin{cases} 
2q + E_{n,q} & \text{if } k = 0, \\
\binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j,q}^{-1} & \text{if } k > 0.
\end{cases}
\]

(3.4)

By (3.1) and Theorem 3.1, we get the following corollary.

**Corollary 3.2.** For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), one has

\[
\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{j} E_{k+j,q}^{-1} = \begin{cases} 
2 + \frac{1}{q} E_{n,q} & \text{if } k = 0, \\
\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{q} E_{n-j,q}^{-1} & \text{if } k > 0.
\end{cases}
\]

(3.5)
For \( m, n, k \in \mathbb{Z}_+ \) with \( m + n > 2k \). Then, we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu_{-1}(x) = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} q^{-x} (1 - x)^{n+m-j} d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} q \int_{\mathbb{Z}_p} (x + 2)^{n+m-j} q^x d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} q \left( \frac{2}{q} + \frac{1}{q^2} E_{n+m-j,q} \right)
\]

\[
= \begin{cases} 
2 + \frac{1}{q} E_{n+m,q} & \text{if } k = 0, \\
\left( \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \frac{1}{q} E_{n+m-j,q} \right) & \text{if } k > 0.
\end{cases}
\]

Therefore, we obtain the following theorem.

**Theorem 3.3.** For \( m, n, k \in \mathbb{Z}_+ \) with \( m + n > 2k \), one has

\[
\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu_{-1}(x) = \begin{cases} 
2q + E_{n+m,q} & \text{if } k = 0, \\
\left( \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q} \right) & \text{if } k > 0.
\end{cases}
\]

(3.7)

By using binomial theorem, for \( m, n, k \in \mathbb{Z}_+ \), we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu_{-1}(x) = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^{j} \int_{\mathbb{Z}_p} x^{j+2k} q^{-x} d\mu_{-1}(x)
\]

\[
= q^{\binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^{j} E_{j+2k,q^{-1}}}
\]

(3.9)

By comparing the coefficients on the both sides of (3.8) and Theorem 3.3, we obtain the following corollary.
Corollary 3.4. Let \( m, n, k \in \mathbb{Z}_+ \) with \( m + n > 2k \). Then, we get

\[
\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k, q} = \begin{cases} 
2 + \frac{1}{q} E_{n+m, q} & \text{if } k = 0, \\
\frac{1}{q} \sum_{j=0}^{2k} \binom{2k}{j} (2k) (-1)^{j+2k} E_{n+m-j, q} & \text{if } k > 0.
\end{cases}
\] (3.10)

For \( s \in \mathbb{N} \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + \cdots + n_s > sk \). By induction, we get

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{-x} d\mu_{-1}(x)
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} x^{sk} (1-x)^{n_1+\cdots+n_s-sk} q^{-x} d\mu_{-1}(x)
\]
\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \int_{\mathbb{Z}_p} (1-x)^{n_1+\cdots+n_s-j} q^{-x} d\mu_{-1}(x)
\]
\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} q \int_{\mathbb{Z}_p} (x+2)^{n_1+\cdots+n_s-j} q^{x} d\mu_{-1}(x)
\]
\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \left( 2q + \frac{1}{q^2} E_{n_1+\cdots+n_s-j, q} \right)
\]
\[
= \begin{cases} 
2 + \frac{1}{q} E_{n_1+\cdots+n_s, q} & \text{if } k = 0, \\
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+\cdots+n_s-j, q} & \text{if } k > 0.
\end{cases}
\] (3.11)

Therefore, we obtain the following theorem.

Theorem 3.5. Let \( s \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + \cdots + n_s > sk \), one has

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{-x} d\mu_{-1}(x) = \begin{cases} 
2q + E_{n_1+n_2+\cdots+n_s, q} & \text{if } k = 0, \\
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j, q} & \text{if } k > 0.
\end{cases}
\] (3.12)
For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ by binomial theorem, we get
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{-x} d\mu_{-1}(x) \]
\[
= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+sk} q^{-x} d\mu_{-1}(x) 
\]
\[
= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q^{-1}}. \tag{3.13}
\]

By using (3.13) and Theorem 3.5, we obtain the following corollary.

**Corollary 3.6.** Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, one has
\[
\sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q^{-1}} = \begin{cases} 2 + \frac{1}{q} E_{n_1+n_2+\cdots+n_s,q} & \text{if } k = 0, \\ \frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s+j,q} & \text{if } k > 0. \end{cases} \tag{3.14}
\]

**References**


