Research Article

Generalized $\psi \rho$-Operations on Fuzzy Topological Spaces

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The aim of this work is to introduce $\psi \rho$-operations on fuzzy topological spaces and to use them to study fuzzy generalized $\psi \rho$-closed sets and fuzzy generalized $\psi \rho$-open sets. Also, we introduce some characterizations and properties for these concepts. Finally we show that certain results of several publications on the concepts of weakness and strength of fuzzy generalized closed sets are considered as corollaries of the results of this research.

1. Preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [1] based on the concept of a fuzzy set introduced by Zadeh in [2]. Since then, various important notions in the classical topology such as generalized closed, generalized open set, and weaker and stronger forms of generalized closed and generalized open sets have been extended to fuzzy topological spaces. The purpose of this paper is to introduce and study the concept of $\psi \rho$-operations, and by using these operations, we will study fuzzy generalized $\psi \rho$-closed sets and fuzzy generalized $\psi \rho$-open sets. Also, we introduce some characterizations and properties for these concepts. Finally we show that certain results of several publications on the concepts of weakness and strength of fuzzy generalized closed sets are considered as corollaries of the results of this research.

For a fuzzy set $\mu$ in $X$, we write $x_r \in \mu$ if and only if $r \leq \mu(x)$. Evidently, every fuzzy set $\mu$ can be expressed as the union of all fuzzy points which belongs to $\mu$. A fuzzy point $x_r$ is said to be quasicoincident [17] with $\mu$ denoted by $x_r \cong \mu$ if and only if $r + \mu(x) > 1$. A
fuzzy set $\mu$ is said to be quasicoincident with $\lambda$, denoted by $\mu q \lambda$, if and only if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If $\mu$ is not quasicoincident with $\lambda$, then we write $\lambda \tilde{q} \mu$.

For a fuzzy set $A$ of an fts $(X, \tau)$, $\text{Cl}(A)$ (resp., $\text{Scl}(A)$, $\text{Pcl}(A)$, $\alpha\text{-cl}(A)$, $\gamma\text{-cl}(A)$, $\beta\text{-cl}(A)$, $\text{Spcl}(A)$, $\delta\text{-cl}(A)$, $\theta\text{-cl}(A)$) denote the fuzzy closure (resp., semiclosure, preclosure, $\alpha$-closure, $\gamma$-closure, $\beta$-closure, semi-preclosure, $\delta$-closure, $\theta$-closure) and $\text{Int}(A)$ (resp., $\text{Sint}(A)$, $\text{Pint}(A)$, $\alpha\text{-int}(A)$, $\gamma\text{-int}(A)$, $\beta\text{-int}(A)$, $\text{Spint}(A)$, $\delta\text{-int}(A)$, $\theta\text{-int}(A)$) denote the fuzzy interior (resp., semi-interior, preinterior, $\alpha$-interior, $\gamma$-interior, $\beta$-interior, semi-preinterior, $\delta$-interior, and $\theta$-interior) of $A$.

**Definition 1.1** (see [17]). A fuzzy set $A$ in an fts $(X, \tau)$ is said to be $q$-neighborhood of a fuzzy point $x$, if there exists a fuzzy open set $U$ with $x, qU \subseteq A$.

**Definition 1.2.** A fuzzy set $\mu$ in an fts $(X, \tau)$ is said to be:

1. fuzzy regular open [18] (ro, for short) if $\text{Int}(\text{Cl}(\mu)) = \mu$,
2. fuzzy regular closed [18] (rc, for short) if $\text{Cl}(\text{Int}(\mu)) = \mu$,
3. fuzzy regular semiopen [19] (rso, for short) if there exists a fuzzy regular open set $\lambda$ such that $\lambda \leq \mu \leq \text{Cl}(\lambda)$,
4. fuzzy $\alpha$-open [20] (ao, for short) if $\mu \leq \text{Int}(\text{Cl}(\mu))$,
5. fuzzy $\alpha$-closed [20] (ac, for short) if $\text{Cl}(\text{Int}(\mu)) \leq \mu$,
6. fuzzy semiopen [18] (so, for short) if $\mu \leq \text{Cl}(\text{Int}(\mu))$,
7. fuzzy semiclosed [18] (sc, for short) if $\text{Int}(\text{Cl}(\mu)) \leq \mu$,
8. fuzzy preopen [20] (po, for short) if $\mu \leq \text{Int}(\text{Cl}(\mu))$,
9. fuzzy preclosed [20] (pc, for short) if $\text{Cl}(\text{Int}(\mu)) \leq \mu$,
10. fuzzy $\gamma$-open [21] (go, for short) if $\mu \leq \text{Cl}(\text{Int}(\mu)) \lor \text{Int}(\text{Cl}(\mu))$,
11. fuzzy $\gamma$-closed [21] (gc, for short) if $\mu \leq \text{Cl}(\text{Int}(\mu)) \land \text{Int}(\text{Cl}(\mu))$,
12. fuzzy semi-preopen [22] (spo, for short) if there exists a fuzzy preopen set $\lambda$ such that $\lambda \leq \mu \leq \text{Cl}(\lambda)$,
13. fuzzy semi-preclosed [22] (spc, for short) if there exists a fuzzy preclosed set $\lambda$ such that $\text{Int}(\lambda) \leq \mu \leq \lambda$,
14. fuzzy $\beta$-open [23] ($\beta_0$, for short) if $\mu \leq \text{Cl}(\text{Int}(\mu))$,
15. fuzzy $\beta$-closed [23] ($\beta_c$, for short) if $\text{Int}(\text{Cl}(\text{Int}(\mu))) \leq \mu$.

**Remark 1.3.** From the above definition we have a diagram, Figure 1, showing all relationships between the classes of open sets. None of the implications shown in Figure 1 can be reversed in general.
Definition 1.4 (see [18, 20–23]). (1) The intersection of all fuzzy \( a \)-closed (resp., semiclosed, preclosed, \( \gamma \)-closed, semi-preclosed, \( \beta \)-closed) sets containing a fuzzy set \( A \) is called a fuzzy \( a \)-closure (resp., semiclosure, preclosure, \( \gamma \)-closure, semi-preclosure, \( \beta \)-closure) of \( A \).

(2) The union of all fuzzy \( a \)-open (resp., semiopen, preopen, \( \gamma \)-open, semi-preopen, \( \beta \)-open) sets contained in a fuzzy set \( A \) is called a fuzzy \( a \)-interior (resp., semi-interior, preinterior, \( \gamma \)-interior, semi-preinterior, \( \beta \)-interior) of \( A \).

Definition 1.5. A fuzzy point \( x, \) in an fts \( X \) is said to be a fuzzy cluster (resp., \( \theta \)-cluster [24], \( \delta \)-cluster [16]) point of a fuzzy set \( A \) if and only if for every fuzzy open (resp., open, regular open) \( q \)-neighborhood \( U \) of \( x, \) \( UqA \) (resp., \( \text{Cl}(U)qA, UqA \)). The set of all fuzzy cluster (resp., fuzzy \( \theta \)-cluster, fuzzy \( \delta \)-cluster) points of \( A \) is called the fuzzy closure (resp., \( \theta \)-closure, \( \delta \)-closure) of \( A \) and is denoted by \( \text{Cl}(A) \) (resp., \( \theta \text{-cl}(A), \delta \text{-cl}(A) \)). A fuzzy set \( A \) is fuzzy \( \theta \)-closed (resp., \( \delta \)-closed) if and only if \( A = \theta \text{-cl}(A) \) (resp., \( A = \delta \text{-cl}(A) \)). The complement of a fuzzy \( \theta \)-closed (resp., \( \delta \)-closed) set is called fuzzy \( \theta \)-open (resp., \( \delta \)-open).

Definition 1.6. A fuzzy set \( A \) of an fts \( (X, \tau) \) is said to be:

1. fuzzy generalized closed [3] (briefly, g-closed) if \( \text{Cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \),

2. fuzzy generalized \( a \)-closed [10] (briefly, ga-closed) if \( a \text{-cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \),

3. fuzzy \( a \)-generalized closed [13] (briefly, ag-closed) if \( a \text{-cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy \( a \)-open set in \( X \),

4. fuzzy generalized semiclosed [12] (briefly, gs-closed) if \( \text{Scl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \),

5. fuzzy semigeneralized closed [5] (briefly, sg-closed) if \( \text{Scl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy semiopen set in \( X \),

6. fuzzy generalized preclosed [8] (briefly, gp-closed) if \( \text{Pcl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \),

7. fuzzy pregeneralized closed [6] (briefly, pg-closed) if \( \text{Pcl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy preopen set in \( X \),

8. fuzzy generalized semi-preclosed [11] (briefly, gsp-closed) if \( \text{Spcl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \),

9. fuzzy semi-pregeneralized closed [14] (briefly, spg-closed) if \( \text{Spcl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy semi-preopen set in \( X \),
(10) fuzzy regular generalized closed [9] (briefly, rg-closed) if \( \text{Cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy regular open set in \( X \).

(11) fuzzy generalized \( \theta \)-closed [4] (briefly, \( g\theta \)-closed) if \( \theta -\text{cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy open set in \( X \).

(12) fuzzy \( \theta \)-generalized closed [7] (briefly, \( \theta g \)-closed) if \( \theta -\text{cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy \( \theta \)-open set in \( X \).

(13) fuzzy \( \delta \theta \)-generalized closed [15] (briefly, \( \delta \theta g \)-closed) if \( \delta -\text{cl}(A) \leq U \), whenever \( A \leq U \) and \( U \) is a fuzzy \( \theta \)-open set in \( X \).

The complement of a fuzzy generalized closed (resp., generalized \( \alpha \)-closed, \( \alpha \)-generalized closed, generalized semiclosed, semigeneralized closed, generalized preclosed, pre generalized closed, generalized semi-preclosed, semi-pregeneralized closed, regular generalized closed, \( \theta \)-generalized closed, generalized \( \theta \)-closed) set is called fuzzy generalized open (\( g \)-open, for short) (resp., generalized \( \alpha \)-open (\( g\alpha \)-open), \( \alpha \)-generalized open (\( g\alpha \)-open), generalized semiopen (\( g\alpha \)-open), semi generalized open (\( g\alpha \)-open), generalized preopen (\( \gamma g \)-open), pre generalized open (\( \gamma g \)-open), generalized semi-preopen (\( \gamma g \)-open), semi-pregeneralized open (\( \gamma g \)-open), regular generalized open (\( \gamma g \)-open), \( \theta \)-generalized open (\( \gamma g \)-open), generalized \( \theta \)-open (\( \gamma g \)-open)).

**Definition 1.7** (see [25]). A fuzzy point \( x_r \) in an fts \( (X, \tau) \) is called weak (resp., strong) if \( r \leq 1/2 \) (resp., \( r > 1/2 \)).

2. \( \psi \)-Operations

In this research, we will denote for a fuzzy open set from type \( \psi \) by fuzzy \( \psi \)-open and the family of all fuzzy \( \psi \)-open sets in an fts \( (X, \tau) \) by \( \psi O(X) \). Also we will denote a fuzzy open (resp., \( \alpha \)-open, semiopen, preopen, semi-preopen, \( \gamma \)-open, \( \beta \)-open, \( \delta \)-open, \( \theta \)-open, and regular open) set by \( \tau \)-open (resp., \( \alpha \)-open, \( s \)-open, \( p \)-open, \( sp \)-open, \( \gamma \)-open, \( \beta \)-open, \( \delta \)-open, \( \theta \)-open, and \( r \)-open). Similarly we will denote a fuzzy closed (resp., \( \alpha \)-closed, semiclosed, preclosed, semi-preclosed, \( \gamma \)-closed, \( \beta \)-closed, \( \delta \)-closed, \( \theta \)-closed, and regular closed) sets by \( \tau \)-closed (resp., \( \alpha \)-closed, \( s \)-closed, \( p \)-closed, \( sp \)-closed, \( \gamma \)-closed, \( \beta \)-closed, \( \delta \)-closed, \( \theta \)-closed, and \( r \)-closed). Let \( \Omega = \{ \tau, \alpha, s, p, sp, \gamma, \beta, \delta, \theta, r \} \).

**Definition 2.1.** A fuzzy set \( A \) in an fts \( (X, \tau) \) is said to be a fuzzy \( \psi -q \)-neighborhood of a fuzzy point \( x_r \) if and only if there exists a fuzzy \( \psi \)-open set \( U \) such that \( x_r, q U \leq A \). The family of all fuzzy \( \psi - q \)-neighborhoods of a fuzzy point \( x_r \) is denoted by \( N^\psi_q(x_r) \).

**Definition 2.2.** A fuzzy point \( x_r \) in an fts \( (X, \tau) \) is said to be a fuzzy \( \psi \)-cluster point of a fuzzy set \( A \) if and only if for every fuzzy \( \psi -q \)-neighborhood \( U \) of a fuzzy point \( x_r \), \( U q A \). The set of all fuzzy \( \psi \)-cluster points of fuzzy set \( A \) is called the fuzzy \( \psi \)-closure of \( A \) and is denoted by \( \psi \text{cl}(A) \). A fuzzy set \( A \) is fuzzy \( \psi \)-closed if and only if \( A = \psi \text{cl}(A) \) and a fuzzy set \( A \) is fuzzy \( \psi \)-open if and only if its complement is fuzzy \( \psi \)-closed.

**Theorem 2.3.** For a fuzzy set \( A \) in an fts \( (X, \tau) \),

\[
\psi \text{cl}(A) = \land \{ F : F \geq A, 1 - F \in \psi O(X) \}.
\]
Theorem 2.4. Let $A$ and $B$ be fuzzy sets in an fts $(X, \tau)$. Then the following statements are true:

1. $\psi cl(0) = 0, \psi cl(1) = 1$;
2. $A \leq \psi cl(A)$ for each fuzzy set $A$ of $X$;
3. if $A \leq B$, then $\psi cl(A) \leq \psi cl(B)$;
4. if $A$ is $\psi$-closed, then $A = \psi cl(A)$, and if one supposes $\psi cl(A)$ is $\psi$-closed, then the converse of (4) is true;
5. if $V \in \psi O(X)$, then $V q A$ if and only if $V q \psi cl(A)$;
6. $\psi cl(\psi cl(A)) = \psi cl(A)$;
7. $\psi cl(A) \lor \psi cl(A) \leq \psi cl(A \lor B)$. If the intersection of two fuzzy $\psi$-open sets is fuzzy $\psi$-open, then $\psi cl(A) \lor \psi cl(A) = \psi cl(A \lor B)$.

Proof. (1), (2), (3), and (4) are easily proved.

5. Let $V q A$. Then $A \leq 1 - V$, and hence $\psi cl(A) \leq \psi cl(1 - V) = 1 - V$, which implies $V q \psi cl(A)$. Hence $V q A$ if and only if $V q \psi cl(A)$.

6. Let $x_r$ be a fuzzy point with $x_r \not\in \psi cl(A)$. Then there is a fuzzy $\psi - q$-neighborhood $U$ of $x_r$ such that $U q A$. From (5) there is a fuzzy $\psi - q$-neighborhood $U$ of $x_r$ such that $U q \psi cl(A)$ and hence $x_r \not\in \psi cl(\psi cl(A))$. Thus $\psi cl(\psi cl(A)) \leq \psi cl(A)$. But $\psi cl(\psi cl(A)) \geq \psi cl(A)$, therefore $\psi cl(\psi cl(A)) = \psi cl(A)$.

7. It is clear.

Definition 2.5. For a fuzzy set $A$ in an fts $(X, \tau)$, we define a fuzzy $\psi$-interior of $A$ as follows:

$$\psi int(A) = \lor\{U : U \leq A, U \in \psi O(X)\}. \quad (2)$$

Theorem 2.6. Let $A$ and $B$ be fuzzy sets in an fts $(X, \tau)$. Then the following statements are true:

1. $\psi int(0) = 0, \psi int(1) = 1$;
2. $A \geq \psi int(A)$ for each fuzzy set $A$ of $X$;
3. if $A \leq B$, then $\psi int(A) \leq \psi int(B)$;
4. if $A$ is $\psi$-open, then $A = \psi int(A)$, if one supposes, $\psi int(A)$ is $\psi$-open, then the converse of (4) is true;
5. if $V \in \psi C(X)$, then $V q A$ if and only if $V q \psi int(A)$;
6. $\psi int(\psi int(A)) = \psi int(A)$;
7. $\psi int(A) \land \psi int(A) \geq \psi int(A \land B)$. If the intersection of two fuzzy $\psi$-open sets is $\psi$-open, then $\psi int(A) \land \psi int(A) = \psi int(A \land B)$.

Proof. It is similar to that of Theorem 2.4.
Theorem 2.7. For a fuzzy set \( A \) in an fts \((X, \tau)\), the following statements are true:

1. \( \psi cl(1 - A) = 1 - \psi int(A) \);
2. \( \psi int(1 - A) = 1 - \psi cl(A) \).

Proof. It follows from the fact that the complement of a fuzzy \( \psi \)-open set is fuzzy \( \psi \)-closed and \( \vee (1 - A_i) = 1 - \land A_i \).

Definition 2.8. Let \( A \) be a fuzzy set of an fts \((X, \tau)\). A fuzzy point \( x_r \) is said to be \( \psi \)-boundary of a fuzzy set \( A \) if and only if \( x_r \in \psi cl(A) \land (1 - \psi cl(A)) \). By \( \psi - \text{Bd}(A) \) one denotes the fuzzy set of all \( \psi \)-boundary points of \( A \).

Theorem 2.9. Let \( A \) be a fuzzy set of an fts \((X, \tau)\). Then

\[
A \lor \psi - \text{Bd}(A) \leq \psi cl(A).
\]

Proof. It follows from Definition 2.8 and Theorem 2.4.

3. Generalized \( \psi \rho \)-Closed and Generalized \( \psi \rho \)-Open Sets

Definition 3.1. Let \((X, \tau)\) be an fts. We define the concepts of fuzzy generalized \( \psi \rho \)-closed and fuzzy generalized \( \psi \rho \)-open sets, where \( \psi \) represents a fuzzy closure operation and \( \rho \) represents a notion of fuzzy openness as follows:

1. A fuzzy set \( A \) is said to be generalized \( \psi \rho \)-closed (\( g \psi \rho \)-closed, for short) if and only if \( \psi cl(A) \leq U \), whenever \( A \leq U \) and \( U \) is fuzzy \( \rho \)-open.

2. The complement of a fuzzy generalized \( \psi \rho \)-closed set is said to be fuzzy generalized \( \psi \rho \)-open (\( g \psi \rho \)-open, for short).

Remark 3.2. Note that each type of generalized closed set in Definition 2.8 is defined to be generalized \( \psi \rho \)-closed for some \( \psi \in \Omega \setminus \{r\} \) and \( \rho \in \Omega \). Namely, a fuzzy set \( A \) is fuzzy \( g \)-closed [3] if it is \( g \tau \)-closed, \( g a \)-closed [10] if it is \( g a a \)-closed, \( a g \)-closed [13] if it is \( a g a a \)-closed, \( g s \)-closed [12] if it is \( g s \tau \)-closed, \( s g \)-closed [5] if it is \( g s - s \)-closed, \( g p \)-closed [8] if it is \( g p \tau \)-closed, \( p g \)-closed [6] if it is \( g p - p \)-closed, \( g s p \)-closed [11] if it is \( g s p \tau \)-closed, \( s p g \)-closed [14] if it is \( g s p - s p \)-closed, \( g \theta \)-closed [4] if it is \( g \theta \tau \)-closed, \( \theta g \)-closed [7] if it is \( g \theta \theta \)-closed, and \( g r \)-closed [9] if it is \( g \tau r \)-closed.

Theorem 3.3. A fuzzy set \( A \) is generalized \( \psi \rho \)-open if and only if \( \psi int(A) \geq F \), whenever \( A \geq F \) and \( F \) is fuzzy \( \rho \)-closed.

Proof. It is clear.

Theorem 3.4. If \( A \) is a fuzzy \( \psi \)-closed set in an fts \((X, \tau)\), then \( A \) is fuzzy generalized \( \psi \rho \)-closed.

Proof. Let \( A \) be a fuzzy \( \psi \)-closed, and let \( U \) be a fuzzy \( \rho \)-open set in \( X \) such that \( A \leq U \). Then \( \psi cl(A) = A \leq U \), and hence \( A \) is fuzzy generalized \( \psi \rho \)-closed.
Remark 3.5. In classical topology, if $A$ is a generalized $\eta\rho$-closed set in a topological space $X$, then $\phi cl(A) \setminus A$ does not contain nonempty $\rho$-closed. But in fuzzy topology this is not true in general as shown by the following example.

Example 3.6. Let $\mu, \nu, \lambda, \eta,$ and $\sigma$ be fuzzy subsets of $X = \{x, y\}$ defined as follows:

$$
\begin{align*}
\mu(x) &= 0.25, & \mu(y) &= 0.70, \\
\nu(x) &= 0.65, & \nu(y) &= 0.35, \\
\lambda(x) &= 0.30, & \lambda(y) &= 0.30, \\
\eta(x) &= 0.30, & \eta(y) &= 0.36, \\
\sigma(x) &= 0.20, & \sigma(y) &= 0.20.
\end{align*}
$$

Let $\tau = \{0, \mu, \nu, \mu \wedge \nu, \mu \vee \nu, 1\}$ be a fuzzy topology on $X$.

One may notice that the following:
1. $\eta$ is a fuzzy $g\theta$-closed set and
   \begin{align*}
   &\theta-cl(\eta)(x) = 0.35, & \theta-cl(\eta)(y) = 0.65, \\
   &\left(\theta-cl(\eta) \setminus \eta\right)(x) = 0.35, & \left(\theta-cl(\eta) \setminus \eta\right)(y) = 0.64.
   \end{align*}

But $\left(\theta-cl(\eta) \setminus \eta\right)$ contains nonempty fuzzy closed $(\mu \vee \nu)^c$.

2. $\lambda$ is a fuzzy generalized closed set and
   \begin{align*}
   & Cl(\lambda)(x) = 0.35, & Cl(\lambda)(y) = 0.30, \\
   &\left(Cl(\lambda) \setminus \lambda\right)(x) = 0.35, & \left(Cl(\lambda) \setminus \lambda\right)(y) = 0.30.
   \end{align*}

But $\left(Cl(\lambda) \setminus \lambda\right)$ contains nonempty closed set $(\mu \vee \nu)^c$.

3. $\lambda$ is a fuzzy $ag$-closed (resp., $sg$-closed, $pg$-closed, $yg$-closed, $spg$-closed) set and
   \begin{align*}
   &\alpha-cl(\lambda)(x) = 0.30, & \alpha-cl(\lambda)(y) = 0.30, \\
   &\left(\alpha-cl(\lambda) \setminus \lambda\right)(x) = 0.30, & \left(\alpha-cl(\lambda) \setminus \lambda\right)(y) = 0.30, \\
   &\alpha-cl(\lambda) = Scl(\lambda) = Pcl(\lambda) = Spcl(\lambda),
   \end{align*}

and hence

$$
\begin{align*}
\left(\alpha-cl(\lambda) \setminus \lambda\right) = (Scl(\lambda) \setminus \lambda) = (Pcl(\lambda) \setminus \lambda) = (Spcl(\lambda) \setminus \lambda).
\end{align*}
$$
But \((\alpha-\text{cl}(\lambda) \setminus \lambda)\) contains nonempty set \(\sigma\) which is fuzzy \(\alpha\)-closed and hence is fuzzy semiclosed, preclosed, and semi-preclosed, and so on.

**Theorem 3.7.** Let \((X, \tau)\) be an fts, and let \(A\) be a fuzzy g\(\psi\)-closed set with \(A \leq B \leq \varrho\text{cl}(A)\). Then \(B\) is a fuzzy g\(\psi\)-closed set.

**Proof.** Let \(H\) be a fuzzy \(\rho\)-open set in \(X\) such that \(B \leq H\). Then \(A \leq H\). Since \(A\) is fuzzy g\(\psi\)-closed, then \(\varrho\text{cl}(A) \leq H\), and hence \(\varrho\text{cl}(B) \leq \varrho\text{cl}(A)\). Thus \(\varrho\text{cl}(B) \leq H\), and hence \(B\) is a fuzzy g\(\psi\)-closed set \(\square\).

**Theorem 3.8.** Let \((X, \tau)\) be an fts, and let \(A\) be a fuzzy g\(\psi\)-open set with \(\varrho\text{int}(A) \leq B \leq A\). Then \(B\) is a fuzzy g\(\psi\)-open set.

**Proof.** It is similar to that of Theorem 3.7. \(\square\)

**Theorem 3.9.** Let \(A\) be a fuzzy set in an fts, \((X, \tau)\) and let \(\varrho\text{cl}(A)\) be \(\rho\)-closed for each fuzzy set \(A\). Then \(A\) is fuzzy g\(\psi\)-closed if and only if for each fuzzy point \(x_r\) with \(x_r, \varrho\text{qcl}(A)\), one has \(\varrho\text{cl}(x_r) \varrho A\).

**Proof.** Let \(x_r, \varrho\text{qcl}(A)\) and suppose that \(\varrho\text{cl}(x_r) \varrho A\). Since \(\varrho\text{cl}(x_r)\) is \(\rho\)-closed, then \((\varrho\text{cl}(x_r))^c\) is fuzzy \(\rho\)-open and \(A \leq (\varrho\text{cl}(x_r))^c\). Since \(A\) is fuzzy g\(\psi\)-closed, then \(\varrho\text{cl}(A) \leq (\varrho\text{cl}(x_r))^c\) and hence \(\varrho\text{cl}(x_r) \varrho\varrho\text{cl}(A)\) which contradict with \(x_r, \varrho\text{qcl}(A)\) and hence \(\varrho\text{cl}(x_r) \varrho A\).

Conversely, let \(B\) be fuzzy \(\rho\)-open set with \(A \leq B\) and let \(x_r, \varrho\text{qcl}(A)\). By hypothesis \(\varrho\text{cl}(x_r) \varrho A\), and hence there is \(y \in X\) such that \(\varrho\text{cl}(x_r)(y) + A(y) \geq 1\). Put \(\varrho\text{cl}(x_r)(y) = s\). Then \(y_s \in \varrho\text{cl}(x_r), y_s A\) and hence \(y_s B\). Since \(y_s \in \varrho\text{cl}(x_r)\), \(B\) is a fuzzy \(\rho\)-open set and \(y_s B\), then \(x_r B\). Hence \(\varrho\text{cl}(A) \leq B\). Thus \(A\) is fuzzy g\(\psi\)-closed. \(\square\)

**Theorem 3.10.** Let \((X, \tau)\) be an fts, and let \(A\) be a fuzzy set in \(X\). Then the following are equivalent:

1. \(A\) is fuzzy g\(\psi\)-closed;

2. if \(A\) is fuzzy \(\rho\)-open, then \(A\) is fuzzy \(\psi\)-closed.

**Proof.** (1) \(\rightarrow\) (2). Let \(A\) be fuzzy g\(\psi\)-closed and fuzzy \(\rho\)-open with \(A \leq A\). Then \(\varrho\text{cl}(A) \leq A\). Since \(A \leq \varrho\text{cl}(A)\), then \(A = \varrho\text{cl}(A)B\). Therefore \(A\) is \(\varrho\)-closed.

(2) \(\rightarrow\) (1). Let \(A\) be a fuzzy set with \(A \leq B\), where \(B\) is fuzzy \(\rho\)-open set in \(X\). From (2) we have \(B\) is \(\varrho\)-closed, and hence \(\varrho\text{cl}(A) = A \leq B\). Thus \(A\) is fuzzy g\(\psi\)-closed. \(\square\)

**Theorem 3.11.** Let \((X, \tau)\) be an fts and suppose that \(x_r\) and \(y_s\) are weak and strong fuzzy points, respectively. If \(x_r\) is fuzzy g\(\psi\)-closed and \(\varrho\text{cl}(y_s)\) is fuzzy \(\rho\)-closed, then

\[y_s \in \varrho\text{cl}(x_r) \implies x_r \in \varrho\text{cl}(y_s).\]  

(6)

**Proof.** Let \(y_s \in \varrho\text{cl}(x_r)\) and \(x_r \notin \varrho\text{cl}(y_s)\). Then \(r > \varrho\text{cl}(y_s)\). Since \(x_r\) is a weak fuzzy point, then \(r \leq 1/2\), and hence \(\varrho\text{cl}(y_s) \leq 1 - r\). Thus \(r \leq (\varrho\text{cl}(y_s))^c(x)\). So \(x_r \in (\varrho\text{cl}(y_s))^c\). Since \(x_r\) is fuzzy g\(\psi\)-closed and \((\varrho\text{cl}(y_s))^c\) is fuzzy \(\rho\)-open, then \(\varrho\text{cl}(x_r) \leq (\varrho\text{cl}(y_s))^c\), and hence \(y_s \in (\varrho\text{cl}(y_s))^c\), which is a contradiction, since if \(y_s \in (\varrho\text{cl}(y_s))^c\), then \((\varrho\text{cl}(y_s))(y) \leq 1 - s\). But since \(y_s \in (\varrho\text{cl}(y_s))\), then \(s \leq (\varrho\text{cl}(y_s))(y)\), and hence \(s \leq 1 - s\), which implies \(s \leq 1/2\). Since \(y_s\) is fuzzy strong point, then \(s > 1/2\), which is a contradiction. Thus \(x_r \in \varrho\text{cl}(y_s)\). \(\square\)
Definition 3.12. Let \((X, \tau)\) be an fts. A fuzzy point \(x_r\) is said to be fuzzy just-\(q\)-closed if the fuzzy set \(\psi cl(x_r)\) is a fuzzy point.

Theorem 3.13. Let \((X, \tau)\) be an fts. If \(x_r, x_s\) are two fuzzy points such that \(r < s\) and \(x_s\) is fuzzy \(\rho\)-open, then \(x_r\) is fuzzy just-\(q\)-closed if it is fuzzy \(gq\rho\)-closed.

Proof. Let \(x_r < x_s, x_s\) be fuzzy \(\rho\)-open, and let \(x_r\) be fuzzy \(gq\rho\)-closed. Then \(\psi cl(x_r) \leq x_s\), and hence \((\psi cl(x_r))(x) \leq s\) and \((\psi cl(x_r))(z) = 0\) for each \(z \in X \setminus \{x\}\). Thus \(\psi cl(x_r)\) is a fuzzy point. Therefore \(x_r\) is fuzzy just-\(q\)-closed. \(\square\)

Definition 3.14. Let \((X, \tau)\) be an fts. A fuzzy set \(U\) of \(X\) is called fuzzy \(q\)-nearly crisp if \(\psi cl(U) \wedge (\psi cl(U))^C = 0\).

Theorem 3.15. If \(A\) is fuzzy \(gq\rho\)-closed and fuzzy \(q\)-nearly crisp of an fts \((X, \tau)\), then \(\psi cl(A) \setminus A\) does not contain any nonempty fuzzy \(\rho\)-closed set in \(X\).

Proof. Suppose that \(A\) is a fuzzy \(gq\rho\)-closed set in \(X\), and let \(F\) be a fuzzy \(\rho\)-closed set such that \(F \leq \psi cl(A) \setminus A\) and \(F \neq 0\). Then \(A \leq F^C\) and \(F^C\) is fuzzy \(\rho\)-open. Since \(A\) is a fuzzy \(gq\rho\)-closed, then \(\psi - cl(A) \leq F^C\), and hence \(F \leq (\psi cl(A))^C\), so \(F \leq \psi cl(A) \wedge (\psi cl(A))^C = 0\). Therefore, \(F = 0\), which is contradiction. Hence \(\psi cl(A) \setminus A\) does not contain any nonempty fuzzy \(\rho\)-closed set in \(X\). \(\square\)

Theorem 3.16. Let \((X, \tau)\) be an fts. Then every fuzzy \(\rho\)-open set is fuzzy \(q\)-closed if and only if every fuzzy subset of \(X\) is fuzzy \(gq\rho\)-closed.

Proof. Suppose that \(U\) be a fuzzy \(\rho\)-open set and \(A\) be any fuzzy subset of \(X\) such that \(A \leq U\). By hypothesis, \(U\) is fuzzy \(q\)-closed, and hence \(\psi cl(A) \leq \psi cl(U) \leq U\). Thus \(A\) is fuzzy \(gq\rho\)-closed.

Conversely, suppose that every fuzzy subset of \(X\) is fuzzy \(gq\rho\)-closed and \(U\) is a fuzzy \(\rho\)-open set. Since \(U \leq U\) and \(U\) is fuzzy \(gq\rho\)-closed, then \(\psi cl(U) \leq U\) and hence \(\psi cl(U) = U\). Thus \(U\) is fuzzy \(q\)-closed. \(\square\)

Theorem 3.17. If \(A\) is fuzzy \(gq\rho\)-open and fuzzy \(q\)-nearly crisp of an fts \((X, \tau)\), then \(G = 1\), where \(G\) is a fuzzy \(\rho\)-open and \(qint(A) \lor A^C \leq G\).

Proof. Suppose that \(A\) is a fuzzy \(gq\rho\)-open set in \(X\), and let \(G\) be a fuzzy \(\rho\)-open set such that \(qint(A) \lor A^C \leq G\). Then \(G^C \leq (qint(A) \lor A^C)^C = (qint(A))^C \land A\). That is \(G^C \leq (qint(A))^C \land A^C\), and hence \(G^C \leq \psi cl(A^C) \setminus A^C\). Since \(G^C\) is fuzzy \(\rho\)-closed and \(A^C\) is fuzzy \(gq\rho\)-closed, then by Theorem 3.15, we have \(G^C = 0\). Hence \(G = 1\). \(\square\)

Definition 3.18. An fts \((X, \tau)\) is said to be fuzzy \(q\rho\)-regular if for each fuzzy point \(x_r\) and a fuzzy \(\rho\)-closed set \(F\) not containing \(x_r\), there is \(U, V \in \psi O(X)\) such that \(x_r \in U, F \leq V,\) and \(U \supseteq V\).

Theorem 3.19. If \((X, \tau)\) is a fuzzy \(q\rho\)-regular space, then for each strong fuzzy \(x_r\) and a fuzzy \(\rho\)-open set \(U\) containing \(x_r\), there is \(V \in \psi O(X)\) such that \(x_r \in V\) and \(\psi cl(V) \leq U\).

Proof. It is clear. \(\square\)
Theorem 3.20. If \((X, \tau)\) is a fuzzy \(\varphi\rho\)-regular space, then each strong fuzzy point in \(X\) is fuzzy \(g\varphi\rho\)-closed.

Proof. Let \(x_r\) be strong fuzzy point in \(X\), and let \(U\) be a fuzzy \(\rho\)-open set such that \(x_r \leq U\). Then by Theorem 3.19, there is a fuzzy \(\varphi\)-open set \(W\) such that \(x_r \in W\) and \(\varphi cl(W) \leq U\), and hence \(\varphi cl(x_r) \leq \varphi cl(W) \leq U\). Thus \(x_r\) is fuzzy \(g\varphi\rho\)-closed.

Theorem 3.21. A fuzzy set \(A\) in an fts \((X, \tau)\) is fuzzy \(g\varphi\rho\)-closed if and only if \(A\bar{q}E \Rightarrow \varphi cl(A)\bar{q}E\) for each fuzzy \(\rho\)-closed set \(E\) of \(X\).

Proof. Let \(E\) be a fuzzy \(\rho\)-closed set of \(X\) and \(A\bar{q}E\). Then \(A \leq 1 - E\), and \(1 - E\) is fuzzy \(\rho\)-open in \(X\). Since \(A\) is fuzzy \(g\varphi\rho\)-closed, then \(\varphi cl(A) \leq 1 - E\) and hence \(\varphi cl(A)\bar{q}E\).

Conversely, let \(B\) be a fuzzy \(\rho\)-open set of \(X\) such that \(A \leq H\). Then \(A\bar{q}(1 \setminus B)\) and \(1 \setminus B\) is fuzzy \(\rho\)-closed in \(X\). By hypothesis \(\varphi cl(A)\bar{q}(1 \setminus B)\), which implies \(\varphi cl(A) \leq B\). Hence \(A\) is fuzzy \(g\varphi\rho\)-closed in \(X\).

Definition 3.22. An fts \((X, \tau)\) is said to be fuzzy quasi-\(\varphi\)-\(T_1\) if for all fuzzy points \(x_r\) and \(y_s\) with \(x \neq y\), there exist two fuzzy \(\varphi\)-open sets \(U, V\) such that \(x_r \in U\) and \(y_s \notin U\), \(x_r \notin V\), and \(y_s \in V\).

Definition 3.23. Let \((X, \tau)\) be an fts. A fuzzy point \(x_r\) is said to be well \(\varphi\)-closed if there exists \(y_s \in \varphi cl(x_r)\) such that \(x \neq y\).

Theorem 3.24. If \((X, \tau)\) is an fts and \(x_r\) is fuzzy \(g\varphi\rho\)-closed, well \(\varphi\)-closed fuzzy point, then \(X\) is not fuzzy quasi-\(\varphi\)-\(T_1\).

Proof. Let \(X\) be a fuzzy quasi-\(\varphi\) - \(T_1\) space and \(x_r\) is well \(\varphi\)-closed. Then there exists a fuzzy point \(y_s\) with \(x \neq y\) such that \(y_s \in \varphi cl(x_r)\), and hence there exists a fuzzy \(\varphi\)-open \(U\) such that \(x_r \in U\) and \(y_s \notin U\). Since \(x_r\) is fuzzy \(g\varphi\rho\)-closed, then \(\varphi cl(x_r) \leq U\), and hence \(y_s \in U\). This is a contradiction, and hence \(X\) is not quasi-\(\varphi\) - \(T_1\).

Theorem 3.25. If \(A\) is a fuzzy \(\rho\)-open and a fuzzy \(g\varphi\rho\)-closed in an fts \((X, \tau)\) and \(\varphi cl(A)\) is \(\varphi\)-closed, then \(A\) is \(\varphi\)-closed.

Proof. Let \(A\) be fuzzy \(\rho\)-open and fuzzy \(g\varphi\rho\)-closed in \(X\). Then \(\varphi cl(A) \leq A\), and hence \(\varphi cl(A) = A\). Therefore \(A\) is \(\varphi\)-closed.

Theorem 3.26. If \(\varphi cl(A) \vee \varphi cl(A) = \varphi cl(A \vee B)\), then the union of two fuzzy \(g\varphi\rho\)-closed sets in an fts \((X, \tau)\) is \(g\varphi\rho\)-closed.

Proof. Let \(A\) and \(B\) be fuzzy \(g\varphi\rho\)-closed sets in an fts \((X, \tau)\), and let \(U\) be fuzzy \(\rho\)-open such that \(A \vee B \leq U\). Then \(A \leq U\) and \(B \leq U\), and hence \(\varphi cl(A) \leq U\) and \(\varphi cl(B) \leq U\). Since \(\varphi cl(A) \vee \varphi cl(A) = \varphi cl(A \vee B)\), then \(\varphi cl(A \vee B) \leq U\), and hence \(A \vee B\) is fuzzy \(g\varphi\rho\)-closed in \(X\).

Remark 3.27. Some results in papers [3–15] can be considered as special results from our results in this paper.
### Table 1

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### 4. Summary

The results are summarized in the following table. Each cell gives the type of generalized closed set which is \( g^\psi \rho \)-closed, where \( \psi \) (closure) is given by the left-hand (zeroth) column and \( \rho \) (openness) is given by the top (zeroth) row.

The table highlights some general relationships between certain groups of generalized closed sets. For example, column 2 implies column 1. (Each type of generalized closed set listed in column 2 implies the type of generalized closed set listed in the same row of column 1.) In fact each column in Table 1 implies each of the preceding column apart from columns 6 and 7. Each of these implications, apart from columns 6 and 7, follows immediately from the definitions, since the types of generalized closed sets in any particular row involve the same notion of closure, and these notions of closure decrease in strength from top to down, apart from rows 5 and 6. Similarly each row implies each subsequent row, apart from rows 5 and 6.

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