Quasimultipliers on $F$-Algebras

Marjan Adib, Abdolhamid Riazi, and Liaqat Ali Khan

1 Department of Mathematics, Payamenoor University-Aligodarz Branch, Aligodarz, Iran
2 Department of Mathematics and Computer Science, Amirkabir University of Technology, P.O. Box 15914, Tehran, Iran
3 Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Liaqat Ali Khan, akliaqat@yahoo.com

Received 7 June 2010; Revised 24 October 2010; Accepted 25 January 2011

Academic Editor: Wolfgang Ruess

Copyright © 2011 Marjan Adib et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the extent to which the study of quasimultipliers can be made beyond Banach algebras. We will focus mainly on the class of $F$-algebras, in particular on complete $k$-normed algebras, $0 < k < 1$, not necessarily locally convex. We include a few counterexamples to demonstrate that some of our results do not carry over to general $F$-algebras. The bilinearity and joint continuity of quasimultipliers on an $F$-algebra $A$ are obtained under the assumption of strong factorability. Further, we establish several properties of the strict and quasistrict topologies on the algebra $QM(A)$ of quasimultipliers of a complete $k$-normed algebra $A$ having a minimal ultra-approximate identity.

1. Introduction

A quasimultiplier is a generalization of the notion of a left (right, double) multiplier and was first introduced by Akemann and Pedersen in [1, Section 4]. The first systematic account of the general theory of quasimultipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon [2] in 1977. Further developments have been made, among others, by Vasudevan and Goel [3], Kassem and Rowlands [4], Lin [5, 6], Dearden [7], Argün and Rowlands [8], Grosser [9], Yilmaz and Rowlands [10], and Kaneda [11, 12].

In this paper, we consider the notion of quasimultipliers on certain topological algebras and give an account, how far one can get beyond Banach algebras, using combination of standard methods. In particular, we are able to establish some results of the above authors in the framework of $F$-algebras or complete $k$-normed algebras.
2. Preliminaries

Definition 2.1. Let $E$ be a vector space over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).

1. A function $q : E \to \mathbb{R}$ is called an $F$-seminorm on $E$ if it satisfies

   $(F_1)$ $q(x) \geq 0$ for all $x \in E$,
   $(F_2)$ $q(x) = 0$ if $x = 0$,
   $(F_3)$ $q(\alpha x) \leq q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$,
   $(F_4)$ $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$,
   $(F_5)$ if $\alpha_n \to 0$ in $\mathbb{K}$, then $q(\alpha_n x) \to 0$ for all $x \in E$.

2. An $F$-seminorm $q$ on $E$ is called an $F$-norm, if for any $x \in E$, $q(x) = 0$ implies that $x = 0$.

3. An $F$-seminorm (or $F$-norm) $q$ on $E$ is called $k$-homogeneous ([13, page 160]; [14, pages 90, 95]), where $0 < k \leq 1$, if it also satisfies

   $(F'_3)$ $q(\alpha x) = |\alpha|^k q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$.

4. A $k$-homogeneous $F$-seminorm (resp., $F$-norm) on $E$ is called, in short, a $k$-seminorm (resp., $k$-norm).

Definition 2.2. (1) A vector space with an $F$-norm $q$ is called an $F$-normed space and is denoted by $(E, q)$; if it is also complete, it is called an $F$-space. Clearly, any $F$-normed space $(E, q)$ is a metrizable TVS with metric given by $d(x, y) = q(x - y)$, $x, y \in E$.

2. An $F$-seminorm (or $F$-norm) $q$ on an algebra $A$ is called submultiplicative if

   $$q(xy) \leq q(x)q(y), \quad \forall x, y \in A.$$  \hfill (2.1)

An algebra with a submultiplicative $F$-norm $q$ is called an $F$-normed algebra; if it is also complete, it is called an $F$-algebra. An algebra with a submultiplicative $k$-norm $q$ is called an $k$-normed algebra. A complete $k$-normed algebra is also called a $k$-Banach algebra in the literature.

Theorem 2.3. (a) If $(E, \tau)$ is TVS, then its topology $\tau$ can be defined by a family of $F$-seminorms (see [15, pages 48–51]; [16, pages 2–3]).

(b) If $(E, \tau)$ is a metrizable TVS, then $\tau$ may be defined by a single $F$-norm (see [13, 15, 17]).

(c) If $(E, \tau)$ is a Hausdorff locally bounded TVS, then $\tau$ may be a single $k$-norm for some $k$, $0 < k \leq 1$ (see [13, 14]).

Note that if $(E, q)$ is an $F$-normed space, then, for any $\varepsilon > 0$, the set $\{x \in E : q(x) \leq \varepsilon\}$ is a neighbourhood of 0 in $E$, but it need not be a bounded set. In case, if $\{x \in E : q(x) \leq \varepsilon\}$ is bounded for some $\varepsilon > 0$, then $(E, q)$ becomes a Hausdorff locally bounded TVS and hence, by Theorem 2.3(c), a $k$-normed space for some $k$, $0 < k \leq 1$.

Definition 2.4. Let $A$ be an algebra over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\tau$ a topology on $A$ such that $(A, \tau)$ is a TVS. Then the pair $(A, \tau)$ is called a topological algebra if it has a separately continuous multiplication. A topological algebra $A$ is said to be locally bounded if it has a bounded...
neighbourhood of 0 (see [18, page 39]). If \((A, \tau)\) is a complete Hausdorff locally bounded topological algebra, then its topology can be defined by a submultiplicative \(k\)-norm \(q\), \(0 < k \leq 1\) [18, page 41].

By a famous result of Arens (see [18, page 24]), every Baire metrizable topological algebra has jointly continuous multiplication; in particular, every \(F\)-algebra has jointly continuous multiplication.

For the general theory and undefined terms, the reader is referred to [13, 15–17, 19] for topological vector spaces, to [13, 14, 20] for \(F\)-normed and \(k\)-normed spaces, and to [18, 21, 22] for various classes of topological algebras.

If \(E\) and \(F\) are topological vector spaces over the field \(\mathbb{K}\) (= \(\mathbb{R}\) or \(\mathbb{C}\)), then the set of all continuous linear mappings \(T : E \to F\) is denoted by \(\text{CL}(E,F)\). Clearly, \(\text{CL}(E,F)\) is a vector space over \(\mathbb{K}\) with the usual pointwise operations. Further, if \(F = E\), \(\text{CL}(E) = \text{CL}(E,E)\) is an algebra under composition (i.e., \((ST)(x) = S(T(x)), x \in E\)) and has the identity \(I : E \to E\) given by \(I(x) = x (x \in E)\).

We now state the following three versions of the uniform boundedness principle for reference purpose.

**Theorem 2.5** (see [23, page 142, principle 33.1]). Let \(X\) be a complete metric space and \(\mathcal{H} = \{f_\alpha : \alpha \in I\}\) a family of continuous real-valued functions on \(X\). If \(\mathcal{H}\) is pointwise bounded from above, then on a certain closed ball \(B \subseteq X\) it is uniformly bounded above, that is, there exists a constant \(C > 0\) such that

\[
f_\alpha(x) \leq C, \quad \forall \alpha \in I, \ x \in B. \tag{2.2}
\]

**Theorem 2.6** (see [14, page 39]; [19, page 465]). Let \(E\) be an \(F\)-space and \(F\) any topological vector space. Let \(\mathcal{H} \subseteq \text{CL}(E,F)\) be a collection such that \(\mathcal{H}\) is pointwise bounded on \(E\). Then \(\mathcal{H}\) is equicontinuous; hence, for any bounded set \(D\) in \(E\), \(\cup \{T(D) : T \in \mathcal{H}\}\) is a bounded set in \(F\).

The following version is for bilinear mappings.

**Theorem 2.7.** Let \(E\) and \(F\) be \(F\)-spaces and \(G\) any TVS.

(a) A collection \(\mathcal{H}\) of bilinear mappings from \(E \times F\) into \(G\) is equicontinuous if and only if each \(f \in \mathcal{H}\) is separately continuous and \(\mathcal{H}\) is pointwise bounded on \(E \times F\). In particular, every separately continuous bilinear map \(f : E \times F \to G\) is jointly continuous (see [13, page 172]; [19, page 489]).

(b) Let \(f_n : E \times F \to G\) be a sequence of separately continuous bilinear mappings such that \(\lim_{n \to \infty} f_n(x, y) = f(x, y)\) exists for each \((x, y) \in E \times F\). Then \(\{f_n\}\) is equicontinuous and \(f\) is bilinear and jointly continuous (see [19, page 490]; [24, page 328]).

**Definition 2.8.** (1) A net \(\{e_\lambda : \lambda \in I\}\) in a topological algebra \(A\) is called an approximate identity if

\[
\lim_{\lambda} e_\lambda a = a = \lim_{\lambda} ae_\lambda, \quad \forall a \in A. \tag{2.3}
\]

(2) An approximate identity \(\{e_\lambda : \lambda \in I\}\) in an \(F\)-normed algebra \((A, q)\) is said to be minimal if \(q(e_\lambda) \leq 1\) for all \(\lambda \in I\).
(3) An algebra $A$ is said to be left (resp., right) faithful if, for any $a \in A$, $aA = \{0\}$ (resp., $Aa = \{0\}$) implies that $a = 0$; $A$ is called faithful if it is both left and right faithful. One mentions that $A$ is faithful in each of the following cases:

(i) $A$ is a topological algebra with an approximate identity (e.g., $A$ is a locally $C^*$-algebra);

(ii) $A$ is a topological algebra with an orthogonal basis [25].

**Definition 2.9.** A topological algebra $A$ is called

1. factorable if, for each $a \in A$, there exist $b, c \in A$ such that $a = bc$,

2. strongly factorable if, for any sequence $\{a_n\}$ in $A$ with $a_n \to 0$, there exist $a \in A$ and a sequence $\{b_n\}$ (resp., $\{c_n\}$) in $A$ with $b_n \to 0$ (resp., $c_n \to 0$) such that $a_n = ab_n$ (resp., $a_n = c_n a$) for all $n \geq 1$.

Clearly, every strongly factorable algebra is factorable. Factorization in Banach and topological algebras plays an important role in the study of multipliers and quasimultipliers. There are several versions of the famous Hewitt-Cohen’s factorization theorem in the literature (see, e.g., the book [26] and its references). Using the terminology of [27], we state the following version in the nonlocally convex case.

**Theorem 2.10** (see [27]). Let $A$ be a fundamental $F$-algebra with a uniformly bounded left approximate identity. Then $A$ is strongly factorable.

**Definition 2.11.** Let $(A, q)$ be an $F$-normed space (in particular, an $F$-normed algebra). For any $T \in \text{CL}(A)$, let

$$
\|T\|_q = \sup \left\{ \frac{q(T(x))}{q(x)} : x \in A, x \neq 0 \right\}.
$$

(\ast)

It is easy to see that if $q$ is a $k$-norm, $0 < k \leq 1$, (resp., a seminorm) on $A$, then $\| \cdot \|_q$ is a $k$-norm (resp., a seminorm) on $\text{CL}(A)$; further, in these cases, we have alternate formulas for $\|T\|_q$ as

$$
\|T\|_q = \sup \{ q(T(x)) : x \in A, q(x) = 1 \}
= \sup \{ q(T(x)) : x \in A, q(x) \leq 1 \},
$$

(2.4)

for each $T \in \text{CL}(A)$ (see [14, pages 101-102]; [28, pages 3-5]; [19, page 87]).

**Remark 2.12.** In an earlier version of this paper, the authors had erroneously made the blank assumption that, for $q$ an $F$-norm on $A$, $\|T\|_q$ given by (\ast) always exists for each $T \in \text{CL}(A)$. We are grateful to referee for pointing out that this assumption cannot be justified in view of the following counterexamples.

1. First, $\|T\|_q$ need not be finite for a general $F$-norm. For example, let $A = \mathbb{R}^2$, $q(x_1, x_2) = |x_1| + |x_2|^{1/2}, T(x_1, x_2) = (x_2, x_1)$. Then $q$ is an $F$-norm on $A$, but $\|T\|_q = \infty$: for any $n \in \mathbb{N}$,

$$
\|T\|_q \geq \frac{q[T(n, n^2)]}{q(n, n^2)} = \frac{q(n^2, n)}{q(n, n^2)} = \frac{n^2 + n^{1/2}}{2n} \to \infty.
$$

(2.5)
(2) Even when considering the subspace of those $T$ for which $\|T\|_q < \infty$, then $\| \cdot \|_q$ need not always be an $F$-norm, since $(F_3)$ need not hold. For example, for a fixed sequence $(p_n)$ with $0 < p_n \leq 1$, $p_n \to 0$, consider the $F$-algebra $A$ of sequences $(x_n) \subseteq \mathbb{R}$ with $|x_n|^{p_n} \to 0$ and $q((x_n)) = \sup_{n \geq 1} |x_n|^{p_n}$. Then $\|T\|_q < \infty$ for all multipliers of $A$, but $\| \cdot \|_q$ is not an $F$-norm, that is, it makes the space $\text{CL}(A)$ into an additive topological group but not into a topological vector space (as it would lack the continuity of scalar multiplication in the absence of $(F_3)$ (cf. [14, Example 1.2.3, page 8]).

In view of the above remark, we will need to assume that $(A, q)$ is a $k$-normed space (or a $k$-normed algebra) whenever $\|T\|_q$ is considered for $T \in \text{CL}(A)$ or $\text{CL}(A \times A, A)$.

Some useful properties of $(\text{CL}(A), \| \cdot \|_q)$ are summarized as follows.

**Theorem 2.13** (see [14, pages 101-102]). Let $(A, q)$ be a $k$-normed space (in particular, a $k$-normed algebra) with $0 < k \leq 1$. Then:

(a) a linear mapping $T : A \to A$ is continuous $\iff$ $\|T\|_q < \infty$;

(b) $\| \cdot \|_q$ is a $k$-norm on $\text{CL}(A)$;

(c) $q(T(x)) \leq \|T\|_q \cdot q(x)$ for all $x \in A$;

(d) for any $S, T \in \text{CL}(A)$, $\|ST\|_q \leq \|S\|_q \|T\|_q$; hence $(\text{CL}(A), \| \cdot \|_q)$ is a $k$-normed algebra;

(e) if $A$ is complete, then $(\text{CL}(A), \| \cdot \|_q)$ is a complete $k$-normed algebra.

**Remark 2.14.** The referee has enquired if the present theory can be considered for the class of locally convex $F$-algebras. It is well known (e.g., [22, page 33]; [21, page 9]) that, for this class of topological algebras, the topology can be generated by an increasing countable family of seminorms $(q_n)$, which need not be submultiplicative but satisfy the weaker condition $q_n(xy) \leq q_{n+1}(x)q_{n+1}(y)$; however, for locally $m$-convex $F$-algebras, the seminorms can be chosen to be submultiplicative. In view of this, we believe that a study of quasimultipliers can possibly be made on locally $m$-convex algebra $F$-algebras parallel to the one given by Phillips (see [29, pages 177–180]) for multipliers.

### 3. Multipliers on $F$-Normed Algebras

In this section, we recall definitions and results on various notions of multipliers on an algebra $A$ (as given in [30–33]) which we shall require later in the study of quasimultipliers (see also [25, 29, 34–40]). In fact, we shall see that the proofs of most of the results on quasimultipliers are based on the properties of left, right, and double multipliers.

**Definition 3.1** (see [31]). Let $A$ be an algebra over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

1. A mapping $T : A \to A$ is called a

   - (i) *multiplier* on $A$ if $aT(b) = T(ab)$ for all $a, b \in A$,
   - (ii) *left multiplier* on $A$ if $T(ab) = T(a)b$ for all $a, b \in A$,
   - (iii) *right multiplier* on $A$ if $T(ab) = aT(b)$ for all $a, b \in A$.

2. A pair $(S, T)$ of mappings $S, T : A \to A$ is called a *double multiplier* on $A$ if $aS(b) = T(ab) = T(a)b$ for all $a, b \in A$. 
Some authors use the term *centralizer* instead of *multiplier* (see, e.g., [30, 31]).

Let $M(A)$ (resp., $M_\ell(A), M_r(A)$) denote the set of all multipliers (resp., left multipliers, right multipliers) on $A$ and $M_d(A)$ the set of all double multipliers on an algebra $A$. For any $a \in A$, let $L_a, R_a : A \to A$ be given by

$$L_a(x) = ax, \quad R_a(x) = xa, \quad x \in A. \quad (3.1)$$

Clearly, $L_a \in M_\ell(A), R_a \in M_r(A)$, and $(L_a, R_a) \in M_d(A)$.

For convenience, we summarize some basic properties of multipliers in the following theorems for later references.

**Theorem 3.2** (see [31]). Let $A$ be an algebra. Then,

(a) $M_\ell(A) \cap M_r(A) \subseteq M(A)$;

(b) if $A$ is faithful, then $M(A) \subseteq M_\ell(A) \cap M_r(A)$ and hence $M(A) = M_\ell(A) \cap M_r(A)$;

(c) if $A$ is commutative and faithful, then $M_\ell(A) = M_r(A) = M(A)$;

(d) $M_\ell(A)$ and $M_r(A)$ are algebras with composition as multiplication (i.e., $(T_1 T_2)(x) = T_1(T_2(x)))$ and have the identity $I : A \to A$, $I(x) = x$ ($x \in A$);

(e) $M(A)$ is a vector space; if, in addition, $A$ is faithful, then $M(A)$ is a commutative algebra (without $A$ being commutative) with identity $I$.

**Theorem 3.3** (see [31]). Let $A$ be a faithful algebra. Then,

(a) if $(S, T) \in M_d(A)$, then (i) $S$ and $T$ are linear and (ii) $S \in M_\ell(A)$ and $T \in M_r(A)$. In particular, every $T \in M(A)$ is linear;

(b) $M_d(A)$ is an algebra with identity $(I, I)$ under the operations

$$\begin{align*}
(S_1, T_1) + (S_2, T_2) &= (S_1 + S_2, T_1 + T_2), \\
\lambda(S_1, T_1) &= (\lambda S_1, \lambda T_1) \quad (\lambda \in \mathbb{K}), \\
(S_1, T_1)(S_2, T_2) &= (S_1 S_2, T_1 T_2) ;
\end{align*} \quad (3.2)$$

(c) let $(S_1, T_1), (S_2, T_2) \in M_d(A)$. If $S_1 = S_2$, then $T_1 = T_2$; if $T_1 = T_2$ then $S_1 = S_2$;

(d) if $A$ is commutative, then $M_\ell(A)$ is commutative and $M_d(A) = M(A)$; in fact, if $(S, T) \in M_d(A)$, then $S = T$.

**Definition 3.4.** One defines mappings $\mu_\ell : A \to M_\ell(A), \mu_r : A \to M_r(A)$, and $\mu_d : A \to M_d(A)$ by

$$\mu_\ell(a) = L_a, \quad \mu_r(a) = R_a, \quad \mu_d(a) = (L_a, R_a), \quad a \in A. \quad (3.3)$$
Theorem 3.5 (see [31]). Let $A$ be an algebra, and let $\mu_\ell$, $\mu_r$, and $\mu_d$ be the mappings as defined above. Then,

(a) $\mu_\ell$, $\mu_r$, and $\mu_d$ are linear;

(b) $\mu_\ell$ and $\mu_d$ are algebra homomorphisms, while $\mu_r$ is an algebra antihomomorphism;

(c) $\mu_\ell$ (resp., $\mu_r$) is 1-1 $\iff$ $A$ is left (resp., right) faithful; $\mu_d$ is 1-1 $\iff$ $A$ is faithful;

(d) $\mu_\ell$ (resp., $\mu_r$) is onto $\iff$ $A$ has left (resp., right) identity; $\mu_d$ is onto $\iff$ $A$ has an identity.

Theorem 3.6 (see [31]). Let $A$ be an algebra.

(a) For any $a \in A$ and $T \in M_\ell(A)$, $TL_a = L_{T(a)} \in \mu_\ell(A)$; hence $\mu_\ell(A)$ is a left ideal in $M_\ell(A)$.

(b) For any $a \in A$ and $T \in M_r(A)$, $TR_a = R_{T(a)} \in \mu_r(A)$; hence $\mu_r(A)$ is a left ideal in $M_r(A)$.

(c) Suppose that $A$ is faithful. Then, for any $a \in A$ and $(S,T) \in M_d(A)$,

$$(L_a, R_a)(S,T) = (L_{T(a)}, R_{T(a)}) \in \mu_d(A), \quad (S,T)(L_a, R_a) = (L_{S(a)}, R_{S(a)}) \in \mu_d(A);$$

hence $\mu_d(A)$ is a two-sided ideal in $M_d(A)$.

Regarding the continuity of multipliers, we state the following.

Theorem 3.7. (a) Suppose that $A$ is a strongly factorable $F$-normed algebra. If $T \in M_\ell(A)$ (resp., $M_r(A)$), then $T$ is linear and continuous (see [32, 33]).

(b) Suppose that $A$ is a faithful F-algebra. If $(S,T) \in M_d(A)$, then $S$ and $T$ are linear and continuous; in particular each $T \in M(A)$ is linear and continuous (see [31, 33]).

Convention 1. In the remaining part of this paper, unless stated otherwise, $A$ is a topological algebra and $M(A)$ (resp., $M_\ell(A)$, $M_r(A)$) denotes the set of all continuous linear multipliers (resp., left multipliers, right multipliers) on $A$ and $M_d(A)$ denotes the set of all double multipliers $(S,T)$ on $A$ with both $S$ and $T$ continuous and linear.

Definition 3.8 (see [31, 33]). Let $A$ be a topological algebra. The uniform operator topology $u$ (resp., the strong operator topology $s$) on $M_d(A)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N(D,W) = \{(S,T) \in M_d(A) : S(D) \subseteq W, T(D) \subseteq W\},$$

where $D$ is a bounded (resp., finite) subset of $A$ and $W$ is a neighborhood of 0 in $A$. Clearly, $s \subseteq u$. Note that the $u$ and $s$ topologies can also be defined on the multiplier algebras $M(A)$, $M_\ell(A)$ and $M_r(A)$ in an analogous way. (The topology $s$ is also called the strict topology in the literature and denoted by $\beta$.) There is an extensive literature on the $s$ and $u$ topologies (see, e.g., [30, 33–35, 37–39, 41–45]).
Theorem 3.9 (see [33]). Let $A$ be a faithful $F$-algebra, and let $M_t(A)$ denote any one of the algebras $M_{\ell}(A), M_r(A), M(A),$ and $M_d(A).$ Then,

(a) $(M_t(A), u)$ and $(M_t(A), s)$ are topological algebras with separately continuous multiplication;

(b) $(M_t(A), u)$ and $(M_t(A), s)$ are complete;

(c) $s$ and $u$ have the same bounded sets;

(d) if $(M_t(A), s)$ is metrizable, then $s = u$ on $M_t(A)$;

(e) if $A$ has a two-sided approximate identity, then $A$ is $s$-dense in $M_t(A).

Remark 3.10. Let $(A, q)$ be an $F$-normed algebra.

(1) If $(A, q)$ is a $k$-normed algebra, the $u$-topology on $M_{\ell}(A), M_r(A),$ and $M(A)$ is given by the $k$-norm

$$
||T||_q = \sup \left\{ \frac{q(T(x))}{q(x)} : x \in A, x \neq 0 \right\}; \quad (3.6)
$$

the $u$-topology on $M_d(A)$ is given by the $k$-norm

$$
||S,T||_q = \max \left\{ ||S||_q, ||T||_q \right\}, \quad (S,T) \in M_d(A). \quad (3.7)
$$

(2) The $s$-topology on $M_{\ell}(A), M_r(A),$ and $M(A)$ is given by the family of $\{p_a : a \in A\}$ of $F$-seminorms, where

$$
p_a(T) = q(T(a)), \quad T \in M_{\ell}(A), M_r(A) \text{ or } M(A); \quad (3.8)
$$

the $s$-topology on $M_d(A)$ is given by the family $\{r_a : a \in A\}$ of $F$-seminorms, where

$$
r_a(S,T) = \max \{q(S(a)), q(T(a))\}, \quad (S,T) \in M_d(A). \quad (3.9)
$$

Theorem 3.11. Let $(A, q)$ be a $k$-normed algebra having a minimal approximate identity $\{e_\lambda : \lambda \in I\}$. Then,

(a) for any $a \in A$, $||L_a||_q = ||R_a||_q = q(a);$ so each of the maps $\mu_{\ell} : A \to (M_{\ell}(A), u)$, $\mu_r : A \to (M_r(A), u), \mu_d : A \to (M_d(A), u)$ is an isometry and hence continuous;

(b) for any $(S,T) \in M_d(A)$, $||S||_q = ||T||_q$;

(c) if $(A, q)$ is complete, then $A$ is a $u$-closed two-sided ideal in $M_d(A)$, under the identification $\mu_d : a \to (L_a, R_a)$.

Proof. (a) Let $a \in A.$ Then

$$
||L_a||_q = \sup_{b \neq 0} \frac{q(L_a(b))}{q(b)} = \sup_{b \neq 0} \frac{q(ab)}{q(b)} \leq \sup_{b \neq 0} \frac{q(a)}{q(b)} = q(a). \quad (3.10)
$$
On the other hand,

\[ \|L_a\|_q = \sup_{b \neq 0} \frac{q(ab)}{q(b)} = \frac{q(ae_1)}{q(e_1)} \geq q(ae_1), \quad \forall a \in I; \]  

so

\[ \|L_a\|_q \geq \lim_{a \to 1} q(ae_1) = q\left(\lim_{a \to 1} ae_1\right) = q(a). \]  

Hence \( \|\mu_e(a)\|_q = \|L_a\|_q = q(a) \). Similarly, \( \|\mu_r(a)\|_q = \|R_a\|_q = q(a) \). Thus

\[ \|\mu_d(a)\|_q = \max\left\{\|L_a\|_q, \|R_a\|_q\right\} = q(a). \]  

(b) Let \((S, T) \in M_d(A)\). Using (a), we have

\[
\|S\|_q = \sup_{a \neq 0} \frac{q(S(a))}{q(a)} = \sup_{a \neq 0} \frac{\|R_{S(a)}\|_q}{q(a)} = \sup_{a \neq 0} \frac{1}{q(a)} \cdot \frac{q[R_{S(a)}(b)]}{q(b)}
\]

\[= \sup_{a \neq 0} \sup_{b \neq 0} \frac{q[bS(a)]}{q(a) \cdot q(b)} = \sup_{a \neq 0} \frac{q[T(a)]}{q(b)} \]

\[\leq \sup_{a \neq 0} \sup_{b \neq 0} \frac{q(T(b)) \cdot q(a)}{q(a) \cdot q(b)} = \sup_{b \neq 0} \frac{q(T(b))}{q(b)} = \|T\|_q.\]

Similarly,

\[
\|T\|_q = \sup_{a \neq 0} \frac{\|L_T(a)\|_q}{q(a)} = \sup_{a \neq 0} \frac{q[L_T(a)]}{q(a) \cdot q(b)}
\]

\[= \sup_{a \neq 0} \sup_{b \neq 0} \frac{q[a \cdot S(b)]}{q(a) \cdot q(b)} \leq \|S\|_q.\]

Thus, \( \|S\|_q = \|T\|_q \).

(c) In view of Theorem 3.6(c), we only need to show that \(\mu_d(A)\) is \(u\)-closed in \(M_d(A)\). Let \((S, T) \in M_d(A)\) with \((S, T) \in u\text{-}\text{cl}(\mu_d(A))\). Choose \(\{a_\alpha : \alpha \in J\} \subseteq A\) such that \(\{L_{a_\alpha}, R_{a_\alpha}\} \overset{\text{un}}{\to} (S, T)\). By part (a), \(\mu_d\) is an isometry. Hence \(\{a_\alpha : \alpha \in J\}\) is a Cauchy net in \(A\). Then

\[ (S, T) = u - \lim_{\alpha} \mu_d(a_\alpha) \in \mu_d(A). \]  

Thus \(\mu_d(A)\) is \(u\)-closed in \(M_d(A)\). \(\square\)
4. Quasimultipliers on \( F \)-Algebras and \( k \)-Normed Algebras

In this section, we consider the notion of quasimultipliers on \( F \)-algebras and complete \( k \)-normed algebra and extend several basic results of McKennon [2], Kassem and Rowlands [4], Argün and Rowlands [8], and Yılmaz and Rowlands [10] from Banach algebras to these classes of topological algebras.

**Definition 4.1** (see [2, 4]). Let \( A \) be an algebra. A mapping \( m : A \times A \to A \) is said to be a quasimultiplier on \( A \) if

\[
m(ab, c) = am(b, c), \quad m(a, bc) = m(a, b)c, \tag{4.1}
\]

for all \( a, b, c \in A \).

The following Lemma shows in particular that every left multiplier, right multiplier, multiplier, and double multiplier on an algebra \( A \) can be viewed as quasimultiplier on \( A \).

**Lemma 4.2.** Let \( A \) be a faithful algebra.

(a) For any \( c \in A \), define \( m = m_c : A \times A \to A \) by

\[
m_c(a, b) = acb, \quad \forall (a, b) \in A \times A. \tag{4.2}
\]

(b) For any \( T \in M_\ell(A) \), define an associated map \( m = m_T : A \times A \to A \) by

\[
m_T(a, b) = aT(b), \quad \forall (a, b) \in A \times A. \tag{4.3}
\]

(c) For any \( T \in M_r(A) \), define an associated map \( m = m_T : A \times A \to A \) by

\[
m_T(a, b) = T(a)b, \quad \forall (a, b) \in A \times A. \tag{4.4}
\]

(d) For any \( T \in M(A) \), define an associated map \( m = m_T : A \times A \to A \) by

\[
m_T(a, b) = aT(b), \quad \forall (a, b) \in A \times A. \tag{4.5}
\]

(e) For any \( (S, T) \in M_d(A) \), define an associated map \( m = m_{(S, T)} : A \times A \to A \) by

\[
m_{(S, T)}(a, b) = aS(b), \quad \forall (a, b) \in A \times A. \tag{4.6}
\]

Then each of the maps \( m : A \times A \to A \) defined above is a quasimultiplier on \( A \).
Abstract and Applied Analysis

Proof. We only prove (e). Let \((S,T) \in M_d(A)\), for any \(a,b,c,d \in A\),

\[
m(ab,c) = m_{(ST)}(ab,c) = (ab)S(c) = a[bS(c)] = a[m_{(ST)}(b,c)] = am(b,c).
\]

(4.7)

In a similar way, \(m(a,bc) = m(a,b)c\).

\(\square\)

Theorem 4.3. Suppose that \((A, \tau)\) is a strongly factorable \(F\)-algebra. Then,

(a) A map \(m : A \times A \to A\) is a quasimultiplier on \(A\) if and only if

\[
m(ab,cd) = am(b,c)d, \quad \forall a,b,c,d \in A;
\]

(4.8)

(b) every quasimultiplier \(m\) on \(A\) is bilinear;

(c) every quasimultiplier \(m\) on \(A\) is jointly continuous.

Proof. (a) If \(m\) is a quasimultiplier on \(A\), then clearly, for any \(a,b,c,d \in A\),

\[
m(ab,cd) = am(a,cd) = am(b,c)d.
\]

(4.9)

Conversely, let \(a,b,c \in A\). Since \(A\) is a strongly factorable and \(a_n = \{b,c,0,0,\ldots\} \to 0\), there exist \(y,z,w \in A\) such that \(b = wy, c = wz\). Then, using (4.8),

\[
m(ab,c) = m(awy,wz) = (aw)[m(y,w)]z = am(aw,ywz) = am(b,c).
\]

(4.10)

Similarly, we obtain \(m(a,bc) = m(a,b)c\).

(b) Let \(a,b,c \in A\) and \(a \in K\). Choose, as above, \(x,y,w \in A\) such that \(a = wx, b = wy\). Then,

\[
m(a+b,c) = m(wx+w,y,c) = (x+y)m(w,c) = xm(w,c) + ym(w,c) = m(w,c) + m(wy,c) = m(a,c) + m(b,c)
\]

(4.11)

Similarly, \(m(c,a+b) = m(c,a) + m(c,b)\). Next,

\[
m(aa,c) = m(awx,c) = (aw)m(x,bc) = am(wx,c) = am(a,c).
\]

(4.12)

First, we show that \(m\) is separately continuous. Let \(a \in A\) and \(\{x_n\} \subseteq A\) with limit \(x\). Then \(\{x_n - x\}\) converges to 0. By strong factorability, there exist a sequence \(\{z_n\}\) and an element \(z\) of \(A\) such that \(z_n \to 0, x_n - x = zz_n\). Thus,

\[
m(a,x_n) - m(a,x) = m(a,x_n - x) = m(a,zz_n) = m(a,z)z_n \to 0.
\]

(4.13)

Now, the joint continuity of \(m\) follows directly from Theorem 2.7(a).
Theorem 4.4. Let $A$ be a commutative algebra. Then,

(a) $am(b, c) = m(b, a)c$ for any quasimultiplier on $A$ and $a, b, c \in A$;

(b) if $A$ is also faithful, then a bilinear map $m : A \times A \to A$ is a quasimultiplier on $A$ if and only if

$$m(a^2, b) = am(a, b), \quad m(a, b^2) = m(a, b)b \quad \forall a, b \in A.$$ (4.14)

Proof. (a) By hypothesis,

$$am(b, c) = m(b, c)a = m(b, ca) = m(b, ac) = m(b, a)c.$$ (4.15)

(b) $\Rightarrow$ This is obvious.

$\Leftarrow$ For all $a, b, c \in A$, using (4.14),

$$m((a + b)^2, c) = (a + b)m(a + b, c)$$

$$= am(a, c) + am(b, c) + bm(a, c) + bm(b, c).$$ (4.16)

On the other hand, we have

$$m((a + b)^2, c) = m(a^2 + b^2 + 2ab, c)$$

$$= m(a^2, c) + m(b^2, c) + 2m(ab, c)$$ (4.17)

$$= am(a, c) + bm(b, c) + 2m(ab, c).$$

Comparing (4.16) and (4.17), we obtain

$$2m(ab, c) = am(b, c) + bm(a, c).$$ (4.18)

Now, for all $a, b, c, d \in A$, using (4.18) twice,

$$2m(abd, c) = abm(d, c) + dm(ab, c)$$

$$= abm(d, c) + \frac{1}{2}d[2m(ab, c)]$$ (4.19)

$$= abm(d, c) + \frac{1}{2}dam(b, c) + \frac{1}{2}dbm(a, c).$$
Definition 4.6. Let \( a \in A \), and using (4.18) twice as above,

\[
2m(abd, c) = 2m(abd, c) = adm(b, c) + bm(ab, c)
= adm(b, c) + \frac{1}{2} [bam(d, c) + bdm(a, c)] 
= adm(b, c) + \frac{1}{2} abm(d, c) + \frac{1}{2} dbm(a, c).
\]

Comparing (4.19) and (4.20), \( abm(d, c) = adm(b, c) \). Since this holds for all \( a \in A \) and \( A \) is faithful, \( bm(d, c) = dm(b, c) \). Hence, for all \( d \in A \),

\[
dm(ab, c) = abm(d, c) = a [bm(d, c)] = adm(b, c) = dam(b, c).
\]  

(4.21)

Since this holds for all \( d \in A \) and \( A \) is faithful, \( m(ab, c) = am(b, c) \).

A similar computation shows that \( m(a, bc) = m(a, b)c \). Hence \( m \) is a quasimultiplier. \( \square \)

Definition 4.5. Let \( QM(A) \) denote the set of all bilinear jointly continuous quasimultipliers on a topological algebra \( A \). Clearly, \( QM(A) \) is a vector space under the usual pointwise operations. Further, \( QM(A) \) becomes an \( A \)-bimodule as follows. For any \( m \in QM(A) \) and \( a \in A \), we can define the products \( a \circ m \) and \( m \circ a \) as mappings from \( A \times A \) into \( A \) given by

\[
(a \circ m)(x, y) = m(xa, y), \quad (m \circ a)(x, y) = m(x, ay), \quad x, y \in A.
\]  

(4.22)

Then \( a \circ m, m \circ a \in QM(A) \), so that \( QM(A) \) is an \( A \)-bimodule.

Definition 4.6. Let \((A, q)\) be an \( F \)-normed algebra. Following [2, 4, 8], we can define mappings

\[
\phi_a : A \rightarrow QM(A), \quad \phi_f : M_f(A) \rightarrow QM(A), \\
\phi_r : M_r(A) \rightarrow QM(A), \quad \phi_d : M_d(A) \rightarrow QM(A).
\]  

(4.23)

by

\[
(\phi_a(a))(x, y) = xay, \quad a \in A, \\
(\phi_f(T))(x, y) = xT(y), \quad T \in M_f(A), \\
(\phi_r(T))(x, y) = T(x)y, \quad T \in M_r(A), \\
(\phi_d(S, T))(x, y) = xS(y), \quad (S, T) \in M_d(A),
\]  

(4.24)

for all \((x, y) \in A \times A\). By Lemma 4.2, these mappings are well defined.
Definition 4.7. (1) A bounded approximate identity \( \{e_\lambda : \lambda \in I\} \) in a topological algebra \( A \) is said to be ultra-approximate if, for all \( m \in \text{QM}(A) \) and \( a \in A \), the nets \( \{m(a,e_\lambda) : \lambda \in I\} \) and \( \{m(e_\lambda,a) : \lambda \in I\} \) are Cauchy in \( A \) (see [2]).

(2) A topological algebra \( A \) is called \( m \)-symmetric if, for each \( S \in M_\ell(A) \cup M_r(A) \), there is a \( T \in M_\ell(A) \cup M_r(A) \) such that either \( (S,T) \in M_q(A) \) or \( (T,S) \in M_d(A) \) (see [4]).

Theorem 4.8. Let \( (A,a) \) be an \( F \)-algebra with a bounded approximate identity \( \{e_\lambda : \lambda \in I\} \). Consider the following conditions.

(a) \( \{e_\lambda : \lambda \in I\} \) is ultra-approximate.

(b) For any \( a \in A, S \in M_\ell(A), \) and \( T \in M_r(A), \) the nets \( \{aS(e_\lambda)\} \) and \( \{T(e_\lambda)a\} \) are Cauchy in \( A \).

(c) \( A \) is \( m \)-symmetric.

Then \( (a) \Rightarrow (b) \Leftrightarrow (c) \). If \( A \) is factorable, then \( (c) \Rightarrow (a) \); hence \( (a), (b), \) and \( (c) \) are equivalent.

Proof. \( (a) \Rightarrow (b) \) Suppose that \( \{e_\lambda : \lambda \in I\} \) in \( A \) is ultra-approximate. Let \( S \in M_\ell(A) \) and \( T \in M_r(A) \). Put \( m_1 = \phi_\lambda(S) \) and \( m_2 = \phi_r(T) \). Then \( m_1, m_2 \in \text{QM}(A) \) and, for any \( a \in A \), \( \{m_1(a,e_\lambda)\} = \{aS(e_\lambda)\} \) and \( \{m_2(e_\lambda,a)\} = \{T(e_\lambda)a\} \) which are Cauchy in \( A \) by hypothesis.

\( (b) \Rightarrow (c) \) Suppose that (b) holds, let \( S \in M_\ell(A) \cup M_r(A) \), and suppose that \( S \in M_\ell(A) \). Since \( A \) is complete, the map \( T : A \rightarrow A \) given by

\[
T(a) = \lim_\lambda aS(e_\lambda), \quad a \in A,
\]

(4.25)

is well-defined. Since \( S \) is continuous, for any \( a,b \in A \),

\[
aS(b) = a \lim_\lambda S(e_\lambda)b \quad \text{and} \quad S(e_\lambda)b = \left[ \lim_\lambda aS(e_\lambda) \right]b = T(a)b.
\]

(4.26)

Since \( A \) is a faithful \( F \)-algebra, by Theorem 3.7(b), \( (S,T) \in M_d(A) \). Hence \( A \) is \( m \)-symmetric.

\( (c) \Rightarrow (b) \) Suppose that (c) holds. Let \( S \in M_\ell(A) \) and \( T \in M_r(A) \). By (c), there exist \( T_1 \in M_r(A), S_1 \in M_\ell(A) \) such that \( (S,T_1),(S_1,T) \in M_d(A) \). Then, for any \( a \in A \),

\[
aS(e_\lambda) = T_1(a)e_\lambda \rightarrow T_1(a),
\]

\[
T(e_\lambda)a = e_\lambda S_1(a) \rightarrow S_1(a).
\]

(4.27)

Thus, both \( \{aS(e_\lambda)\} \) and \( \{T(e_\lambda)a\} \), being convergent, are Cauchy in \( A \).

Suppose that \( A \) is factorable. Then \( (c) \Rightarrow (a) \), as follows. Let \( m \in \text{QM}(A) \) and \( a \in A \). By factorability, \( a = bc \) for some \( b,c \in A \). Define the mappings \( S,T : A \rightarrow A \) by

\[
S(x) = m(c,x), \quad T(x) = m(x,b), \quad x \in A.
\]

(4.28)
Abstract and Applied Analysis

Then, for any \( x, y \in A \),

\[
S(xy) = m(c, xy) = m(c, x)y = S(x)y, \\
T(xy) = m(xy, b) = xm(y, b) = xT(y),
\]

(4.29)

and so \( S \in M_\ell(A) \) and \( T \in M_r(A) \). By (c), there exist \( T_1 \in M_r(A) \), \( S_1 \in M_\ell(A) \) such that \((S, T_1), (S_1, T) \in M_d(A)\). Then

\[
m(a, e_\lambda) = m(bc, e_\lambda) = bm(c, e_\lambda) = bS(e_\lambda) = T_1(b)e_\lambda \rightarrow T_1(b), \\
m(e_\lambda, a) = m(e_\lambda, bc) = m(e_\lambda, b)c = T(e_\lambda)c = e_\lambda S_1(c) \rightarrow S_1(c).
\]

(4.30)

Hence \( \{m(a, e_\lambda) : \lambda \in I\} \) and \( \{m(e_\lambda, a) : \lambda \in I\} \) are Cauchy in \( A \). So \( \{e_\lambda : \lambda \in I\} \) is ultra-
approximate.

**Theorem 4.9.** Let \((A, q)\) be an \( F \)-algebra having an ultra-approximate identity \( \{e_\lambda : \lambda \in I\} \). Then,

(a) each of the maps \( \phi_A, \phi_\ell, \phi_r, \) and \( \phi_d \) is a bijection;

(b) \( \phi_d | M(A) = \phi_\ell | M(A) = \phi_r | M(A) \);

(c) \( \phi_d | \mu_d(A) = \phi_A \).

**Proof.** (a) We give the proof only for \( \phi_d : M_d(A) \rightarrow QM(A) \). To show that \( \phi_d \) is onto, let \( m \in QM(A) \). Since \( \{e_\lambda\} \) is ultra-approximate, for each \( x \in A \), the nets \( \{m(x, e_\lambda) : \lambda \in I\} \) and \( \{m(e_\lambda, x) : \lambda \in I\} \) are convergent. For each \( x \in A \), define \( S, T : A \rightarrow A \) by

\[
S(x) = \lim_{\lambda} m(x, e_\lambda), \quad T(x) = \lim_{\lambda} m(x, e_\lambda).
\]

(4.31)

Then \((S, T) \in M_d(A)\) since, for any \( x, y \in A \),

\[
xS(y) = x \lim_{\lambda} m(e_\lambda, y) = \lim_{\lambda} m(xe_\lambda, y) = m(x, y) \\
= \lim_{\lambda} m(x, e_\lambda y) = \lim_{\lambda} m(x, e_\lambda y) = T(x)y.
\]

(4.32)

Further, we have for any \( (a, b) \in A \times A \),

\[
[\phi_d(S, T)](a, b) = aS(b) = a \lim_{\lambda} m(e_\lambda, b) = \lim_{\lambda} m(ae_\lambda, b) = m(a, b);
\]

(4.33)

that is, \( \phi_d(S, T) = m \).

To show that \( \phi_d \) is one to one, let \((S_1, T_1), (S_2, T_2) \in M_d(A)\) with \( \phi_d(S_1, T_1) = \phi_d(S_2, T_2) \). Then, for any \( a, b \in A \),

\[
\phi_d(S_1, T_1)(a, b) = \phi_d(S_2, T_2)(a, b), \quad \text{or} \quad aS_1(b) = aS_2(b).
\]

(4.34)
Since $A$ is faithful, $S_1(b) = S_2(b), b \in A$. So $S_1 = S_2$. Consequently, by Theorem 3.3(c), also $T_1 = T_2$. Thus $(S_1, T_1) = (S_2, T_2)$.

(b) Let $T \in M(A)$ and $x, y \in A$. Then, since $T \in M_r(A) \cap M_r(A)$ and $(T, T) \in M_d(A)$,
\[ \phi_d(T)(x, y) = \phi_d(T, T)(x, y) = xT(y) = T(xy). \quad (4.35) \]

Also
\[ \phi_e(T)(x, y) = xT(y) = T(xy); \quad \phi_r(T)(x, y) = T(x)y = T(xy). \quad (4.36) \]

(c) For any $a \in A$ and $(x, y) \in A \times A$,
\[ \phi_d(\mu_d(a))(x, y) = \phi_d(L_a, R_a)(x, y) = xL_a(y) = xay = \phi_A(a)(x, y). \quad (4.37) \]

Thus $\phi_d | \mu_d(A) = \phi_A$. $\square$

We obtain the following lemma for later use.

**Lemma 4.10.** (a) If $(A, q)$ is a factorable $F$-normed algebra having an approximate identity $\{e_\lambda : \lambda \in I\}$, then
\[ \lim_{\lambda} q(e_\lambda a e_\lambda - a) = 0, \quad \forall a \in A. \quad (4.38) \]

(b) If $(A, q)$ is an $F$-algebra having a minimal ultra-approximate identity $\{e_\lambda : \lambda \in I\}$, then, for any $m \in QM(A)$,
\[ \lim_{\lambda} e_\lambda m(e_\lambda, x) = \lim_{\lambda} m(e_\lambda, x) \quad (\text{exists}), \quad \forall x \in A. \quad (4.39) \]

*Compare with [10, page 124].*

**Proof.** (a) Let $a \in A$. Since $A$ is factorable, there exist $x, y \in A$ such that $a = xy$. Then
\[ \lim_{\lambda} q(e_\lambda a e_\lambda - a) = \lim_{\lambda} q[(e_\lambda xy e_\lambda - e_\lambda xy) + (e_\lambda xy - xy)] \leq \lim_{\lambda} q(e_\lambda x) \cdot q(ye_\lambda - y) + \lim_{\lambda} q(e_\lambda xy - xy) \]
\[ = q(x) \cdot q(0) + q(0) = 0. \quad (4.40) \]

Since $A$ is complete and $\{e_\lambda\}$ is ultra-approximate, for any $x \in A$, $\lim_{\lambda} m(e_\lambda, x) = y$ (say) exists. Since $q(e_\lambda) \leq 1$,
\[ q[y - e_\lambda m(e_\lambda, x)] \leq q(y - e_\lambda y) + q[e_\lambda y - e_\lambda m(e_\lambda, x)] \leq q(y - e_\lambda y) + q(e_\lambda)q(y - m(e_\lambda, x)) \leq q(y - e_\lambda y) + q(y - m(e_\lambda, x)) \rightarrow 0. \quad (4.41) \]

$\square$
Abstract and Applied Analysis

Definition 4.11. Let \((A, q)\) be a \(k\)-normed algebra. For any \(m \in QM(A)\), we define

\[
\|m\|_q = \sup \left\{ \frac{q[m(x, y)]}{q(x)q(y)} : x, y \in A, \; x, y \neq 0 \right\}. \tag{4.42}
\]

Clearly,

\[
q[m(x, y)] \leq \|m\|_q q(x)q(y), \quad \forall x, y \in A. \tag{4.43}
\]

Theorem 4.12. Let \((A, q)\) be a \(k\)-normed algebra. Then,

(a) \(\| \cdot \|_q\) is a \(k\)-norm on \(QM(A)\);

(b) if \((A, q)\) is complete, then so is \((QM(A), \| \cdot \|_q)\).

Proof. (a) For any \(m \in QM(A)\),

\[
\|m\|_q = 0 \iff \sup_{x \neq 0, y \neq 0} \frac{q[m(x, y)]}{q(x)q(y)} = 0 \\
\iff q[m(x, y)] = 0, \quad \forall x, y \in A, \; x \neq 0, \; y \neq 0 \tag{4.44}
\]

\[
\iff q[m(x, y)] = 0, \quad \forall x, y \in A
\]

\[
\iff m = 0.
\]

Also, for any \(a \in \mathbb{K}\), \(q(ax) = |a|^k q(x)\) and so it follows that \(\|am\|_q = |a|^k \|m\|_q\).

Next, let \(m, u \in QM(A)\), and let \(\varepsilon > 0\). Choose \(x, y \in A\) such that

\[
\|m + u\|_q \leq \frac{q[(m + u)(x, y)]}{q(x)q(y)} - \varepsilon. \tag{4.45}
\]

Then,

\[
\|m + u\|_q \leq \frac{q[m(x, y) + u(x, y)]}{q(x)q(y)} - \varepsilon \\
\leq \frac{q[m(x, y)]}{q(x)q(y)} + \frac{q[u(x, y)]}{q(x)q(y)} - \varepsilon \tag{4.46}
\]

\[
\leq \|m\|_q + \|u\|_q - \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, \(\|m + u\|_q \leq \|m\|_q + \|u\|_q\). Thus \(\| \cdot \|_q\) is a \(k\)-norm on \(QM(A)\).
(b) Let \( \{m_i : i \in \mathbb{N}\} \) be a \( \| \cdot \|_q \)-Cauchy sequence in \( QM(A) \). Then, for any \( x, y \in A \),

\[
\lim_{i,j} q[m_i(x, y) - m_j(x, y)] = \lim_{i,j} q[(m_i - m_j)(x, y)]
\]

\[
\leq \lim_{i,j} \|m_i - m_j\|_q \cdot q(x)q(y) = 0.
\]

Therefore, for any \( x, y \) in \( A \), \( \{m_i(x, y)\} \) is a Cauchy sequence in \( A \). Since \( A \) is complete, the map \( m : A \times A \to A \) given by

\[
m(x, y) = \lim_{i} m_i(x, y), \quad x, y \in A,
\]

is welldefined. Clearly, \( m \) is bilinear and, by Theorem 2.7(b), \( m \) is jointly continuous. Further, for any \( a, b, x, y \in A \),

\[
m(ax, yb) = \lim_{i} m_i(ax, yb) = \lim_{i} [am_i(x, y)b] = a\left[ \lim_{i} m_i(x, y) \right]b = am(x, y)b.
\]

Hence, \( m \in QM(A) \). Next, \( \|m_i - m\|_q \to 0 \) as follows. Let \( \varepsilon > 0 \). Since \( \{m_i\} \) is \( \| \cdot \|_q \)-Cauchy, there exists an integer \( N \geq 1 \) such that

\[
\|m_i - m_j\|_q < \frac{\varepsilon}{2}, \quad \forall \text{ pairs } i, j \geq N,
\]

that is,

\[
\frac{q[(m_i - m_j)(x, y)]}{q(x)q(y)} < \frac{\varepsilon}{2}, \quad \forall \text{ pairs } i, j \geq N, \quad x, y \in A, \quad x, y \neq 0.
\]

Let \( x, y \in A, \ x, y \neq 0 \). Fix any \( i_o \geq N \) in (4.51); since \( m_i(x, y) \to m(x, y) \) in \( A \), letting \( j \to \infty \) in (4.51),

\[
\frac{q[m_i(x, y) - m(x, y)]}{q(x)q(y)} \leq \frac{\varepsilon}{2}.
\]

Then, for any \( i \geq N \), using (4.51) and (4.52),

\[
\frac{q[m_i(x, y) - m(x, y)]}{q(x)q(y)} \leq \frac{q[m_i(x, y) - m_{i_o}(x, y)]}{q(x)q(y)} + \frac{q[m_{i_o}(x, y) - m(x, y)]}{q(x)q(y)}
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \( m_i \xrightarrow{\| \cdot \|_q} m \). \( \square \)
**Theorem 4.13.** Let \((A, q)\) be a factorable \(k\)-normed algebra having a minimal approximate identity \(\{e_\lambda : \lambda \in I\}\). Then

(a) each of the maps \(\phi_A, \phi_E, \phi_r, \) and \(\phi_d\) defined above is a linear isometry;

(b) for any \(a, b \in A\) and \(m \in QM(A)\), \(\|a \circ m \circ b\|_q = q(m(a, b))\).

**Proof.** (a) We give the proof only for \(\phi_d : M_d(A) \to QM(A)\). Clearly, \(\phi_d\) is linear. Let \((S, T) \in M_d(A)\). Then

\[
\|\phi_d(S, T)\|_q = \sup_{x \neq 0, y \neq 0} \frac{q[\phi_d(S, T)(x, y)]}{q(x)q(y)} = \sup_{x \neq 0, y \neq 0} \frac{q[S(y)]}{q(x)q(y)} \leq \sup_{y \neq 0} \frac{q(S(y))}{q(y)} = \|S\|_q.
\]

To prove the reverse inequality, let \(\varepsilon > 0\). There exists \((y \neq 0) \in A\) such that \(\|S\|_q < q(S(y))/q(y) + \varepsilon\). For any \(\lambda \in I\), since \(0 < q(ye_\lambda) \leq q(y)q(e_\lambda) \leq q(y)\),

\[
\|\phi_d(S, T)\|_q \geq \frac{q[\phi_d(S, T)(e_\lambda, ye_\lambda)]}{q(e_\lambda)q(ye_\lambda)} \geq \frac{q[\phi_d(S)(ye_\lambda)]}{q(y)} \geq \frac{q[S(y)e_\lambda]}{q(y)};
\]

hence, in view of factorability, using Lemma 4.10(a),

\[
\|\phi_d(S, T)\|_q \geq \lim_{\lambda} \frac{q[S(y)e_\lambda]}{q(y)} = \frac{q(S(y))}{q(y)} > \|S\|_q - \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, we obtain \(\|\phi_d(S, T)\|_q = \|S\|_q = \|(S, T)\|_q\), and so \(\phi_d\) is an isometry.

(b) By (a), \(\phi_A\) is an isometry and so

\[
\|a \circ m \circ b\|_q = \sup_{x \neq 0, y \neq 0} \frac{q[(a \circ m \circ b)(x, y)]}{q(x)q(y)} = \sup_{x \neq 0, y \neq 0} \frac{q[m(xa, by)]}{q(x)q(y)} = \sup_{x \neq 0, y \neq 0} \frac{q[\phi_A(m(a, b))(x, y)]}{q(x)q(y)} = \|\phi_A(m(a, b))\|_q = q(m(a, b)).
\]

We next consider multiplication in \(QM(A)\) in various equivalent ways.

**Definition 4.14** (see [2, 4]). Let \(A\) be an \(F\)-algebra with an ultra-approximate identity \(\{e_\lambda : \lambda \in I\}\) and \(m_1, m_2 \in QM(A)\). Since \(\phi_d\) is onto, there exist \((S_1, T_1), (S_2, T_2) \in M_d(A)\) such that

\[
\phi_d(S_1, T_1) = m_1, \quad \phi_d(S_2, T_2) = m_2.
\]
By the definitions of $\phi_\ell$ and $\phi_r$,
\[
\phi_\ell(S_1) = m_1 = \phi_r(T_1), \quad \phi_\ell(S_2) = m_2 = \phi_r(T_2).
\] (4.59)

Therefore, the product of $m_1, m_2$ can be defined in any of the following ways:

(i) $m_1 \circ_{\phi_\ell} m_2 = \phi_d(S_1T_1) \circ_{\phi_\ell} \phi_d(S_2T_2) = \phi_d((S_1, T_1)(S_2, T_2)) = \phi_d(S_1S_2, T_2T_1)$,

(ii) $m_1 \circ_{\phi_\ell} m_2 = \phi_\ell(S_1) \circ_{\phi_\ell} \phi_\ell(S_2) = \phi_\ell(S_1S_2)$,

(iii) $m_1 \circ_{\phi_\ell} m_2 = \phi_r(T_1) \circ_{\phi_r} \phi_r(T_2) = \phi_r(T_2T_1)$.

Note that, for any $(x, y) \in A \times A$,
\[
[\phi_d(S_1S_2, T_2T_1)](x, y) = x(S_1S_2)(y) = [\phi_\ell(S_1S_2)](x, y),
\] (4.60)

also
\[
x(S_1S_2)(y) = (T_2T_1)(x)y = [\phi_r(T_2T_1)](x, y).
\] (4.61)

Hence, $m_1 \circ_{\phi_\ell} m_2 = m_1 \circ_{\phi_\ell} m_2 = m_1 \circ_{\phi_r} m_2$.

Remark 4.15. (1) If $m = \phi_\ell(T)$ with $T \in M_\ell(A)$ and $a \in A$, then, by Theorem 3.6(a),
\[
m \circ_{\phi_\ell} \phi_A(a) = \phi_\ell(T) \circ_{\phi_\ell} \phi_\ell(L_a) = \phi_\ell(TL_a) = \phi_\ell(L_T(a)) \in \phi_A(A). \tag{4.62}
\]

(2) If $m = \phi_r(T)$ with $T \in M_r(A)$ and $a \in A$, then, by Theorem 3.6(b),
\[
\phi_A(a) \circ_{\phi_r} m = \phi_r(R_a) \circ_{\phi_r} \phi_r(T) = \phi_r(TR_a) = \phi_r(R_T(a)) = \phi_A(T(a)) \in \phi_A(A). \tag{4.63}
\]

(3) If $m = \phi_d(S, T)$ and $a \in A$, then, by Theorem 3.6(c),
\[
m \circ_{\phi_d} \phi_A(a) = \phi_d[(S, T)(L_a, R_a)] = \phi_d(SL_a, R_aT)
= \phi_d(L_{S(a)}, R_{S(a)}) = \phi_A(S(a)) \in \phi_A(A). \tag{4.64}
\]

In the sequel, we denote the product on $QM(A)$ arising from (i), (ii), or (iii) by $\circ$. Some properties of this product are given as follows.

Theorem 4.16. Let $A$ be a factorable complete k-normed algebra with a minimal ultra-approximate identity $\{e_\lambda : \lambda \in I\}$. Then,

(a) for any $m_1, m_2 \in QM(A)$,
\[
(m_1 \circ m_2)(x, y) = m_1 \left( x, \lim_{\lambda \to 1} m_2(e_\lambda, y) \right) \quad ((x, y) \in A \times A),
\] (4.65)

defines a product $\circ$ on $QM(A)$ so that $(QM(A), \| \cdot \|_a)$ is a complete k-normed algebra with identity $m_a = \phi_d(I, I)$;
Abstract and Applied Analysis

(b) for any \( m \in QM(A) \) and \( a \in A \),

\[
\phi_A(a) \circ m = a \circ m, \quad m \circ \phi_A(a) = m \circ a; \tag{4.66}
\]

(c) \( \phi_A(A) \) is a two-sided ideal in \( QM(A) \);

(d) If \( A \) is factorable, then both \( M_e(A) \) and \( M_d(A) \) are isometrically algebraically isomorphic to \( QM(A) \), while \( M_r(A) \) is isometrically algebraically anti-isomorphic to \( QM(A) \);

Proof. (a) Let \( m_1, m_2 \in QM(A) \) and \( (x, y) \in A \times A \). Choose \( (S_1, T_1), (S_2, T_2) \in M_d(A) \) such that

\[
\phi_d(S_1, T_1) = m_1, \quad \phi_d(S_2, T_2) = m_2, \tag{4.67}
\]

that is,

\[
xS_1(y) = m_1(x, y), \quad xS_2(y) = m_2(x, y). \tag{4.68}
\]

Then,

\[
(m_1 \circ m_2)(x, y) = [\phi_d(S_1S_2, T_2T_1)](x, y) = x(S_1S_2)(y)
\]

\[
= x \left[ S_1(\lim_{\lambda} e_{\lambda}S_2(y)) \right] = S_1 \left( x \lim_{\lambda} e_{\lambda}S_2(y) \right)
\]

\[
= m_1 \left( x, \lim_{\lambda} e_{\lambda}S_2(y) \right) = m_2 \left( x, \lim_{\lambda} m_2(e_{\lambda}, y) \right). \tag{4.69}
\]

It is easy to verify that \( \|m_1 \circ m_2\|_q \leq \|m_1\|_q \|m_2\|_q \), so that \( (QM(A), \| \cdot \|_q) \) is an \( F \)-normed algebra. Further, by Theorem 4.12, \( (QM(A), \| \cdot \|_q) \) is also complete.

(b) Let \( m \in QM(A) \) and \( a \in A \). Using (a), for any \( (x, y) \in A \times A \),

\[
[m \circ \phi_A(a)](x, y) = m \left( x, \lim_{\lambda} \phi_A(a)(e_{\lambda}, y) \right) = m \left( x, \lim_{\lambda} m e_{\lambda} ay \right)
\]

\[
= m(x, ay) = (m \circ a)(x, y). \tag{4.70}
\]

Similarly, \( [\phi_A(a) \circ m](x, y) = (a \circ m)(x, y) \).

(c) To show that \( \phi_A(A) \) is a two-sided ideal, let \( m \in QM(A) \) and \( a \in A \). By Theorem 4.9(a), there exists \( (S, T) \in M_d(A) \) such that \( \phi_d(S, T) = m \). Using (b), for any \( (x, y) \in A \times A \),

\[
[m \circ \phi_A(a)](x, y) = (m \circ a)(x, y) = m(x, ay) = \phi_d(S, T)(x, ay)
\]

\[
= xS(ay) = xS(a)y = \phi_A(S(a))(x, y). \tag{4.71}
\]

Hence, \( m \circ \phi_A(a) = \phi_A(S(a)) \in \phi_A(A) \). Similarly, \( \phi_A(a) \circ m = \phi_A(T(a)) \in \phi_A(A) \).
(d) Suppose that $A$ is factorable. Then, by Theorem 4.13(a), each of the maps $\phi_d$, $\phi_e$, $\phi_r$, and $\phi_A$ is a linear isometry. If $(S_1, T_1), (S_2, T_2) \in M_d(A)$, then by definition

$$\phi_d(S_1, T_1) \circ \phi_d(S_2, T_2) = \phi_d[(S_1, T_1)(S_2, T_2)].$$

If $S_1, S_2 \in M_e(A)$, then by definition

$$\phi_e(S_1) \circ \phi_e(S_2) = \phi_e(S_1 S_2).$$

If $T_1, T_2 \in M_r(A)$, then by definition

$$\phi_r(T_1) \circ \phi_r(T_2) = \phi_r(T_2 T_1).$$

Hence, $\phi_d : M_d(A) \to QM(A)$ and $\phi_e : M_e(A) \to QM(A)$ are algebraic isomorphisms and $\phi_r : M_r(A) \to QM(A)$ is algebraic anti-isomorphism.

Remark 4.17. If $A$ has an identity $e$, then $A$ may be identified with $QM(A)$ as follows. Let $m \in QM(A)$. Then $m(e,e) \in A$ and, for any $x, y \in A$,

$$\phi_A(m(e,e))(x,y) = xm(e,e)y = m(xe, ey) = m(x, y).$$

5. Quasistrict and Strict Topologies on $QM(A)$

In this section, we consider the quasistrict and strict topologies on $QM(A)$ and extend several results from [2, 4, 8]. Throughout we will assume, unless stated otherwise, that $(A, q)$ is a factorable complete $k$-normed algebra having a minimal ultra-approximate identity $\{e_{\lambda} : \lambda \in I\}$.

Definition 5.1. For any $m \in QM(A)$ and $a, b \in A$, we define mappings $a \circ m, m \circ a, a \circ m \circ b : A \times A \to A$ by

$$(a \circ m)(x, y) = m(xa, y), \quad (m \circ a)(x, y) = m(x, ay), \quad (a \circ m \circ b)(x, y) = m(xa, by), \quad (x, y) \in A \times A. \quad (5.1)$$

Lemma 5.2. Let $m \in QM(A)$ and $a, b \in A$. Then

(a) $\|a \circ m\|_q \leq \|m\|_q q(a)$, $\|m \circ a\|_q \leq \|m\|_q q(a)$,

(b) $\|a \circ m \circ b\|_q = q(m(a, b)) \leq \|m\|_q q(a)q(b)$. 

Abstract and Applied Analysis

Proof. (a) By definition,

\[
\|a \circ m\|_q = \sup_{x \neq 0, y \neq 0} \frac{q(a \circ m)(x, y)}{q(x)q(y)} = \sup_{x \neq 0, y \neq 0} \frac{q[x \cdot m(a, y)]}{q(x)q(y)}
\]

\[
\leq \sup_{x \neq 0, y \neq 0} \frac{q(x)q(m(a, y))}{q(x)q(y)} = \sup_{y \neq 0} \frac{q(m(a, y))}{q(y)}
\]

(5.2)

\[
= \sup_{y \neq 0} \frac{\|m\|_q q(a)q(y)}{q(y)} = \|m\|_q q(a).
\]

Similarly, \(\|m \circ a\|_q \leq \|m\|_q q(a)\).

(b) By Theorem 4.13(b), \(q(m(a, b)) = \|a \circ m \circ b\|_q\). Further, using (a),

\[
\|a \circ m \circ b\|_q \leq \|a \circ m\|_q q(b) \leq \|m\|_q q(a)q(b).
\]

(5.3)

Definition 5.3. (1) The quasistrict topology \(\gamma\) on \(QM(A)\) is determined by the family \(\{\xi_{a,b}(m) : a, b \in A\}\) of \(k\)-seminorms, where

\[
\xi_{a,b}(m) = \|a \circ m \circ b\|_q = q(m(a, b)), \quad m \in QM(A).
\]

(5.4)

Compare with [2, page 109]; [4, page 558].

(2) The strict topology \(\beta\) on \(QM(A)\) is determined by the family \(\{\eta_a(m) : a \in A\}\) of \(k\)-seminorms, where

\[
\eta_a(m) = \max\left\{\|a \circ m\|_q, \|m \circ a\|_q\right\}, \quad m \in QM(A).
\]

(5.5)

Compare with [8, page 227].

Let \(\tau\) denote the topology on \(QM(A)\) generated by the \(k\)-norm \(\|\cdot\|_q\).

Lemma 5.4. \(\gamma \subseteq \beta \subseteq \tau\) on \(QM(A)\).

Proof. To show that \(\gamma \subseteq \beta\), let \(a, b \in A\). Then

\[
\xi_{a,b}(m) = \|a \circ m \circ b\|_q \leq \|a \circ m\|_q q(b) \leq \eta_a(m)q(b), \quad m \in QM(A),
\]

(5.6)

also

\[
\xi_{a,b}(m) = \|a \circ m \circ b\|_q \leq \|m \circ b\|_q q(a) \leq \eta_b(m)q(a), \quad m \in QM(A).
\]

(5.7)
Hence,

\[ \xi_{a,b}(m) \leq \max \{ \eta_a(m), \eta_b(m) \} q(a)q(b), \quad m \in QM(A). \] (5.8)

Let \( \{ m_a : a \in J \} \) be a net in \( QM(A) \) with \( m_a \xrightarrow{\beta} m \in QM(A) \). Then, for any \( a, b \in A \),

\[ \eta_a(m_a - m) \to 0 \quad \text{and} \quad \eta_b(m_a - m) \to 0. \]

Hence,

\[ \xi_{a,b}(m_a - m) \leq \max \{ \eta_a(m_a - m), \eta_b(m_a - m) \} q(a)q(b) \to 0. \] (5.9)

Thus \( m_a \xrightarrow{\gamma} m \), and so \( \gamma \subseteq \beta \).

To show that \( \beta \subseteq \tau \), let \( a \in A \). Note that, for any \( x, y \in A \),

\[
\begin{align*}
(a \circ m)(x, y) &= m(xa, y) = x \cdot m(a, y), \quad m \in QM(A), \\
(m \circ a)(x, y) &= m(x, ay) = m(x, a)y, \quad m \in QM(A).
\end{align*}
\] (5.10)

By Lemma 5.2(a), \( \| a \circ m \|_q \leq \| m \|_q q(a) \) and \( \| m \circ a \|_q \leq \| m \|_q q(a) \); hence,

\[ \eta_a(m) = \max \left\{ \| a \circ m \|_q, \| m \circ a \|_q \right\} \leq \| m \|_q q(a), \quad m \in QM(A). \] (5.11)

Consequently, if \( \{ m_a \} \) is a net in \( QM(A) \) with \( m_a \xrightarrow{\gamma} m \in QM(A) \), then \( \| m_a - m \|_q \to 0 \), and so, for any \( a, b \in A \),

\[ \eta_a(m_a - m) \leq \| m_a - m \|_q q(a) \to 0. \] (5.12)

Thus \( m_a \xrightarrow{\beta} m \); that is, \( \beta \subseteq \tau \). \(\square\)

**Theorem 5.5.** \( \phi_A(A) \) is \( \beta \)-dense in \( QM(A) \) and hence \( \gamma \)-dense in \( QM(A) \).

**Proof.** Let \( m \in QM(A) \). Clearly \( \{ m(e_\lambda, e_\lambda) \}_{\lambda \in I} \subseteq A \). We claim that \( \phi_A(m(e_\lambda, e_\lambda)) \xrightarrow{\beta} m \). Let \( a \in A \). We need to show that

\[ \eta_a[\phi_A(m(e_\lambda, e_\lambda)) - m] \to 0. \] (5.13)

For any \( x, y \in A \), by joint continuity of \( m \),

\[
q[a \circ \{ \phi_A(m(e_\lambda, e_\lambda)) - m \}(x, y)] \\
= q[\{ \phi_A(m(e_\lambda, e_\lambda)) - m \}(xa, y)] = q[xa \circ (e_\lambda, e_\lambda) - m(xa, y)] \\
= q[m(xae_\lambda, e_\lambda y) - m(xa, y)] \to q[m(xa, y) - m(xa, y)] = 0.
\] (5.14)
Hence \( \|a \circ (\phi_{\ell}(m(e_1, e_\lambda)) - m)\|_q \to 0 \). Similarly, \( \|\{\phi_{\ell}(m(e_1, e_\lambda)) - m\} \circ a\|_q \to 0 \). Thus,

\[
\eta_a[\phi_{\ell}(m(e_1, e_\lambda)) - m] \to 0; \tag{5.15}
\]

that is, \( \phi_{\ell}(A) \) is \( \beta \)-dense in \( QM(A) \). Since \( \gamma \subseteq \beta \), it follows that \( \phi_{\ell}(A) \) is \( \gamma \)-dense in \( QM(A) \).

\[\square\]

**Theorem 5.6.** (a) \((QM(A), \gamma)\) and \((QM(A), \beta)\) are sequentially complete.

(b) If, in addition, \((A, q)\) is strongly factorable, then \((QM(A), \gamma)\) and \((QM(A), \beta)\) are complete.

**Proof.** (a) Let \( \{m_i : i \in \mathbb{N}\} \) be a \( \gamma \)-Cauchy sequence in \( QM(A) \). For any \( x, y \in A \), using Theorem 4.13(b),

\[
q[m_i(x, y) - m_i(x, y)] = q[(m_i - m_i)(x, y)] = \|x \circ (m_i - m_i) \circ y\|_q = \xi_{x,y}(m_i - m_i),
\]

which implies that \( \{m_i(x, y)\} \) is a Cauchy sequence in \( A \). Define \( m : A \times A \to A \) by \( m(x, y) = \lim_i m_i(x, y) \). Clearly, \( m \) is bilinear and, by Theorem 2.7(b), \( m \) is jointly continuous. Further, for any \( a, b, x, y \in A \),

\[
m(ax, yb) = \lim_i m_i(ax, yb) = a \left[ \lim_i m_i(x, y) \right] b = am(x, y)b,
\]

and so \( m \in QM(A) \). Further, for any \( a, b \in A \),

\[
\xi_{a,b}(m_i - m) = \|a \circ m_i \circ b - a \circ m \circ b\|_q = q[(m_i - m)(a, b)] \to 0. \tag{5.18}
\]

Hence \( m_i \xrightarrow{\gamma} m \). So \( (QM(A), \gamma) \) is sequentially complete.

Next we show that \((QM(A), \beta)\) is sequentially complete. We first note that, if \( m \in QM(A) \), then, for each \( c \in A \), the mappings \( S_c, T_c : A \to A \) given by

\[
S_c(x) = m(c, x), \quad T_c(x) = m(x, c), \quad x \in A,
\]

define elements in \( M_{\ell}(A) \) and \( M_r(A) \), respectively, and it is easy to see that

\[
\phi_{\ell}(S_c) = c \circ m, \quad \phi_{r}(T_c) = m \circ c. \tag{5.20}
\]

Let \( \{m_i : i \in \mathbb{N}\} \) be a \( \beta \)-Cauchy sequence in \( QM(A) \), and let \( c \in A \). It follows from the definition of the \( \beta \)-topology that the sequences \( \{\phi_{\ell}(S_c)_i\} \) and \( \{\phi_{r}(T_c)_i\} \), where \( (S_c)_i(x) = m_i(c, x) \) and \( (T_c)_i(x) = m_i(x, c) \), are \( \tau \)-Cauchy in \( QM(A) \). Since \( \phi_{\ell} \) and \( \phi_{r} \) are topological embeddings, the sequences \( \{(S_c)_i\} \) and \( \{(T_c)_i\} \) are \( \| \cdot \|_{\gamma} \)-Cauchy in \( M_{\ell}(A) \) and
\( M_r(A) \), respectively. Both \( M_r(A) \) and \( M_r(A) \) are complete (Theorem 3.9) and so there exist \( S^{(c)} \) in \( M_r(A) \) and \( T^{(c)} \) in \( M_r(A) \) such that

\[
\left\| (S_{c})_i - S^{(c)} \right\|_q \rightarrow 0, \quad \left\| (T_{c})_i - T^{(c)} \right\|_q \rightarrow 0.
\] (5.21)

Since \( \gamma \subseteq \beta \), the sequence \( \{m_i\} \) is \( \gamma \)-Cauchy. As proved above, the space \( QM(A) \) is \( \gamma \)-complete and so there exists an element \( m_o \) in \( QM(A) \) such that

\[
\lim_i m_i(x,y) = m_o(x,y), \quad \forall x,y \in A.
\] (5.22)

For any \( a,b \in A \),

\[
\left[ \phi_r \left( S^{(c)} \right) \right](a,b) = \lim_i \left[ \phi_r \left( S^{(c)} \right) \right](a,b) = \lim_i am_i(c,b) = (c \circ m_o)(a,b),
\] (5.23)

which implies that \( \phi_r(S^{(c)}) = c \circ m_o \). Similarly, we can prove that \( \phi_r(T^{(c)}) = m_o \circ c \). Thus, by (5.21),

\[
\left\| c \circ m_i - c \circ m_o \right\|_q = \left\| \phi_r \left( S^{(c)} \right)_i - \phi_r \left( S^{(c)} \right) \right\|_q = \left\| \left( S^{(c)} \right)_i - S^{(c)} \right\|_q \rightarrow 0,
\]

\[
\left\| m_i \circ c - m_o \circ c \right\|_q = \left\| \phi_r \left( T^{(c)} \right)_i - \phi_r \left( T^{(c)} \right) \right\|_q = \left\| \left( T^{(c)} \right)_i - T^{(c)} \right\|_q \rightarrow 0,
\] (5.24)

which implies that \( m_o \) is the \( \beta \)-limit of the sequence \( \{m_i\} \) that is, \( QM(A) \) is \( \beta \)-complete.

(b) Suppose that \( A \) is strongly factorable. Let \( \{m_a : a \in J\} \) be a \( \gamma \)-Cauchy net in \( QM(A) \). Replacing the sequence \( \{m_i : i \in \mathbb{N}\} \) by the net \( \{m_a : a \in J\} \) in part (a), we obtain a map \( m : A \times A \rightarrow A \) given by \( m(x,y) = \lim_a m_a(x,y) \). Then \( m \) is bilinear; further, for any \( a,b,x,y \in A \),

\[
m(ax,by) = \lim_a m_a(ax,by) = a \left[ \lim_a m_a(x,y) \right] b = am(x,y)b.
\] (5.25)

Hence, using strong factorability as in Theorem 4.3(c), it follows that \( m \) is jointly continuous and so \( m \in QM(A) \). Again, as in part (a), it follows that \( m_a \xrightarrow{\gamma} m \) and consequently \( (QM(A),\gamma) \) is complete. That \( QM(A) \) is \( \beta \)-complete also follows by the argument similar to the above one.

\[ \square \]

Remark 5.7. The authors do not know whether part (b) of the above theorem can be proved without the assumption of the strong factorability of \( A \).
Theorem 5.8. \((QM(A), \gamma), (QM(A), \beta), \) and \((QM(A), \tau)\) have the same bounded sets.

Proof. (a) Since \(\gamma \subseteq \tau\), every \(\tau\)-bounded set is \(\gamma\)-bounded. Let \(H\) be any \(\gamma\)-bounded set in \(QM(A)\). Then, for each \(a, b \in A\), there exists a constant \(r = r(a, b) > 0\) such that

\[
\|a \circ m \circ b\|_{q} \leq r, \quad \forall m \in H,
\]

or

\[
q[m(a, b)] \leq r, \quad \forall m \in H \quad \text{(using Theorem 4.13(b))}.
\]

For each \(a \in A\) and \(m \in H\), define \(M_{a} : A \rightarrow A\) by

\[
M_{a}(x) = m(a, x), \quad x \in A.
\]

Then, \(\mathcal{F} = \{M_{a} : m \in H\} \subseteq CL(A)\). By (5.26), for any \(x \in A\)

\[
q[M_{a}(x)] = q[m(a, x)] \leq r(a, x), \quad \forall m \in H;
\]

hence \(\mathcal{F}\) is pointwise bounded. Then, by the uniform boundedness principle (Theorem 2.6), there exists \(c = c(a) > 0\) such that

\[
\|M_{a}\|_{q} \leq c, \quad \forall m \in H.
\]

Consider now the family \(P = \{p_{m} : m \in H\}\) of \(k\)-seminorms on \(A\) defined by

\[
p_{m}(a) = \|M_{a}\|_{q} = \sup_{b \neq 0} \frac{q[M_{a}(b)]}{q(b)} = \sup_{b \neq 0} \frac{q[m(a, b)]}{q(b)}, \quad a \in A.
\]

For each \(m \in H\), \(p_{m}\) is continuous on \(A\) since, if \(\{a_{n}\} \subseteq A\) with \(a_{n} \rightarrow a_{o}\) in \(A\), then

\[
|p_{m}(a_{n}) - p_{m}(a_{o})| \leq p_{m}(a_{n} - a_{o}) = \sup_{b \neq 0} \frac{q[M_{a_{n} - a_{o}}(b)]}{q(b)}
\]

\[
= \sup_{b \neq 0} \frac{q[m(a_{n} - a_{o}, b)]}{q(b)} \leq \sup_{b \neq 0} \frac{\|m\|_{q}q(a_{n} - a_{o})q(b)}{q(b)}
\]

\[
= \|m\|_{q}q(a_{n} - a_{o}) \rightarrow 0.
\]

Then, by (5.29), the family \(P\) is pointwise bounded. Applying Theorem 2.5, there exists a ball \(B = B(x_{o}, r) = \{x \in A : q(x - x_{o}) \leq r\}\) and a constant \(C > 0\) such that

\[
p_{m}(x) \leq C, \quad \forall m \in H, \ x \in B(x_{o}, r).
\]
For any fixed $a \in A$, we claim that

$$p_m(a) \leq \frac{2C \cdot q(a)}{r}. \quad (5.33)$$

If $a = 0$, this is obvious. Suppose that $a \neq 0$. For simplification, put $t = \left(\frac{r}{q(a)}\right)^{1/k}$. Then, $q$ is $k$-homogeneous, and we have $ta + x_o, x_o \in B(x_o, r)$, as follows:

$$q(ta + x_o - x_o) = q(ta) = t^k \cdot q(a) \leq \frac{r}{q(a)} q(a) = r,$$

$$q(x_o - x_o) = q(0) = 0 < r. \quad (5.34)$$

So, by (5.32),

$$p_m(ta + x_o) \leq C, \quad p_m(x_o) \leq C. \quad (5.35)$$

Now, using (5.35) and the properties of $k$-norm again,

$$p_m(a) = p_m\left(\frac{1}{t} ta\right) = \left(\frac{1}{t}\right)^k p_m(ta) \leq \frac{q(a)}{r} \left[p_m(ta + x_o) + p_m(x_o)\right] \leq \frac{q(a)}{r} \left[C + C\right] = \frac{2C \cdot q(a)}{r}. \quad (5.36)$$

This proves our claim. Hence, using (5.33), for any $m \in H$,

$$\|m\|_q = \sup_{a,b \neq 0} \frac{q[m(a,b)]}{q(a)q(b)} = \sup_{a \neq 0} \frac{1}{q(a)} \sup_{b \neq 0} \frac{q[m(a,b)]}{q(b)} = \sup_{a \neq 0} \frac{1}{q(a)} p_m(a) \leq \sup_{a \neq 0} \frac{1}{q(a)} \frac{2C \cdot q(a)}{r} \leq \frac{2C}{r}. \quad (5.37)$$

Consequently, $H$ is $\tau$-bounded.

(b) This follows from (a) since $\gamma \subseteq \beta \subseteq \tau$. \hfill $\square$

**Acknowledgment**

The authors are grateful to the referees for their several useful suggestions, including those mentioned in Remarks 2.12 and 2.14, which improved significantly the quality of this paper.
Abstract and Applied Analysis

References


Submit your manuscripts at http://www.hindawi.com