Research Article

Non-Self-Adjoint Singular Sturm-Liouville Problems with Boundary Conditions Dependent on the Eigenparameter

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Let \( A \) denote the operator generated in \( L^2(\mathbb{R}_+) \) by the Sturm-Liouville problem:
\[
-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ = [0, \infty),
\]
with the boundary condition \( y(0) = 0 \), where \( q \) is a complex valued function and \( \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{C} \), with \( \alpha_0 \beta_1 - \alpha_1 \beta_0 \neq 0 \). In this paper, using the uniqueness theorems of analytic functions, we investigate the eigenvalues and the spectral singularities of \( A \). In particular, we obtain the conditions on \( q \) under which the operator \( A \) has a finite number of the eigenvalues and the spectral singularities.

1. Introduction

Let \( L \) denote the non-self-adjoint Sturm-Liouville operator generated in \( L^2(\mathbb{R}_+) \) by the differential expression
\[
I(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+
\]
and the boundary condition \( y(0) = 0 \), where \( q \) is a complex valued function. The spectral analysis of \( L \) with continuous and discrete spectrum was studied by Naimark [1]. In this article, the spectrum of \( L \) was investigated and shown that it is composed of the eigenvalues, the continuous spectrum and the spectral singularities. The spectral singularities of \( L \) are poles of the resolvent which are imbedded in the continuous spectrum and are not the eigenvalues.
If the function $q$ satisfies the Natmark condition, that is,

$$
\int_0^\infty e^{\varepsilon x} |q(x)| \, dx < \infty
$$

for some $\varepsilon > 0$, then $L$ has a finite number of the eigenvalues and spectral singularities with finite multiplicities.

The results of Natmark were extended to the Sturm-Liouville operators on the entire real axis by Kemp [2] and to the differential operators with a singularity at the zero point by Gasymov [3]. The spectral analysis of dissipative Sturm-Liouville operators with spectral singularities was considered by Pavlov [4]. A very important development in the spectral analysis of $L$ was made by Lyance [5, 6]. He showed that the spectral singularities play an important role in the spectral theory of $L$. He also investigated the effect of the spectral singularities in the spectral expansion. The spectral singularities of the non-self-adjoint Sturm-Liouville operator generated in $L_2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$
\int_0^\infty K(x)y(x)\, dx + \alpha y'(0) - \beta y(0) = 0,
$$

in which $K \in L_2(\mathbb{R}_+)$ is a complex valued function and $\alpha, \beta \in \mathbb{C}$, was studied in detail by Krall [7–9].

Some problems of spectral theory of differential and difference operators with spectral singularities were also investigated in [10–16]. Note that, the boundary conditions used in [1–17] are independent of spectral parameter. In recent years, various problems of the spectral theory of regular Sturm-Liouville problem whose boundary conditions depend on spectral parameter have been examined in [18–22].

Let us consider the boundary value problem

$$
-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+,
$$

$$
\frac{y'}{y}(0) = \frac{\beta_1 \lambda + \beta_0}{\alpha_1 \lambda + \alpha_0},
$$

where $q$ is a complex valued function and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are complex numbers such that $\alpha_0 \beta_1 - \alpha_1 \beta_0 \neq 0$. By $A$ we will denote the operator generated in $L_2(\mathbb{R}_+)$ by (1.1) and (1.5). In this paper we discuss the discrete spectrum of $A$ and prove that the operator $A$ has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity if

$$
\lim_{x \to \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} |q'(x)| \, dx < \infty
$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$. We also show that the analogue of the Natmark condition for $A$ is the form

$$
\lim_{x \to \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} |q'(x)| \, dx < \infty
$$

for some $\varepsilon > 0$. 

\[ \text{Abstract and Applied Analysis} \]
2. Jost Solution of (1.4)

We will denote the solution of (1.4) satisfying the condition
\[
\lim_{x \to \infty} y(x, \lambda) e^{-i\lambda x} = 1, \quad \lambda \in \mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \Im \lambda \geq 0\},
\]
by \( e(x, \lambda) \). The solution \( e(x, \lambda) \) is called the Jost solution of (1.4). Under the condition
\[
\int_0^\infty x |q(x)| dx < \infty,
\]
the Jost solution has a representation
\[
e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t) e^{i\lambda t} dt
\]
for \( \lambda \in \mathbb{C}_+ \), where the kernel \( K(x, t) \) satisfies
\[
K(x, t) = \frac{1}{2} \int_{(x+t)/2}^\infty q(\xi) d\xi + \frac{1}{2} \int_x^{(x+t)/2} K(\xi, \eta) q(\xi) d\eta d\xi
\]
\[
+ \frac{1}{2} \int_{(x+t)/2}^{(x+t)/2} K(\xi, \eta) q(\xi) d\eta d\xi.
\]
Moreover, \( K(x, t) \) is continuously differentiable with respect to its arguments and
\[
|K(x, t)| \leq c \int_{(x+t)/2}^\infty |q(\xi)| d\xi,
\]
\[
|K_x(x, t)|, |K_t(x, t)| \leq \frac{1}{4} |q\left(\frac{x + t}{2}\right)| + c \int_{(x+t)/2}^\infty |q(\xi)| d\xi,
\]
where \( c > 0 \) is a constant [23, Chapter 3].

The solution \( e(x, \lambda) \) is analytic with respect to \( \lambda \) in \( \mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \Im \lambda > 0\} \) and continuous on the real axis.

Let \( \mathcal{AC}(\mathbb{R}_+) \) denote the class of complex valued absolutely continuous functions in \( \mathbb{R}_+ \).

In the sequel we will need the following.
Lemma 2.1. If

\[ q \in \mathcal{A}(R_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \int_{0}^{\infty} x^2 |q'(x)| \, dx < \infty, \tag{2.7} \]

then \( K_{xt}(x, t) := (\partial^2 / \partial t \partial x) K(x, t) \) exists and

\[
K_{xt}(x, t) = -\frac{1}{8} q'(\frac{t}{2}) - \frac{1}{4} K\left(\frac{t}{2}, \frac{t}{2}\right) q\left(\frac{x + t}{2}\right) - \frac{1}{2} \int_{x}^{\infty} K(\xi, t + x - \xi) + K(\xi, t - x + \xi) \, d\xi \tag{2.8}
\]

\[
- \frac{1}{2} \int_{(x+t)/2}^{\infty} K(\xi, t - x + \xi) q(\xi) \, d\xi.
\]

The proof of the lemma is the direct consequence of (2.4).

From (2.5)–(2.8) we find that

\[
|K_{xt}(0, t)| \leq c \left[ |q\left(\frac{t}{2}\right)| + \left| q\left(\frac{t}{2}\right) \right| + \int_{t/2}^{\infty} |q(\xi)| \, d\xi \right], \tag{2.9}
\]

where \( c > 0 \) is a constant.

3. The Green Function and the Continuous Spectrum

Let \( \varphi(x, \lambda) \) denote the solution of (1.4) subject to the initial conditions \( \varphi(0, \lambda) = \alpha_0 + \alpha_1 \lambda, \ \varphi'(0, \lambda) = \beta_0 + \beta_1 \lambda \). Therefore \( \varphi(x, \lambda) \) is an entire function of \( \lambda \).

Let us define the following functions:

\[
D_\pm(\lambda) = (\alpha_0 + \alpha_1 \lambda) e_x(0, \pm \lambda) - (\beta_0 + \beta_1 \lambda) e(0, \pm \lambda) \quad \lambda \in \mathcal{C}_\pm, \tag{3.1}
\]

where \( \mathcal{C}_\pm = \{ \lambda : \lambda \in \mathcal{C}, \pm \text{Im} \lambda \geq 0 \} \). It is obvious that the functions \( D_+ \) and \( D_- \) are analytic in \( \mathcal{C}_+ \) and \( \mathcal{C}_- := \{ \lambda : \lambda \in \mathcal{C}, \text{Im} \lambda < 0 \} \), respectively and continuous on the real axis.

Let

\[
G(x, t; \lambda) = \begin{cases} G_+(x, t; \lambda), & \lambda \in \mathcal{C}_+, \\ G_-(x, t; \lambda), & \lambda \in \mathcal{C}_- \end{cases} \tag{3.2}
\]

be the Green function of \( A \) (obtained by the standard techniques), where

\[
G_\pm(x, t; \lambda) = \begin{cases} \frac{e(x, \pm \lambda) \varphi(t, \lambda)}{D_\pm(\lambda)}, & 0 \leq t \leq x, \\ \frac{e(t, \pm \lambda) \varphi(x, \lambda)}{D_\pm(\lambda)}, & x \leq t < \infty. \tag{3.3}\end{cases}
\]
We will denote the continuous spectrum of \( A \) by \( \sigma_c \). Using (3.1)–(3.3) in a way similar to Theorem 2 [17, page 303], we get the following:

\[
\sigma_c = \mathbb{R}.
\tag{3.4}
\]

### 4. The Discrete Spectrum of the Operator \( A \)

Let us denote the eigenvalues and the spectral singularities of the operator \( A \) by \( \sigma_d \) and \( \sigma_{ss} \) respectively. From (2.3) and (3.1)–(3.4) it follows that

\[
\begin{align*}
\sigma_d &= \{ \lambda : \lambda \in \mathbb{C}_+, D_+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{C}_-, D_-(\lambda) = 0 \}, \\
\sigma_{ss} &= \{ \lambda : \lambda \in \mathbb{R}^*, D_+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{R}^*, D_-(\lambda) = 0 \},
\end{align*}
\tag{4.1}
\]

where \( \mathbb{R}^* = \mathbb{R} - \{0\} \).

**Definition 4.1.** The multiplicity of a zero of \( D_+ \) (or \( D_- \)) in \( \mathbb{C}_+ \) (or \( \mathbb{C}_- \)) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of \( A \).

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of \( A \) we need to discuss the quantitative properties of the zeros of \( D_+ \) and \( D_- \) in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively. For the sake of simplicity we will consider only the zeros of \( D_+ \) in \( \mathbb{C}_+ \). A similar procedure may also be employed for zeros of \( D_- \) in \( \mathbb{C}_- \).

Let us define

\[
\begin{align*}
M_1^+ &= \{ \lambda : \lambda \in \mathbb{C}_+, D_+(\lambda) = 0 \}, \\
M_2^+ &= \{ \lambda : \lambda \in \mathbb{R}, D_+(\lambda) = 0 \},
\end{align*}
\tag{4.2}
\]

So we have, by (4.1), that

\[
\begin{align*}
\sigma_d &= M_1^+ \cup M_2^-, \\
\sigma_{ss} &= M_2^+ \cup M_2^- - \{0\}.
\end{align*}
\tag{4.3}
\]

**Theorem 4.2.** Under the conditions in (2.7):

(i) the discrete spectrum \( \sigma_d \) is a bounded, at most countable set and its limit points lie on the bounded subinterval of the real axis;

(ii) the set \( \sigma_{ss} \) is a bounded and its linear Lebesgue measure is zero.

**Proof.** From (2.3) and (3.1) we obtain that \( D_+ \) is analytic in \( \mathbb{C}_+ \), continuous on the real axis and has the form

\[
D_+(\lambda) = i\alpha_1 \lambda^2 + a\lambda + b + \int_0^\infty f(t)e^{i\lambda t}dt,
\tag{4.4}
\]
Proof. From (2.5), (2.6), and (2.9) we get that $f \in L_1(R)$. So

$$D_+(\lambda) = i\alpha_1\lambda^2 + a\lambda + b + o(1), \quad \lambda \in \overline{C}_+, \quad |\lambda| \to \infty. \quad (4.6)$$

From (4.3), (4.6) and uniqueness theorem for analytic functions [24], we get (i) and (ii). \qed

**Theorem 4.3.** If

$$q \in \mathcal{A}C(R_+), \quad \lim_{x \to -\infty} q(x) = 0, \quad \int_{-\infty}^\infty x^3|q'(x)|dx < \infty, \quad (4.7)$$

then

$$\sum_{\nu}|l_\nu| \ln \frac{1}{|l_\nu|} < \infty, \quad (4.8)$$

where $|l_\nu|$ is the lengths of the boundary complementary intervals of $\sigma_{ss}$.

**Proof.** From (2.5), (2.6), (2.9), (4.4) and (4.7) we see that $D_+$ is continuously differentiable on $R$. Since the function $D_+$ is not identically equal to zero, by Beurling’s theorem we obtain (4.8) [25]. \qed

**Theorem 4.4.** Under the conditions

$$q \in \mathcal{A}C(R_+), \quad \lim_{x \to -\infty} q(x) = 0, \quad \int_{-\infty}^\infty e^{\varepsilon x}|q'(x)|dx < \infty, \quad \varepsilon > 0, \quad (4.9)$$

the operator $A$ has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

**Proof.** (2.5), (2.7), (2.9), (4.4) and (4.9) imply that the function $D_+$ has an analytic continuation to the half-plane $\text{Im}\lambda > -\varepsilon/2$. Hence the limit points of its zeros on $\overline{C}_+$ cannot lie in $R$. Therefore using Theorem 4.2, we have the finiteness of zeros of $D_+$ in $\overline{C}_+$. Similarly we find that the function $D_-$ has a finite number of zeros with finite multiplicity in $\overline{C}_-$. Then the proof of the theorem is the direct consequence of (4.3).

Note that the conditions in (4.9) are analogous to the Natmark condition (1.2) for the operator $A$. 

\[ a = i\alpha_0 - \alpha_1K(0,0) - \beta_1, \]
\[ b = -(a_0 + i\beta_1)K(0,0) - \beta_0 + i\alpha_1K_x(0,0), \quad (4.5) \]
\[ f(t) = -\beta_0K(0,t) - i\beta_1K_t(0,t) + \alpha_0K_x(0,t) + i\alpha_1K_{xt}(0,t). \]
It is clear that the condition (4.9) guarantees the analytic continuation of $D_+$ and $D_-$ from the real axis to the lower and the upper half-planes respectively. So the finiteness of the eigenvalues and the spectral singularities of $A$ are obtained as a result of these analytic continuations.

Now let suppose that

$$q \in \mathcal{AC}(\mathbb{R}_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x^2} |q'(x)| \, dx < \infty, \quad (4.10)$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$, which is weaker than (4.9). It is obvious that under the condition (4.10) the function $D_+$ is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But $D_+$ does not have analytic continuation from the real axis to the lower half-plane. Similarly, $D_-$ does not have analytic continuation from the real axis to the upper half-plane either. Consequently, under the conditions in (4.10) the finiteness of the eigenvalues and the spectral singularities of $A$ cannot be shown in a way similar to Theorem 4.4.

Let us denote the sets of limit points of $M^+_1$ and $M^+_2$ by $M^+_0$ and $M^+_1$ respectively and the set of all zeros of $D_+$ with infinite multiplicity in $\mathbb{C}_+$ by $M^+_\infty$. Analogously define the sets $M^-_1, M^-_2$ and $M^-_\infty$.

It is clear from the boundary uniqueness theorem of analytic functions that [24]

$$M^+_1 \cap M^+_\infty = \emptyset, \quad M^+_3 \subset M^+_2, \quad M^+_4 \subset M^+_3,$$

$$M^+_2 \subset M^+_4, \quad M^+_5 \subset M^+_4, \quad M^+_6 \subset M^+_5, \quad (4.11)$$

and $\mu(M^+_2) = \mu(M^+_3) = \mu(M^+_4) = 0$, where $\mu$ denote the Lebesgue measure on the real axis.

**Theorem 4.5.** If (4.10) holds, then $M^+_\infty = M^-_\infty = \emptyset$.

**Proof.** We will prove that $M^+_\infty = \emptyset$. The case $M^-_\infty = \emptyset$ is similar. Under the condition (4.10) $D_+$ is analytic in $\mathbb{C}_+$ all of its derivatives are continuous on the real axis and there exists $N > 0$ such that

$$\left| \frac{d^n}{dx^n} D_+ (\lambda) \right| \leq B_n, \quad n = 0, 1, 2, \ldots, \lambda \in \overline{\mathbb{C}_+}, \quad |\lambda| < 2N,$$

$$B_0 = 4|\alpha_1|N^2 + 2|\alpha|N + |b| + \int_0^{\infty} |f(t)| \, dt,$$

$$B_1 = 4|\alpha_1|N + |a| + \int_0^{\infty} t |f(t)| \, dt,$$

$$B_2 = 2|\alpha_1| + \int_0^{\infty} t^2 |f(t)| \, dt,$$

$$B_n = \int_0^{\infty} t^n |f(t)| \, dt, \quad n \geq 3. \quad (4.12)$$
From Theorem 4.2, we get that
\[ \left| \int_{-\infty}^{-N} \frac{\ln|D_*(\lambda)|}{1 + \lambda^2} d\lambda \right| < \infty, \quad \left| \int_{N}^{\infty} \frac{\ln|D_*(\lambda)|}{1 + \lambda^2} d\lambda \right| < \infty. \tag{4.13} \]

Let us define the function
\[ T(s) = \inf \frac{B_n s^n}{n!}. \tag{4.14} \]

Since the function \( D_+ \) is not equal to zero identically, by Pavlov’s theorem [4],
\[ \int_{0}^{h} \ln T(s) d\mu(M_{\infty,s}^+) > -\infty \tag{4.15} \]
holds, where \( h > 0 \) is a constant and \( \mu(M_{\infty,s}^+) \) is the Lebesgue measure of \( s \)-neighborhood of \( M_{\infty}^+ \). Using (2.5), (2.6), (2.9) and (4.4) we obtain that
\[ B_n \leq Bd^n n!n^{n(1/\delta - 1)}, \tag{4.16} \]
where \( B \) and \( d \) are constants depending on \( \varepsilon \) and \( \delta \). Substituting (4.16) in the definition of \( T(s) \) we get
\[ T(s) \leq B \exp\left\{ -\left( \frac{1}{\delta} - 1 \right) e^{-1} d^{-\delta/(1-\delta)} s^{-\delta/(1-\delta)} \right\}. \tag{4.17} \]

Now (4.15) and (4.17) imply that
\[ \int_{0}^{h} s^{-\delta/(1-\delta)} d\mu(M_{\infty,s}^+) < \infty. \tag{4.18} \]

Since \( \delta/(1-\delta) \geq 1 \), consequently (4.18) holds for arbitrary \( s \) if and only if \( \mu(M_{\infty,s}^+) = 0 \) or \( M_{\infty}^+ = \emptyset \).

**Theorem 4.6.** Under the condition (4.10) the operator \( A \) has a finite number of the eigenvalues and the spectral singularities and each of them is of a finite multiplicity.

**Proof.** To be able to prove the theorem we have to show that the functions \( D_+ \) and \( D_- \) have finite number of zeros with finite multiplicities in \( \overline{C}_+ \) and \( \overline{C}_- \), respectively. We will prove it only for \( D_+ \). The case of \( D_- \) is similar.

It follows from (4.11) that \( M_+^B = M_+^I = \emptyset \). So the bounded sets \( M_+^B \) and \( M_+^I \) have no limit points, that is, the \( D_+ \) has only a finite number of zeros in \( \overline{C}_+ \). Since \( M_{\infty}^+ = \emptyset \) these zeros are of a finite multiplicity.

**Theorem 4.7.** If the condition (2.7) is satisfied then the set \( \sigma_{ss} \) is of the first category.
Proof. From the continuity of $D_+$ it is clear that the set $M^+_2$ is closed and is a set of Lebesgue measure zero which is of type $F_\sigma$. According to Martin’s theorem [26] there is measurable set whose metric density exists and is different from 0 and 1 at every point of $M^+_2$. So, $M^+_2$ is of the first category from the theorem due to Goffman [27]. We also have obviously same things for $M^+_2$. Consequently $\sigma_{ss}$ is of the first category by (4.3).

References


