Necessary and Sufficient Conditions for the Boundedness of Dunkl-Type Fractional Maximal Operator in the Dunkl-Type Morrey Spaces

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We consider the generalized shift operator, associated with the Dunkl operator
\[ \Lambda_\alpha(f)(x) = (d/dx)f(x) + ((2\alpha + 1)/x)((f(x) - f(-x))/2), \]
where \( \alpha > -1/2 \). We study the boundedness of the Dunkl-type fractional maximal operator \( M_\beta \) in the Dunkl-type Morrey space \( L_{p,1,\lambda}(\mathbb{R}) \), \( 0 \leq \lambda < 2\alpha + 2 \). We obtain necessary and sufficient conditions on the parameters for the boundedness \( M_\beta \), \( 0 \leq \beta < 2\alpha + 2 \) from the spaces \( L_{p,1,\lambda}(\mathbb{R}) \) to the spaces \( L_{q,1,\lambda}(\mathbb{R}) \), \( 1 < q < \infty \), and from the spaces \( L_{1,1,\lambda}(\mathbb{R}) \) to the weak spaces \( WL_{q,1,\lambda}(\mathbb{R}) \), \( 1 < q < \infty \). As an application of this result, we get the boundedness of \( M_\beta \) from the Dunkl-type Besov-Morrey spaces \( B^s_{p,q,1,\lambda}(\mathbb{R}) \) to the spaces \( B^s_{q,1,1,\lambda}(\mathbb{R}) \), \( 1 < p \leq q < \infty \), \( 0 \leq \lambda < 2\alpha + 2 \), \( 1/p - 1/q = \beta/(2\alpha + 2 - \lambda) \), \( 1 \leq \theta \leq \infty \), and \( 0 < s < 1 \).

1. Introduction

On the real line, the Dunkl operators \( \Lambda_\alpha \) are differential-difference operators introduced in 1989 by Dunkl [1]. For a real parameter \( \alpha > -1/2 \), we consider the Dunkl operator, associated with the reflection group \( \mathbb{Z}_2 \) on \( \mathbb{R} \):
\[ \Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha + 1}{x}\left(\frac{f(x) - f(-x)}{2}\right). \] (1.1)

In the theory of partial differential equations, together with weighted \( L_{p,w}(\mathbb{R}^n) \) spaces, Morrey spaces \( L_{p,1}(\mathbb{R}^n) \) play an important role. Morrey spaces were introduced by Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [2]).
The Hardy-Littlewood maximal function, fractional maximal function, and fractional integrals are important technical tools in harmonic analysis, theory of functions, and partial differential equations. In the works [3–5], the maximal operator and in [6, 7] the fractional maximal operator associated with the Dunkl operator on $\mathbb{R}$ were studied. In this work, we study the boundedness of the fractional maximal operator $M_\beta$ (Dunkl-type fractional maximal operator) in Morrey spaces $L_{p,1,\alpha}(\mathbb{R})$ (Dunkl-type Morrey spaces) associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the operator $M_\beta$ from the spaces $L_{p,1,\alpha}(\mathbb{R})$ to $L_{q,1,\alpha}(\mathbb{R})$, $1 < p \leq q < \infty$, and from the spaces $L_{1,1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,1,\alpha}(\mathbb{R})$, $1 < q < \infty$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give our main result on the boundedness of the operator $M_\beta$ in $L_{p,1,\alpha}(\mathbb{R})$. We obtain necessary and sufficient conditions on the parameters for the boundedness of the operator $M_\beta$ from the spaces $L_{p,1,\alpha}(\mathbb{R})$ to the spaces $L_{q,1,\alpha}(\mathbb{R})$, $1 < p \leq q < \infty$, and from the spaces $L_{1,1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,1,\alpha}(\mathbb{R})$, $1 < q < \infty$. As an application of this result, in Section 4 we prove the boundedness of the operator $M_\beta$ from the Dunkl-type Besov-Morrey spaces $B^s_{p0,1,\alpha}(\mathbb{R})$ to the spaces $B^s_{q0,1,\alpha}(\mathbb{R})$, $1 < p \leq q < \infty$, $0 \leq \lambda < 2\alpha + 2$, $1/p - 1/q = \beta/(2\alpha + 2 - \lambda)$, $1 \leq \theta \leq \infty$, and $0 < s < 1$.

Finally, we mention that, $C$ will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

2. Preliminaries

Let $\alpha > -1/2$ be a fixed number and $\mu_\alpha$ be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha + 1)\right)^{-1}|x|^{2\alpha+1}dx.$$  \hfill (2.1)

For every $1 \leq p \leq \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(d\mu_\alpha)(\mathbb{R})$ the spaces of complex-valued functions $f$, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if } p \in [1, \infty),$$

$$\|f\|_{\infty,\alpha} := \text{ess sup}_{x \in \mathbb{R}} |f(x)| \quad \text{if } p = \infty.$$  \hfill (2.2)

For $1 \leq p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} := \sup_{r > 0} r^{\alpha} \mu_\alpha\{x \in \mathbb{R} : |f(x)| > r\}^{1/p}.$$  \hfill (2.3)

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha}, \quad \|f\|_{WL_{p,\alpha}} \leq \|f\|_{p,\alpha} \quad \forall f \in L_{p,\alpha}(\mathbb{R}).$$  \hfill (2.4)
For all $x, y, z \in \mathbb{R}$, we put

$$W_{\alpha}(x, y, z) := (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}) \Delta_{\alpha}(x, y, z),$$

(2.5)

where

$$\sigma_{x,y,z} := \begin{cases} \frac{x^2 + y^2 - z^2}{2xy} & \text{if } x, y \in \mathbb{R} \setminus 0, \\ 0 & \text{otherwise} \end{cases}$$

(2.6)

and $\Delta_{\alpha}$ is the Bessel kernel given by

$$\Delta_{\alpha}(x, y, z) := \begin{cases} d_{\alpha} \left( \frac{[(|x| + |y|)^2 - z^2][z^2 - (|x| - |y|)^2]}{|xyz|^{2\alpha}} \right)^{\alpha^{-1}/2} & \text{if } |z| \in A_{x,y}, \\ 0 & \text{otherwise}, \end{cases}$$

(2.7)

where $d_{\alpha} = (\Gamma(\alpha + 1))^2 / (2^{n-1}\sqrt{\pi} \Gamma(\alpha + 1/2))$ and $A_{x,y} = [|x| - |y|, |x| + |y|]$. In the sequel we consider the signed measure $\nu_{x,y}$ on $\mathbb{R}$, given by

$$\nu_{x,y} := \begin{cases} W_{\alpha}(x, y, z)d\mu_{\alpha}(z) & \text{if } x, y \in \mathbb{R} \setminus 0, \\ d\delta_{x}(z) & \text{if } y = 0, \\ d\delta_{y}(z) & \text{if } x = 0. \end{cases}$$

(2.8)

For $x, y \in \mathbb{R}$ and $f$ being a continuous function on $\mathbb{R}$, the Dunkl translation operator $\tau_x$ is given by

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z).$$

(2.9)

Using the change of variable $z = \Psi(x,y,\theta) = \sqrt{x^2 + y^2 - 2xy \cos \theta}$, we have also (see [8])

$$\tau_x f(y) = C_\alpha \int_0^{\pi} \left[ f(\Psi) + f(-\Psi) + \frac{x+y}{\Psi} \left( f(\Psi) - f(-\Psi) \right) \right] d\nu_{\alpha}(\theta),$$

(2.10)

where $d\nu_{\alpha}(\theta) = (1 - \cos \theta) \sin^{2\alpha} \theta d\theta$ and $C_\alpha = \Gamma(\alpha + 1)/2\sqrt{\pi} \Gamma(\alpha + 1/2)$.

**Proposition 2.1** (see Soltani [9]). For all $x \in \mathbb{R}$ the operator $\tau_x$ extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{L_{p,\alpha}} \leq 4\|f\|_{L_{p,\alpha}}.$$  

(2.11)
Let $B(x,r) = \{ y \in \mathbb{R} : |y| \leq \max\{0,|x|-r\}, |x| + r \geq 0 \}$, and $b_\alpha = [2^{\alpha+1}(\alpha + 1) \Gamma(\alpha + 1)]^{-1}$. Then $B(0,r) = ]-r,r[$. Then $b_\alpha B(0,r) = b_\alpha r^{2\alpha+2}$.

Now we define the Dunkl-type fractional maximal function (see [3–5]) by

$$M_\beta f(x) = \sup_{r>0} (b_\alpha B(0,r))^{-1+\beta/(2\alpha+2)} \int_{B(0,r)} \tau_x |f(y)| \, d\mu_\alpha(y), \quad 0 \leq \beta < 2\alpha + 2. \quad (2.12)$$

If $\beta = 0$, then $M = M_0$ is the Dunkl-type maximal operator.

In [3–5] was proved the following theorem (see also [10]).

**Theorem 2.2.** (1) If $f \in L_{1,\alpha}(\mathbb{R})$, then for every $\beta > 0$

$$\mu_\alpha \{ x \in \mathbb{R} : Mf(x) > \beta \} \leq \frac{C}{\beta} \| f \|_{L_{1,\alpha}}, \quad (2.13)$$

where $C > 0$ is independent of $f$.

(2) If $f \in L_{p,\alpha}(\mathbb{R})$, $1 < p \leq \infty$, then $Mf \in L_{p,\alpha}(\mathbb{R})$ and

$$\| Mf \|_{L_{p,\alpha}} \leq C_p \| f \|_{L_{p,\alpha}}, \quad (2.14)$$

where $C_p > 0$ is independent of $f$.

**Definition 2.3.** Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha + 2$. We denote by $L_{p,\lambda,\alpha}(\mathbb{R})$ Morrey space (equiv. Dunkl-type Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm

$$\| f \|_{p,\lambda,\alpha} = \sup_{x \in \mathbb{R}, r > 0} \left( r^{-1} \int_{B(0,r)} \tau_x |f(y)|^p \, d\mu_\alpha(y) \right)^{1/p}. \quad (2.15)$$

Note that $L_{p,0,\alpha}(\mathbb{R}) = L_{p,\alpha}(\mathbb{R})$, and if $\lambda < 0$ or $\lambda > 2\alpha + 2$, then $L_{p,\lambda,\alpha}(\mathbb{R}) = \emptyset$, where $\emptyset$ is the set of all functions equivalent to 0 on $\mathbb{R}$ (see also [7]).

**Definition 2.4.** Let $1 \leq p < \infty$ and $0 \leq \lambda \leq 2\alpha + 2$. We denote by $WL_{p,\lambda,\alpha}(\mathbb{R})$ a weak Dunkl-type Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with finite norm

$$\| f \|_{WL_{p,\lambda,\alpha}} = \sup_{t > 0} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-1} \int_{\{ y \in B(0,r) : \tau_x |f(y)| > t \}} d\mu_\alpha(y) \right)^{1/p}. \quad (2.16)$$

We note that

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset WL_{p,\lambda,\alpha}(\mathbb{R}), \quad \| f \|_{WL_{p,\lambda,\alpha}} \leq \| f \|_{p,\lambda,\alpha}. \quad (2.17)$$
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3. Main Results

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional maximal operator $M_\beta$ to be bounded from the spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p < q < \infty$ and from the spaces $L_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$.

**Theorem 3.1.** Let $0 \leq \beta < 2\alpha + 2$, $0 \leq \lambda < 2\alpha + 2$, and $1 \leq p \leq (2\alpha + 2 - \lambda)/\beta$.

1. If $p = 1$, then the condition $1 - 1/q = \beta/(2\alpha + 2 - \lambda)$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{q,\lambda,\alpha}(\mathbb{R})$.

2. If $1 < p < (2\alpha + 2 - \lambda)/\beta$, then the condition $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$.

3. If $p = (2\alpha + 2 - \lambda)/\beta$, then $M_\beta$ is bounded from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{\infty}(\mathbb{R})$.

For $1 \leq p \leq \infty$, $0 \leq \lambda < 2\alpha + 2$, and $0 < s < 2$, the Dunkl-type Besov-Morrey $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ consists of all functions $f$ in $L_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$
\|f\|_{B^s_{p\theta,\lambda,\alpha}} = \|f\|_{L_{p,\lambda,\alpha}} + \left( \int_\mathbb{R} \left( \frac{\|\tau_x f(\cdot) - f(\cdot)\|_{L_{p,\lambda,\alpha}}}{|x|^{2\alpha + 2 + s\theta}} \right) d\mu_\alpha(x) \right)^{1/\theta} < \infty. \quad (3.1)
$$

Besov spaces in the setting of the Dunkl operators were studied by Abdelkefi and Sifi [11], Bouguila et al. [12], Guliyev and Mammadov [10], and Kamoun [13]. In the following theorem, we prove the boundedness of the Dunkl-type fractional maximal operator in the Dunkl-type Besov-Morrey spaces.

**Theorem 3.2.** For $1 < p \leq q < \infty$, $0 \leq \lambda < 2\alpha + 2$, $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$, $1 \leq \theta \leq \infty$, and $0 < s < 1$, the Dunkl-type fractional maximal operator $M_\beta$ is bounded from $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that

$$
\|M_\beta f\|_{B^s_{q\theta,\lambda,\alpha}} \leq C \|f\|_{B^s_{p\theta,\lambda,\alpha}} \quad (3.2)
$$

hold for all $f \in B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$.

**Remark 3.3.** Note that Theorem 3.2 in the case $\lambda = 0$ was proved in [10].

4. Boundedness of the Dunkl-Type Fractional Maximal Operator in the Dunkl-Type Morrey Spaces

In the following theorem, we obtain the boundedness of the Dunkl-type fractional maximal operator $M_\beta$ in the Dunkl-type Morrey spaces $L_{p,\lambda,\alpha}(\mathbb{R})$. 
**Theorem 4.1.** Let $0 \leq \beta < 2\alpha + 2$, $0 \leq \lambda < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$, and $1 \leq p \leq (2\alpha + 2 - \lambda)/\beta$.

1. If $p = 1$ and $1 - 1/q = \beta/(2\alpha + 2 - \lambda)$, then $M_\beta f \in WL_{q,\lambda,\alpha}(\mathbb{R})$ and

$$
\|M_\beta f\|_{WL_{q,\lambda,\alpha}} \leq C\|f\|_{1,\lambda,\alpha'},
$$

where $C > 0$ is independent of $f$.

2. If $1 < p < (2\alpha + 2 - \lambda)/\beta$ and $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$, then $M_\beta f \in L_{q,\lambda,\alpha}(\mathbb{R})$ and

$$
\|M_\beta f\|_{q,\lambda,\alpha} \leq C\|f\|_{p,\lambda,\alpha'},
$$

where $C > 0$ is independent of $f$.

3. If $p = (2\alpha + 2 - \lambda)/\beta$ and $q = \infty$, then $M_\beta f \in L_{\infty}(\mathbb{R})$ and

$$
\|M_\beta f\|_{\infty} \leq b_\alpha^{-1/p(2\alpha+2)}\|f\|_{p,\lambda,\alpha'}.
$$

*Proof.* The maximal function $Mf(x)$ may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space $X$ equipped with a continuous pseudometric $\rho$ and a positive measure $\mu$ satisfying

$$
\mu(E,2r) \leq C_0 \mu(E,r)
$$

with a constant $C_0$ being independent of $x$ and $r > 0$. Here $E(x,r) = \{y \in X : \rho(x,y) < r\}$, $\rho(x,y) = |x - y|$. Let $(X,\rho,\mu)$ be a space of homogeneous type, where $X = \mathbb{R}$, $\rho(x,y) = |x - y|$, and $d\mu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (4.4).

Define

$$
M_\mu f(x) = \sup_{r>0} (\mu E(x,r))^{-1} \int_{E(x,r)} |f(y)| d\mu(y).
$$

It is well known that the maximal operator $M_\mu$ is bounded from $L_{1,\lambda}(X,\mu)$ to $WL_{1,\lambda}(X,\mu)$ and is bounded on $L_{p,\lambda}(X,\mu)$ for $1 < p < \infty$, $0 \leq \lambda < 2\alpha + 2$ (see [14, 15]).

The following inequality was proved in [6]

$$
Mf(x) \leq CM_\mu f(x),
$$

where $C > 0$ is independent of $f$.

Then from (4.6) we get the boundedness of the operator $M$ from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{1,\lambda,\alpha}(\mathbb{R})$ and on $L_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$. Thus in the case $\beta = 0$ we complete the proof of (1) and (2).
Let $t > 0$, $0 < \beta < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$, $1 \leq p \leq (2\alpha + 2 - \lambda)/\beta$ and $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$. Applying the Hölders inequality we have

$$
M_\beta f(x) = \max \left\{ \sup_{r \geq t} (\mu_B(0, r))^{\beta/(2\alpha+2)-1} \int_{B(0, r)} \tau_x |f(y)| \, d\mu_\alpha(y), \sup_{r < t} (\mu_B(0, r))^{\beta/(2\alpha+2)-1} \int_{B(0, r)} \tau_x |f(y)| \, d\mu_\alpha(y) \right\}
$$

(4.7)

$$
\leq b_\alpha^{\beta/(2\alpha+2)} \max \left\{ b_\alpha^{1-1/p} \beta^{(2\alpha+2-\lambda)/p} f \|_{p,\lambda,\alpha} \| \right\}.$$

Therefore, for all $t > 0$, we get

$$
M_\beta f(x) \leq b_\alpha^{\beta/(2\alpha+2)} \left( b_\alpha^{1-1/p} \beta^{(2\alpha+2-\lambda)/p} + f \|_{p,\lambda,\alpha} \right) Mf(x).
$$

(4.8)

The minimum value of the right-hand side (4.8) is attained at

$$
t = \left( \frac{2\alpha + 2 - \lambda}{p} \beta^{1-1/p} f \|_{p,\lambda,\alpha} \right)^{p/(2\alpha+2-\lambda)}
$$

(4.9)

and hence

$$
M_\beta f(x) \leq b_\alpha^{\beta/(2\alpha+2) - \beta/(2\alpha+2-\lambda)} \| f \|_{p,\lambda,\alpha}^{1-p/q} (Mf(x))^{p/q}.
$$

(4.10)

Then for $1 < p \leq (2\alpha + 2 - \lambda)/\beta$ from (4.10), we have

$$
\| M_\beta f \|_{q,\lambda,\alpha} = \sup_{r > 0} \left( r^{-1} \int_{B(0, r)} \tau_x (M_\beta f(y))^q d\mu_\alpha(y) \right)^{1/q}
$$

(4.11)

$$
\leq b_\alpha^{\beta/(2\alpha+2) - \beta/(2\alpha+2-\lambda)} \| f \|_{p,\lambda,\alpha}^{1-p/q} \left( r^{-1} \int_{B(0, r)} \tau_x (Mf(y))^p d\mu_\alpha(y) \right)^{1/q}
$$

$$
\leq b_\alpha^{\beta/(2\alpha+2) - \beta/(2\alpha+2-\lambda)} \| f \|_{p,\lambda,\alpha}^{1-p/q} Mf \|_{p,\lambda,\alpha}^{p/q}
$$

$$
\leq C \| f \|_{p,\lambda,\alpha},
$$

where $C > 0$ is independent of $f$. 

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Also for \( p = 1 \) from (4.10) we have

\[
\|M_\beta f\|_{WL_{L,q,\lambda}} = \sup_{t > 0} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-1} \int_{y \in B(0,r) : \tau_x M_\beta f(y) > t} d\mu_\alpha(y) \right)^{1/q} 
\]

\[
\leq \sup_{t > 0} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-1} \int_{y \in B(0,r) : \tau_x Mf(y) > b_\alpha^{-\beta/(2\alpha+2)} \|f\|_{1,\lambda,\alpha}^{q/(2\alpha+2)-1} \|f\|_{1,\lambda,\alpha}^{-q/(2\alpha+2)} d\mu_\alpha(y) \right)^{1/q} 
\]

\[
\leq b_\alpha^{\beta/(2\alpha+2) - \beta/(2\alpha+2)} \|f\|_{1,\lambda,\alpha} \|Mf\|_{WL_{L,q,\lambda}}^{1/q} 
\]

\[
\leq C \|f\|_{1,\lambda,\alpha}, 
\]

(4.12)

where \( C > 0 \) is independent of \( f \).

Therefore, the case \( \beta > 0 \) complete the proof of (1) and (2).

(3) Let \( p = (2\alpha + 2 - \lambda)/\beta, f \in L_{p,\lambda,\alpha}(\mathbb{R}) \); then applying Hölder's inequality, we obtain

\[
\left( \mu_\alpha B(0, r) \right)^{-1+\beta/(2\alpha+2)} \int_{B(0,r)} \tau_x |f(y)| d\mu_\alpha(y) 
\]

\[
\leq \left( \mu_\alpha B(0, r) \right)^{-1+\beta/(2\alpha+2)+1/p} \left( \int_{B(0,r)} \tau_x |f(y)|^p d\mu_\alpha(y) \right)^{1/p} 
\]

\[
= b_\alpha^{\beta/(2\alpha+2)-1/p(2\alpha+2)} \left( r^{-1} \int_{B(0,r)} \tau_x |f(y)|^p d\mu_\alpha(y) \right)^{1/p} 
\]

\[
\leq b_\alpha^{\beta/(2\alpha+2)-1/p(2\alpha+2)} \|f\|_{p,\lambda,\alpha}. 
\]

Thus the case \( \beta > 0 \) completes the proof of (3).

Theorem 4.1 has been proved. \( \square \)

**Proof of Theorem 3.1.** Sufficiency part of the proof follows from Theorem 4.1.

**Necessity.** (1) Let \( 1 < p \leq (2\alpha + 2 - \lambda)/\alpha \) and \( M_\beta \) be bounded from \( L_{p,\lambda,\alpha}(\mathbb{R}) \) to \( L_{q,\lambda,\alpha}(\mathbb{R}) \).

Define \( f_t(x) := f(tx), t > 0 \). Then

\[
\|f_t\|_{p,\lambda,\alpha} = t^{-(2\alpha+2)/p} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-1} \int_{B(0,tr)} \tau_{tx} |f(y)|^p d\mu_\alpha(y) \right)^{1/p} 
\]

\[
= t^{-(2\alpha+2-1)/p} \|f\|_{p,\lambda,\alpha} 
\]

(4.14)
and $M_\beta f_i(x) = t^{-\beta} M_\beta f(tx)$,

$$\|M_\beta f f_i\|_{L_{q,\lambda,\alpha}} = t^{-\beta} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-\lambda} \int_{B(0,r)} \tau(x) |M_\beta f(y)|^q \, d\mu_\alpha(y) \right)^{1/q}$$

$$= t^{-\beta(2\alpha + 2\lambda)/q} \sup_{x \in \mathbb{R}, r > 0} \left( r^{-\lambda} \int_{B(0,t_\lambda r)} \tau(x) |M_\beta f(y)|^q \, d\mu_\alpha(y) \right)^{1/q}$$

$$= t^{-\beta(2\alpha + 2\lambda)/q} \|M_\beta f f\|_{L_{q,\lambda,\alpha}}$$

(4.15)

By the boundedness of $M_\beta$ from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$,

$$\|M_\beta f\|_{L_{q,\lambda,\alpha}} = r^{(2\alpha + 2\lambda)/q} \|M_\beta f f_r\|_{L_{q,\lambda,\alpha}}$$

$$\leq C r^{(2\alpha + 2\lambda)/q} \|f_r\|_{L_{p,\lambda,\alpha}}$$

$$= C r^{(2\alpha + 2\lambda)/q - (2\alpha + 2\lambda)/p} \|f\|_{L_{p,\lambda,\alpha}}$$

(4.16)

where $C$ depends only on $p$, $\beta$, $\lambda$, and $\alpha$.

If $1/p > 1/q + \beta/(2\alpha + 2\lambda)$, then for all $f \in L_{p,\lambda,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{L_{q,\lambda,\alpha}} = 0$ as $r \to 0$, which is impossible. Similarly, if $1/p < 1/q + \beta/(2\alpha + 2\lambda)$, then for all $f \in L_{p,\lambda,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{L_{q,\lambda,\alpha}} = 0$ as $r \to \infty$, which is also impossible.

Therefore, we get $1/p = 1/q + \beta/(2\alpha + 2\lambda)$.

Necessity. Let $M_\beta$ be bounded from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{q,\lambda,\alpha}(\mathbb{R})$. We have

$$\|M_\beta f f_r\|_{WL_{q,\lambda,\alpha}} = r^{-\beta(2\alpha + 2\lambda)/q} \|M_\beta f\|_{WL_{q,\lambda,\alpha}}$$

(4.17)

By the boundedness of $M_\beta$ from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{q,\lambda,\alpha}(\mathbb{R})$ it follows that

$$\|M_\beta f\|_{WL_{q,\lambda,\alpha}} = r^{(2\alpha + 2\lambda)/q} \|M_\beta f f_r\|_{WL_{q,\lambda,\alpha}}$$

$$\leq C r^{(2\alpha + 2\lambda)/q} \|f_r\|_{L_{1,\lambda,\alpha}}$$

$$= C r^{(2\alpha + 2\lambda)/q - (2\alpha + 2\lambda)/p} \|f\|_{L_{1,\lambda,\alpha}}$$

(4.18)

where $C$ depends only on $\beta$, $\lambda$, and $\alpha$.

If $1 < 1/q + \beta/(2\alpha + 2\lambda)$, then for all $f \in L_{1,\lambda,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{WL_{q,\lambda,\alpha}} = 0$ as $r \to 0$. Similarly, if $1 > 1/q + \beta/(2\alpha + 2\lambda)$, then for all $f \in L_{1,\lambda,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{WL_{q,\lambda,\alpha}} = 0$ as $r \to \infty$.

Hence we get $1 = 1/q + \beta/(2\alpha + 2\lambda)$. Thus the proof of Theorem 3.1 is completed. □
Proof of Theorem 3.2. For $x \in \mathbb{R}$, let $\tau_x$ be the generalized translation by $x$. By definition of the Besov spaces, it suffices to show that

$$\|\tau_x M_\beta f - M_\beta f\|_{L^q_{\lambda,\alpha}} \leq C_2 \|\tau_x f - f\|_{L^p_{\lambda,\alpha}}. \quad (4.19)$$

It is easy to see that $\tau_x$ commutes with $M_\beta$, that is, $\tau_x M_\beta f = M_\beta (\tau_x f)$. Hence we have

$$|\tau_x M_\beta f - M_\beta f| = |M_\beta (\tau_x f) - M_\beta f| \leq M_\beta(|\tau_x f - f|). \quad (4.20)$$

Taking $L^p_{\lambda,\alpha}(\mathbb{R})$ norm on both ends of the above inequality, by the boundedness of $M_\beta$ from $L^p_{\lambda,\alpha}(\mathbb{R})$ to $L^q_{\lambda,\alpha}(\mathbb{R})$, we obtain the desired result. Theorem 3.2 is proved. \qedema

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References


