Research Article

Product of Extended Cesàro Operator and Composition Operator from Lipschitz Space to $F(p,q,s)$ Space on the Unit Ball

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Received 16 January 2011; Accepted 16 March 2011

Academic Editor: Ljubisa Kocinac

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This paper characterizes the boundedness and compactness of the product of extended Cesàro operator and composition operator from Lipschitz space to $F(p,q,s)$ space on the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{B}$ be the unit ball in the $n$-dimensional complex space $\mathbb{C}^n$, the closure of $\mathbb{B}$ will be written as $\overline{\mathbb{B}}$. By $d\nu$ we denote the Lebesgue measure on $\overline{\mathbb{B}}$ normalized so that $\nu(\overline{\mathbb{B}}) = 1$ and by $d\sigma$ the normalized rotation invariant measure on the boundary $S = \partial \mathbb{B}$ of $\mathbb{B}$. Let $H(\mathbb{B})$ be the class of all holomorphic functions on $\mathbb{B}$ and $S(\mathbb{B})$ the collection of all the holomorphic self-mappings of $\mathbb{B}$. Denote by $A(\mathbb{B})$ the unit ball algebra of all continuous functions on $\overline{\mathbb{B}}$ that are holomorphic on $\mathbb{B}$.

For $f \in H(\mathbb{B})$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of $f$.

We recall that the $\alpha$-Bloch space $\mathcal{B}^\alpha$ for $\alpha \geq 0$ consists of all $f \in H(\mathbb{B})$ such that

$$\mathcal{B}^\alpha(f) = \sup_{z \in \mathbb{B}} \left(1 - |z|^2\right)^\alpha |\Re f(z)| < \infty.$$
The expression $\mathcal{B}_a(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}_a} = |f(0)| + \mathcal{B}_a(f)$. This norm makes $\mathcal{B}_a$ into a Banach space. When $a = 1$, $\mathcal{B}_1 = \mathcal{B}$ is the well known Bloch space.

For $a \in (0, 1)$, $\mathcal{L}_{a}(\mathbb{B})$ denotes the holomorphic Lipschitz space of order $a$ which is the set of all $f \in H(\mathbb{B})$ such that, for some $C > 0$,

$$|f(z) - f(w)| \leq C|z - w|^a$$

for every $z, w \in \mathbb{B}$. It is clear that each space $\mathcal{L}_{a}(\mathbb{B})$ contains the polynomials and is contained in the ball algebra $A(\mathbb{B})$. It is well known that $\mathcal{L}_{a}(\mathbb{B})$ is endowed with a complete norm $\|\cdot\|_{\mathcal{L}_{a}}$ that is given by

$$\|f\|_{\mathcal{L}_{a}} = |f(0)| + \sup_{z \neq w, z, w \in \mathbb{B}} \left\{ \frac{|f(z) - f(w)|}{|z - w|^a} \right\}.$$  

(1.4)

See [1, 2] for more information of the Lipschitz spaces on $\mathbb{B}$.

For $a \in \mathbb{B}$, let $g(z, a) = \log |\varphi_a(z)|^{-1}$ be Green’s function on $\mathbb{B}$ with logarithmic singularity at $a$, where $\varphi_a$ is the Möbius transformation of $\mathbb{B}$ with $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a = \varphi_a^{-1}$.

Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, a function $f \in H(\mathbb{B})$ is said to belong to $F(p, q, s)$ if (see, e.g., [3–5])

$$\|f\|_{F(p, q, s)}^p = |f(0)|^p + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\Re f(z)|^p \left(1 - |z|^2\right)^{q} g^s(z, a) d\nu(z) < \infty.$$  

(1.5)

If $X$ is a Banach space of holomorphic functions on a domain $\Omega$ and if $\varphi$ is a (holomorphic) self-map of $\Omega$, the composition operator of symbol $\varphi$ is defined by $C_{\varphi}(f) = f \circ \varphi$. The study of composition operators consists in the comparison of the properties of the operator $C_{\varphi}$ with that of the function $\varphi$ itself, which is called the symbol of $C_{\varphi}$. One can characterize boundedness and compactness of $C_{\varphi}$ and many other properties. We refer to the books in [6, 7] and to some recent papers in [4, 5, 8] to learn much more on this subject.

Let $h \in H(\mathbb{B})$, the following integral-type operator was first introduced in [9]

$$T_h f(z) = \int_{0}^{1} f(tz) \Re h(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}.$$  

(1.6)

This operator is called generalized Cesàro operator. It has been well studied in many papers, see, for example, [3, 9–24] as well as the related references therein.

It is natural to discuss the product of extended Cesàro operator and composition operator. For $h \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$, the product can be expressed as

$$T_h C_{\varphi} f(z) = \int_{0}^{1} f(\varphi(tz)) \Re h(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}.$$  

(1.7)
It is interesting to characterize the boundedness and compactness of the product operator on all kinds of function spaces. Even on the disk of \(\mathbb{C}\), some properties are not easily managed; see some recent papers in [18, 25–28].

Building on those foundations, the present paper continues this line of research and discusses the operator in high dimension. The remainder is assembled as follows: in Section 2, we state a couple of lemmas. In Section 3, we characterize the boundedness and compactness of the product \(T_hC_{\varphi}\) of extended Cesàro operator and composition operator from Lipschitz spaces to \(F(p,q,s)\) spaces on the unit ball of \(\mathbb{C}^n\).

Throughout the remainder of this paper, \(C\) will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation \(A \asymp B\) means that there is a positive constant \(C\) such that \(B/C \leq A \leq CB\).

2. Some Lemmas

To begin the discussion, let us state a couple of lemmas, which are used in the proofs of the main results.

Lemma 2.1. Suppose that \(f, h \in H(\mathbb{B})\). Then,

\[
\mathcal{N}[T_hC_{\varphi}(f)](z) = f(\varphi(z))\mathcal{N}h(z).
\] (2.1)

Proof. The proof of this Lemma follows by standard arguments (see, e.g., [9, 29, 30]). \(\square\)

Lemma 2.2 (see [2, 31]). If \(0 < \alpha < 1\), then \(\mathcal{B}^{1-\alpha} = \mathcal{L}_{\alpha}(\mathbb{B})\); furthermore,

\[
\|f\|_{\mathcal{B}^{1-\alpha}} \asymp \|f\|_{\mathcal{L}_{\alpha}}
\] (2.2)

as \(f\) varies through \(\mathcal{L}_{\alpha}(\mathbb{B})\).

The following criterion for compactness follows from standard arguments similar to the corresponding lemma in [6]. Hence, we omit the details.

Lemma 2.3. Assume that \(h \in H(\mathbb{B})\) and \(\varphi \in S(\mathbb{B})\). Suppose that \(X\) or \(Y\) is one of the following spaces \(\mathcal{L}_{\alpha}(\mathbb{B})\), \(F(p,q,s)\). Then, \(T_hC_{\varphi} : X \to Y\) is compact if and only if \(T_hC_{\varphi} : X \to Y\) is bounded, and for any bounded sequence \(\{f_k\}_{k\in\mathbb{N}}\) in \(X\) which converges to zero uniformly on compact subsets of \(\mathbb{B}\) as \(k \to \infty\), one has \(\|T_hC_{\varphi}f_k\|_Y \to 0\) as \(k \to \infty\).

Lemma 2.4 (see [4, 5]). If \(f \in \mathcal{B}^\alpha\), then

\[
|f(z)| \leq C\|f\|_{\mathcal{B}^\alpha}, \quad 0 < \alpha < 1,
\] (2.3)

\[
|f(z)| \leq C\|f\|_{\mathcal{B}^1} \ln \frac{e}{1-|z|^2}, \quad \alpha = 1,
\] (2.3′)

\[
|f(z)| \leq C\frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, \quad \alpha > 1.
\] (2.3″)

The next lemma was obtained in [32].
Lemma 2.5. If \( a > 0, b > 0 \), then the elementary inequality holds

\[
(a + b)^p \leq \begin{cases} 
  a^p + b^p, & 0 < p < 1, \\
  2^{p-1} (a^p + b^p), & p \geq 1.
\end{cases}
\]  

(2.4)

It is obvious that Lemma 2.5 holds for the sum of finite number \( k \), that is,

\[
(a_1 + \cdots + a_k)^p \leq C \left( a_1^p + \cdots + a_k^p \right),
\]  

(2.5)

where \( a_1, \ldots, a_k > 0 \) and \( C \) is a positive constant.

Lemma 2.6 (see [4, 5]). For \( 0 < p, s < +\infty, -n - 1 < q < +\infty, q + s > -1 \), there exists \( C > 0 \) such that

\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^p}{|1 - (z, w)|^{n+1+q+p}} \left( 1 - |z|^2 \right)^q |g^s(z, a)| d\nu(z) \leq C
\]  

(2.6)

for every \( \omega \in \mathbb{B} \).

Lemma 2.7 (see [4]). There is a constant \( C > 0 \) so that, for all \( t > -1 \) and \( z \in \mathbb{B} \), one has

\[
\int_{\mathbb{B}} \left| \ln \frac{1}{1 - (z, w)} \right|^2 \frac{(1 - |w|^2)^t}{|1 - (z, w)|^{n+1+q+p}} d\nu(z) \leq C \left( \ln \frac{1}{1 - |z|^2} \right)^2.
\]  

(2.7)

Lemma 2.8 (see [4, 5]). Suppose that \( 0 < p, s < \infty, -n - 1 < q < \infty, \) and \( q + s > -1 \). If \( f \in F(p, q, s) \), then \( f \in \mathcal{B}^{(n+1+q)/p} \), and \( \|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C \|f\|_{F(p, q, s)} \).

Lemma 2.9. Let \( \{f_k\}_{k \in \mathbb{N}} \) be a bounded sequence in \( F(p, q, s) \) which converges to zero uniformly on compact subsets of the unit ball \( \mathbb{B} \), where \( (n + 1 + q)/p < 1 \). Then, \( \lim_{k \to \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0 \).

Proof. It follows from Lemma 2.8 that \( F(p, q, s) \subseteq \mathcal{B}^{(n+1+q)/p} \) and \( \|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C \|f\|_{F(p, q, s)} \) for any \( f \in F(p, q, s) \). So, when \( (n + 1 + q)/p < 1 \), the proof of this lemma is similar to that of Lemma 3.6 of [33], hence the proof is omitted.

3. The Boundedness and Compactness of the Operator \( T_h \mathcal{C}_\varphi : \mathcal{L}_\alpha(\mathbb{B}) \to F(p, q, s) \)

Theorem 3.1. Assume that \( \alpha \in (0, 1), 0 < p, s < \infty, -n - 1 < q < \infty, q + s > -1, \varphi \in S(\mathbb{B}), \) and \( h \in H(\mathbb{B}) \). Then, \( T_h \mathcal{C}_\varphi : \mathcal{L}_\alpha \to F(p, q, s) \) is bounded if and only if \( h \in F(p, q, s) \).

Proof. Assume that \( h \in F(p, q, s) \). Since \( 0 < 1 - \alpha < 1 \), by Lemmas 2.2 and 2.4, for any \( f \in \mathcal{L}_\alpha \), we have

\[
|f(z)| \leq C \|f\|_{\mathcal{L}_\alpha} \leq C \|f\|_{\mathcal{L}_\alpha}.
\]  

(3.1)
Since $|T_h C_\psi f(0)| = 0$, by using Lemma 2.1 and relations (2.3) and (3.1), we have

\[
\|T_h C_\psi f\|_{F(p,q,s)}^p = \sup_{a \in B} \int_B |f(\varphi(z))\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z)
\]

\[
\leq C \sup_{a \in B} \int_B |\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z) \|f\|_{L_1}^p - \alpha \leq C \|h\|_{F(p,q,s)}^p \|f\|_{L_s}^p < \infty.
\]

Thus $T_h C_\psi : L_\alpha \to F(p, q, s)$ is bounded.

Conversely, suppose that $T_h C_\psi : L_\alpha \to F(p, q, s)$ is bounded. Taking the function $f(z) = 1 \in L_\alpha$, then

\[
\|T_h C_\psi f\|_{F(p,q,s)}^p = |T_h C_\psi f(0)|^p + \sup_{a \in B} \int_B |\Re (T_h C_\psi f)(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z)
\]

\[
= \sup_{a \in B} \int_B |f(\varphi(z))\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z)
\]

\[
= \sup_{a \in B} \int_B |\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z) = \|h\|_{F(p,q,s)}^p.
\]

From which, the boundedness of $T_h C_\psi$ implies that $h \in F(p, q, s)$. This completes the proof of this theorem.

Next, we characterize the compactness of $T_h C_\psi : L_\alpha \to F(p, q, s)$.

**Theorem 3.2.** Assume that $\alpha \in (0, 1)$, $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$, $\varphi \in S(B)$, and $h \in H(B)$. Then, $T_h C_\psi : L_\alpha \to F(p, q, s)$ is compact if and only if $T_h C_\psi : L_\alpha \to F(p, q, s)$ is bounded, and

\[
\lim_{r \to 1} \sup_{\alpha \in B} \int_{|\varphi(z)| > r} |\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z) = 0.
\]

**Proof.** Assume that $T_h C_\psi : L_\alpha \to F(p, q, s)$ is bounded and (3.4) holds. It follows from Theorem 3.1 that $h \in F(p, q, s)$.

Now, let $\{f_j\}_{j \in \mathbb{N}}$ be a bounded sequence of functions in $L_\alpha$ such that $f_j \to 0$ uniformly on the compact subsets of $B$ as $j \to \infty$. Suppose that $\sup_{j \in \mathbb{N}} \|f_j\|_{L_\alpha} \leq L$. It follows from (3.4) that, for any $\varepsilon > 0$, there exists $r_0 \in (0, 1)$ such that, for every $r_0 < r < 1$,

\[
\sup_{\alpha \in B} \int_{|\varphi(z)| > r} |\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z,a) d\nu(z) < \varepsilon.
\]
Set \( r_0 < r < 1 \), then

\[
\| T_h C_f \|_{F(p,q,s)}^p = \sup_{a \in B} \int_B |f_j(\varphi(z))|^p |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z)
\]

\[
\leq \sup_{a \in B} \int_{|\varphi(z)| \leq r} \left| f_j(\varphi(z)) \right|^p |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z)
\]

\[
+ \sup_{a \in B} \int_{|\varphi(z)| > r} \left| f_j(\varphi(z)) \right|^p |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z)
\]

\[
= I_1 + I_2,
\]

(3.6)

where

\[
I_1 := \sup_{a \in B} \int_{|\varphi(z)| \leq r} \left| f_j(\varphi(z)) \right|^p |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z),
\]

\[
I_2 := \sup_{a \in B} \int_{|\varphi(z)| > r} \left| f_j(\varphi(z)) \right|^p |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z).
\]

(3.7)

Let \( K = \{ w : |w| \leq r \} \), then \( K \) is a compact subset of \( B \). Since \( f_j \to 0 \) uniformly on compact subsets of \( B \) as \( j \to \infty \) and \( h \in F(p,q,s) \), we get

\[
I_1 \leq \sup_{w \in K} |f_j(w)|^p \sup_{a \in B} \int_{|\varphi(z)| \leq r} |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z)
\]

\[
\leq \|h\|_{F(p,q,s)}^p \sup_{w \in K} |f_j(w)|^p \leq C \sup_{w \in K} |f_j(w)|^p \to 0, \quad j \to \infty.
\]

(3.8)

On the other hand, by (3.5) and Lemmas 2.2 and 2.4, it follows that

\[
I_2 \leq C \| f_j \|_{L_a}^p \sup_{a \in B} \int_{|\varphi(z)| > r} |\Re h(z)|^p \left(1 - |z|^2\right)^q g^a(z,a) \, dv(z)
\]

\[
\leq C \| f_j \|_{L_a}^p \varepsilon \leq C \| f_j \|_{L_a}^p \varepsilon.
\]

(3.9)

Since \( \varepsilon \) is arbitrary, from the above inequalities, we get

\[
\lim_{j \to \infty} \| T_h C_f \|_{F(p,q,s)} = 0.
\]

(3.10)

Hence, by (3.10) and Lemma 2.3, we conclude that \( T_h C_f : \mathcal{L}_a \to F(p,q,s) \) is compact.

For the converse direction, we suppose that \( T_h C_f : \mathcal{L}_a \to F(p,q,s) \) is compact. It is obvious that \( T_h C_f : \mathcal{L}_a \to F(p,q,s) \) is bounded.

Now, we prove (3.4). Setting the test functions \( f^{(m)}_l(z) = z^m_l \) for fixed \( l \in \{1, \ldots, n\} \), where \( z = (z_1, \ldots, z_n) \) and \( m = 1,2,\ldots \). It is easy to check that \( \| f^{(m)}_l \|_{\mathcal{L}_a} \leq C \), and \( f^{(m)}_l \to 0 \)
uniformly on the compact subsets of \( \mathbb{B} \) as \( m \to \infty \). Write \( \varphi = (\varphi_1, \ldots, \varphi_n) \), since \( T_\alpha C_\varphi : \mathcal{L}_\alpha \to F(p, q, s) \) is compact, by Lemma 2.3, it follows that, as \( m \to \infty \),

\[
\left\| T_\alpha C_\varphi f^{(m)}_1 \right\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi_i(z)|^{mp} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z) \to 0.
\] (3.11)

Note that \(|\varphi(z)|^2 = |\varphi_1(z)|^2 + \cdots + |\varphi_n(z)|^2 \leq (|\varphi_1(z)| + \cdots + |\varphi_n(z)|)^2\); by the relation (3.11) and Lemma 2.5, we have

\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^{mp} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\]
\[
\leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{i=1}^n |\varphi_i(z)|^p \right)^m |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\] (3.12)

\[
\leq C \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{i=1}^n |\varphi_i(z)|^{mp} \right)^m |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z) \to 0, \quad m \to \infty.
\]

This means that, for every \( \varepsilon > 0 \), there is \( m_0 \in \mathbb{N} \) such that, for every \( r \in (0, 1) \),

\[
r^{mp} \sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > r} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\]
\[
= \sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > r} r^{mp} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\]
\[
\leq \sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > r} |\varphi(z)|^{mp} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\] (3.13)

\[
\leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^{mp} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z)
\]
\[
< \varepsilon.
\]

Thus, when \( r > 2^{-(1/mp)} \), by the above inequality, we obtain

\[
\sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > r} |\Re h(z)|^p \left( 1 - |z|^2 \right)^q g^a(z, a) dv(z) < 2\varepsilon.
\] (3.14)

From which, the desired result (3.4) holds. This completes the proof of this theorem. \( \square \)

Remark 3.3. When \( \varphi(z) = z \), the product of extended Cesàro operator \( T_\alpha C_\varphi \) is the generalized extended Cesàro operator \( T_\alpha \); thus, by Theorems 3.1 and 3.2, we have the following two corollaries.

**Corollary 3.4.** Assume that \( \alpha \in (0, 1) \), \( 0 < p, s < \infty, -n - 1 < q < \infty, q + s > -1 \), and \( h \in H(\mathbb{B}) \). Then, \( T_\alpha : \mathcal{L}_\alpha \to F(p,q,s) \) is bounded if and only if \( h \in F(p,q,s) \).
Corollary 3.5. Assume that \( \alpha \in (0, 1) \), \( 0 < p, s < \infty \), \( -n - 1 < q < \infty \), \( q + s > -1 \), and \( h \in H(\mathbb{B}) \). Then, \( T_h : \mathcal{L}_\alpha \rightarrow F(p, q, s) \) is compact if and only if \( T_h : \mathcal{L}_\alpha \rightarrow F(p, q, s) \) is bounded, and

\[
\lim_{r \to 1^-} \sup_{a \in \mathbb{B}} \int_{|z| > r} |\Re h(z)|^p \left(1 - |z|^2\right)^q g^s(z, a) = 0. \tag{3.15}
\]

Acknowledgments

The authors would like to thank the editor and referees for carefully reading the paper and providing corrections and suggestions for improvements. Z.-H. Zhou was supported in part by the National Natural Science Foundation of China (Grant nos. 10971153 and 10671141).

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