Existence and Asymptotic Behavior of Boundary Blow-Up Solutions for Weighted $p(x)$-Laplacian Equations with Exponential Nonlinearities

Li Yin, Yunrui Guo, Jing Yang, Bibo Lu, and Qihu Zhang

1 Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China
2 Department of Mathematics, Henan Institute of Science and Technology, Xinxiang, Henan 453003, China
3 School of Computer Science and Technology, Henan Polytechnic University, Jiaozuo, Henan 454000, China
4 School of Mathematics and Statistics, Huazhong Normal University, Wuhan, Hubei 430079, China

Correspondence should be addressed to Qihu Zhang, zhangqh1999@yahoo.com.cn

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This paper investigates the following $p(x)$-Laplacian equations with exponential nonlinearities:

$$\Delta_{p(x)} u + \rho(x)e^{f(x,u)} = 0 \quad \text{in} \quad \Omega,$$

$$u(x) \to +\infty \quad \text{as} \quad d(x, \partial \Omega) \to 0,$$

where $\rho(x) \in C(\Omega)$. The asymptotic behavior of boundary blow-up solutions is discussed, and the existence of boundary blow-up solutions is given.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. On the background of this class of problems, we refer to [1–3]. Many results have been obtained on this kind of problems, for example, [4–18]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [4, 5, 8]. On the existence of solutions for $p(x)$-Laplacian equation Dirichlet problems in bounded domain, we refer to [7, 9, 15, 18]. In this paper, we consider the following $p(x)$-Laplacian equations with exponential nonlinearities

$$\Delta_{p(x)} u + \rho(x)e^{f(x,u)} = 0, \quad \text{in} \quad \Omega,$$

$$u(x) \to +\infty, \quad \text{as} \quad d(x, \partial \Omega) \to 0,$$

(P)
where \(-\Delta_{p(x)} u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)\) and \(\Omega = B(0,R) \subset \mathbb{R}^N\) is a bounded radial domain \((B(0,R) = \{x \in \mathbb{R}^N \mid |x| < R\})\). Our aim is to give the asymptotic behavior and the existence of boundary blow-up solutions for problem (P).

Throughout the paper, we assume that \(p(x), \rho(x), \) and \(f(x,u)\) satisfy the following.

\((H_1)\) \(p(x) \in C^1(\bar{\Omega})\) is radial and satisfies

\[1 < p^- \leq p^+ < +\infty, \quad \text{where } p^- = \inf_{\Omega} p(x), \quad p^+ = \sup_{\Omega} p(x). \quad (1.1)\]

\((H_2)\) \(f(x,u)\) is radial with respect to \(x\), \(f(x,\cdot)\) is increasing, and \(f(x,0) = 0\) for any \(x \in \Omega\).

\((H_3)\) \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is continuous and satisfies

\[|f(x,t)| \leq C_1 + C_2|t|^{\gamma}(x), \quad \forall (x,t) \in \Omega \times \mathbb{R}, \quad (1.2)\]

where \(C_1, C_2\) are positive constants and \(0 \leq \gamma \in C(\bar{\Omega})\).

\((H_4)\) \(\rho(x) \in C(\Omega)\) is a radial nonnegative function, and there exists a constant \(\sigma \in [R/2, R)\) such that

\[\rho_0(R-r)^{-\beta(r)} \leq \rho(r) \leq \rho_1(R-r)^{-\beta_1(r)} \quad \text{for } r \in [\sigma, R) \text{ uniformly,} \quad (1.3)\]

where \(\rho_0\) and \(\rho_1\) are positive constants and \(\beta(r)\) and \(\beta_1(r)\) are Lipschitz continuous on \([\sigma, R]\), which satisfy \(\beta(r) \leq \beta_1(r) < p(r)\) for any \(r \in [\sigma, R]\).

The operator \(-\Delta_{p(x)} u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)\) is called \(p(x)\)-Laplacian. Specifically, if \(p(x) \equiv p\) (a constant), (P) is the well-known \(p\)-Laplacian problem. If \(f(x,u)\) can be represented as \(h(x)f(u)\), on the boundary blow-up solutions for the following \(p\)-Laplacian equations (\(p\) is a constant):

\[-\Delta_p u + h(x)f(u) = 0, \quad \text{in } \Omega, \quad (1.4)\]

we refer to [19–26], and the following generalized Keller-Osserman condition is crucial

\[\int_1^{\infty} \frac{1}{(F(t))^{1/p}} dt < +\infty, \quad \text{where } F(t) = \int_0^t f(s)ds, \quad (1.5)\]

but the typical form of \(p(x)\)-Laplacian equation is

\[-\Delta_{p(x)} u + |u|^{p(x)-2}u = 0, \quad \text{in } \Omega, \quad (1.6)\]

and there are some differences between the results of (1.4) and (1.6) (see [16]).

On the boundary blow-up solutions for the following \(p\)-Laplacian equations with exponential nonlinearities (\(p\) is a constant):

\[-\Delta_p u + e^{h(x)f(u)} = 0, \quad \text{in } \Omega, \quad (1.7)\]
we refer to [20–22], but the results on the boundary blow-up solutions for \( p(x) \)-Laplacian equations are rare (see [16]).

In [16], the present author discussed the existence and asymptotic behavior of boundary blow-up solutions for the following \( p(x) \)-Laplacian equations:

\[
-\Delta_{p(x)} u + f(x, u) = 0, \quad \text{in } \Omega, \\
\frac{u(x)}{u(x) \to +\infty}, \quad \text{as } d(x, \partial \Omega) \to 0,
\]

on the condition that \( f(x, \cdot) \) satisfies polynomial growth condition.

If \( p(x) \) is a function, the typical form of (P) is the following:

\[
-\Delta_{p(x)} u + p(x) e^{[u(p(x) - 2)]u} = 0,
\]

and the method to construct subsolution and supersolution in [16] cannot give the exact asymptotic behavior of solutions for (P). Our results partially generalized the results of [20–22].

Because of the nonhomogeneity of \( p(x) \)-Laplacian, \( p(x) \)-Laplacian problems are more complicated than those of \( p \)-Laplacian ones (see [10]); another difficulty of this paper is that \( f(x, u) \) cannot be represented as \( h(x) f(u) \).

2. Preliminary

In order to deal with \( p(x) \)-Laplacian problems, we need some theories on the spaces \( L^{p(x)}(\Omega) \), \( W^{1,p(x)}(\Omega) \) and properties of \( p(x) \)-Laplacian, which we will use later (see [6, 11]). Let

\[
L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.
\]

We can introduce the norm on \( L^{p(x)}(\Omega) \) by

\[
|u|_{p(x)} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.
\]

The space \( (L^{p(x)}(\Omega), |\cdot|_{p(x)}) \) becomes a Banach space. We call it generalized Lebesgue space. The space \( (L^{p(x)}(\Omega), |\cdot|_{p(x)}) \) is a separable, reflexive, and uniform convex Banach space (see [6, Theorems 1.10, 1.14]).

The space \( W^{1,p(x)}(\Omega) \) is defined by

\[
W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\},
\]
and it can be equipped with the norm

\[ \|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}^p, \quad \forall u \in W^{1,p(x)}(\Omega). \]  

\( W^{1,p(x)}(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p(x)}(\Omega) \). \( W^{1,p(x)}(\Omega) \) and \( W_0^{1,p(x)}(\Omega) \) are separable, reflexive, and uniform convex Banach spaces (see [6, Theorem 2.1]).

If \( u \in W_0^{1,p(x)}(\Omega) \cap C(\Omega) \), \( u \) is called a blow-up solution of (P) when it satisfies

\[
\int_Q |\nabla u|^{p(x)-2} \nabla u \nabla q \, dx + \int_Q \rho(x) f(x,u)q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(Q),
\]

for any domain \( Q \subseteq \Omega \), and \( \max(k-u,0) \in W_0^{1,p(x)}(\Omega) \) for every positive integer \( k \).

Let \( W_0^{1,p(x)}(\Omega) = \{ u \mid \text{there is an open domain } Q \subseteq \Omega \text{ such that } u \in W_0^{1,p(x)}(Q) \} \), and define \( A : W_0^{1,p(x)}(\Omega) \cap C(\Omega) \to (W_0^{1,p(x)}(\Omega))^* \) as

\[
\langle Au, \varphi \rangle = \int_\Omega \left( |\nabla u|^{p(x)-2} \nabla u \nabla \varphi + \rho(x) e^{f(x,u)} \varphi \right) \, dx, \quad \forall u \in W_0^{1,p(x)}(\Omega) \cap C(\Omega), \forall \varphi \in W_0^{1,p(x)}(\Omega).
\]  

**Lemma 2.1** (see [9, Theorem 3.1]). Let \( h \in W^{1,p(x)}(\Omega) \cap C(\Omega) \), and \( X = h + W_0^{1,p(x)}(\Omega) \cap C(\Omega) \). Then, \( A : X \to (W_0^{1,p(x)}(\Omega))^* \) is strictly monotone.

Letting \( g \in (W_0^{1,p(x)}(\Omega))^* \), if \( \langle g, \varphi \rangle \geq 0 \) for all \( \varphi \in W_0^{1,p(x)}(\Omega) \) with \( \varphi \geq 0 \) a.e. in \( \Omega \), then denote \( g \geq 0 \) in \( (W_0^{1,p(x)}(\Omega))^* \); correspondingly, if \( -g \geq 0 \) in \( (W_0^{1,p(x)}(\Omega))^* \), then denote \( g \leq 0 \) in \( (W_0^{1,p(x)}(\Omega))^* \).

**Definition 2.2.** Let \( u \in W_0^{1,p(x)}(\Omega) \cap C(\Omega) \). If \( Au \geq 0 (Au \leq 0) \) in \( (W_0^{1,p(x)}(\Omega))^* \), then \( u \) is called a weak supersolution (weak subsolution) of (P).

Copying the proof of [14], we have the following.

**Lemma 2.3** (comparison principle). Let \( u, v \in W_0^{1,p(x)}(\Omega) \cap C(\Omega) \) satisfy

\[ Au - Av \geq 0, \quad \text{in } (W_0^{1,p(x)}(\Omega))^*. \]  

Let \( \varphi(x) = \min\{u(x) - v(x), 0\} \). If \( \varphi(x) \in W_0^{1,p(x)}(\Omega) \) (i.e., \( u \geq v \) on \( \partial \Omega \)), then \( u \geq v \) a.e. in \( \Omega \).

**Lemma 2.4** (see [8, Theorem 1.1]). Under the conditions \((H_1)\) and \((H_3)\), if \( u \in W^{1,p(x)}(\Omega) \) is a bounded weak solution of \(-\Delta_{p(x)}u + \rho(x)e^{f(x,u)} = 0 \) in \( \Omega \), then \( u \in C^{1,\delta}_{loc}(\Omega) \), where \( \delta \in (0,1) \) is a constant.
3. Asymptotic Behavior of Boundary Blow-Up Solutions

If $u$ is a radial solution for (P), then (P) can be transformed into

$$\left( r^{N-1} |u'|^{p(r)-2} u' \right)' = r^{N-1} \rho(r) e^{f(r,u)}, \quad r \in (0, R),$$

$$u(0) = u_0, \quad u'(0) = 0, \quad u'(r) \geq 0, \quad \text{for } 0 < r < R. \quad (3.1)$$

It means that $u(r)$ is increasing.

**Theorem 3.1.** If $f(r,u)$ satisfies

$$f(r,u) \geq au^s \quad (\text{as } u \to +\infty) \text{ for } r \in [\sigma, R] \text{ uniformly}, \quad (3.2)$$

where $\sigma$ is defined in (H$_4$) and $a$ and $s$ are positive constants, then there exists a supersolution $\Phi_1(x)$ which satisfies $\Phi_1(x) \to +\infty$ (as $d(x, \partial \Omega) \to 0$), such that for every solution $u$ of problem (P), one has $u(x) \leq \Phi_1(x)$.

**Proof.** Define the function $g(r,a,\lambda)$ on $[0, R_3]$ as

$$g(r,a,\lambda) = \begin{cases} 
\left( a \ln \left( \frac{1}{(R-r)^{1-\theta} - \lambda} \right) \right)^{1/s} + k, & R_0 \leq r < R_3, \\
\left( a \ln \left( \frac{1}{(R-r)^{1-\theta} - \lambda} \right) \right)^{1/s} + k - \int_r^{R_0} \left[ a^{1/s} (1-\theta)(R-R_0)^{-\theta} \left( \ln \frac{1}{(R-R_0)^{1-\theta} - \lambda} \right) \right]^{(1/s)-1} \left( \frac{R_0}{(R-R_0)^{1-\theta}} \right)^{1/(p(\theta)-1)} \sin \varepsilon(t-\sigma) \, dt, & \sigma < r < R_0, \\
\left( a \ln \left( \frac{1}{(R-R_0)^{1-\theta} - \lambda} \right) \right)^{1/s}, & r \leq \sigma, \end{cases} \quad (3.3)$$

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where \( \theta < \beta(R)/p(R) \), \( a > (1/\alpha)\sup_{|x| \geq R_0} p(x) \) are constants, \( R_0 \in (\sigma, R) \), \( R - R_0 \) is small enough, parameter \( \lambda \in [0, (R - R_0)^{1-\theta}/2] \), \( R_1 \) satisfies \((R - R_1)^{1-\theta} - \lambda = 0, \varepsilon = \pi/2(R_0 - \sigma)\)

\[
k = \left[ \frac{2p^*((1 + s)/s + 1/(1 - \theta)) + |\beta\|^*/(1 - \theta)}{a} \ln \frac{2}{(R - R_0)^{1-\theta}} \right]^{1/s} \]

\[
+ \int_{\sigma}^{R_0} \left[ \frac{2a^{1/s}(1 - \theta)}{s(R - R_0)} \left( \ln \frac{2}{(R - R_0)^{1-\theta}} \right)^{(1/s)-1} \right]^{(p(R_0)-1)/(p(r)-1)} \left( R - R_0 \right)^{-\theta(p(r)-1)} \sin \varepsilon(t - \sigma)^{1/(p(r)-1)} \frac{(R_0)^{N-1}}{t^{N-1}} \ dt.
\]

(3.4)

Obviously, for any positive constant \( a \), we have \( g(r, a, \lambda) \in C^1[0, R_1) \).

When \( R_0 < r < R_1 < R \), we have

\[
g' = g'(r, a, \lambda) = \frac{a^{1/s}}{s} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s)-1} \frac{(1 - \theta)(R - r)^{-\theta}}{(R - r)^{1-\theta} - \lambda},
\]

\[
|g'|^{p(r)-2} g' = \left[ \frac{(1 - \theta)a^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s)-1} \frac{(R - r)^{-\theta(p(r)-1)}}{(R - r)^{1-\theta} - \lambda}^{p(r)-1},
\]

\[
\left( r^{N-1} |g'|^{p(r)-2} g' \right)' = r^{N-1} \left[ \frac{(1 - \theta)a^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s)-1} \frac{(R - r)^{-\theta(p(r)-1)}}{(R - r)^{1-\theta} - \lambda}^{p(r)-1} \left[ (1 - \theta) + \Pi(r) \right],
\]

(3.5)

where

\[
\Pi(r) = \frac{\left\{ r^{N-1} [(1 - \theta)a^{1/s} / s]^{p(r)-1} \right\}'}{(p(r) - 1)r^{N-1} [(1 - \theta)a^{1/s} / s]^{p(r)-1}} \frac{(R - r)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \left( \frac{1}{(R - r)^{1-\theta} - \lambda} \right) \left( R - r \right)^{1-\theta} - \lambda + \frac{(1/s)-1}{\ln \left( 1 / \left( (R - r)^{1-\theta} - \lambda \right) \right)}
\]

\[
+ \frac{\left( R - r \right)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \left( R - r \right) \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \left( R - r \right) \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{\partial p(r)}{p(r) - 1} \frac{\left( R - r \right)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{\partial p(r)}{p(r) - 1} \frac{\left( R - r \right)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{(R - r)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} p(r) - 1 \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{(R - r)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{(R - r)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

\[
+ \frac{(R - r)^{1-\theta} - \lambda}{(R - r)^{1-\theta}} \ln \frac{1}{(R - r)^{1-\theta} - \lambda}
\]

(3.6)
If \((R - R_0)\) is small enough, it is easy to see that

\[
|\Pi(r)| \leq \ln \frac{1}{(R - r)^{1-\theta} - \lambda}, \quad \text{for } \lambda \in \left[0, \frac{(R - R_0)^{1-\theta}}{2}\right] \text{ uniformly,}
\]

(3.7)

and then

\[
\left(r^{N-1}\left|g'\right|^{|p(r)-2}g\right)' \leq r^{N-1}\left[\frac{(1 - \theta)a^{1/s}}{s}\right]^{p(r)-1}\left(\ln \frac{1}{(R - r)^{1-\theta} - \lambda}\right)^{(1/s)-1(p(r)-1)+1}
\]

\[
\times \frac{(p(r) - 1)(R - r)^{-\varphi(p(r))}}{\left[(R - r)^{1-\theta} - \lambda\right]^{p(r)}}, \quad \forall r \in (R_0, R_\lambda).
\]

(3.8)

Thus, when \(0 < R - R_0\) is small enough, from (3.5) and (3.8), for \(\lambda \in [0, (R - R_0)^{1-\theta}/2]\) uniformly, we have

\[
\left(r^{N-1}\left|g'\right|^{|p(r)-2}g\right)' \leq 2r^{N-1}\left[\frac{(1 - \theta)a^{1/s}}{s}\right]^{p(r)-1}\left(\ln \frac{1}{(R - r)^{1-\theta} - \lambda}\right)^{(1/s)-1(p(r)-1)+1}\frac{(p(r) - 1)(R - r)^{-\varphi(p(r))}}{\left[(R - r)^{1-\theta} - \lambda\right]^{p(r)}}
\]

\[
\leq r^{N-1}\rho(r)\left(\frac{1}{(R - r)^{1-\theta} - \lambda}\right)^{\alpha a} = r^{N-1}\rho(r)e^{ag} \leq r^{N-1}\rho(r)e^{f(r,g)}, \quad \forall r \in (R_0, R_\lambda).
\]

(3.9)

Thus, when \(0 < R - R_0\) is small enough, the following inequality is valid for \(\lambda \in [0, (R - R_0)^{1-\theta}/2]\) uniformly:

\[
\left(r^{N-1}\left|g'\right|^{|p(r)-2}g\right)' \leq r^{N-1}\rho(r)f(r,g), \quad \forall r \in (R_0, R_\lambda).
\]

(3.10)
Obviously, if \( R - R_0 \) is small enough, then 
\[
g \geq \left[ \left(2p^\ast((s + 1)/s + 1/(1 - \theta)) + |\beta|^\ast/(1 - \theta))\right) / \alpha \right) \ln(2/ (R - R_0)^{1-\theta}) \right]^{1/s} \text{ is large enough. Since } \lambda \in (0, (R - R_0)^{-\theta}/2],
\]

\[
\left( r^{N-1}|g'|^{p(r)-2}g' \right)'
\]

\[
\leq \epsilon(R_0)^{N-1} \left[ \frac{a^{1/s}(1 - \theta)(R - R_0)^{-\theta}}{s(1/2)(R - R_0)^{1-\theta}} \left( \ln \frac{2}{(R - R_0)^{1-\theta}} \right)^{(1/s) + 1} \right]^{p(R_0)-1}
\]

\[
\leq \epsilon(R_0)^{N-1} \left[ \frac{2a^{1/s}(1 - \theta)}{s(R - R_0)} \left( \frac{2}{(R - R_0)^{1-\theta}} \right)^{(1/s) + 1} \right]^{p(R_0)-1}
\]

\[
\leq r^{N-1} \rho(r) e^{\epsilon g} \leq r^{N-1} \rho(r) e^{f(r,g)}, \quad \sigma < r < R_0.
\]

(3.11)

Thus,

\[
\left( r^{N-1}|g'|^{p(r)-2}g' \right) \leq r^{N-1} \rho(r) e^{f(r,g)}, \quad \sigma < r < R_0.
\]

(3.12)

Obviously,

\[
\left( r^{N-1}|g'|^{p(r)-2}g' \right)' = 0 \leq r^{N-1} \rho(r) e^{f(r,g)}, \quad 0 \leq r < \sigma.
\]

(3.13)

Since \( g(x, a, \lambda) = g(|x|, a, \lambda) \) is a \( C^1 \) function on \( B(0, R_1) \), if \( 0 < R - R_0 \) is small enough \((R_0 \text{ depends on } R, p, s, a, \text{ from (3.10), (3.12), and (3.13), for any } \lambda \in (0, (R - R_0)^{-\theta}/2], \text{ we can see that } g(|x|, a, \lambda) \text{ is a supersolution for (P) on } B(0, R_1), \text{ and then } g(|x|, a, 0) \text{ is a supersolution for (P).})\)

Defining the function \( g_m(|x|, a - e) = g(r, a - e, 1/m) \text{ on } [0, R_{1/m}] \), where \( a - e > (1/\alpha) \sup_{|x| \geq R} p(x) \text{, then } g_m(|x|, a - e) \text{ is a supersolution for (P) on } B(0, R - (1/m)). \text{ If } u \text{ is a solution for (P), according to the comparison principle, we get that } g_m(|x|, a - e) \geq u(x) \text{ for any } x \in B(0, R_{1/m}). \text{ For any } x \in B(0, R) \setminus B(0, R_0), \text{ we have } g_m(|x|, a - e) \geq g_{m+1}(|x|, a - e), \text{ when } m \text{ is large enough. Thus}

\[
u(x) \leq \lim_{m \to +\infty} g_m(|x|, a - e), \quad \forall x \in B(0, R) \setminus B(0, R_0).
\]

(3.14)
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When \( d(x, \partial \Omega) > 0 \) is small enough, we have

\[
\lim_{m \to +\infty} g_m(|x|, a - \varepsilon) < \left( a \ln \frac{1}{(R - r)^{1 - \sigma}} \right)^{1/\alpha} + k \leq g(|x|, a, 0). \tag{3.15}
\]

According to the comparison principle, we get that \( g(|x|, a, 0) \geq u(x) \), for all \( x \in B(0, R) \); then \( \Phi_1(x) = \Phi_1(|x|) = g(|x|, a, 0) \) is a radial upper control function of all of the solutions for (P), and \( \Phi_1(x) = \Phi_1(|x|) \) is a radial supersolution for (P). The proof is completed. \( \square \)

**Theorem 3.2.** If \( f(r, u) \) satisfies

\[
f(r, u) \to -\infty \quad (as \ u \to -\infty) \quad for \ r \in [\sigma, R] \ \text{uniformly},
\]

\[
f(r, u) \leq \delta u^s \quad (as \ u \to +\infty) \quad for \ r \in [\sigma, R] \ \text{uniformly},
\]

where \( \sigma \) is defined in (H_4) and \( \delta \) and \( s \) are positive constants, then there exists a subsolution \( \Phi_2(x) \) which satisfies \( \Phi_2(x) \to +\infty \) (as \( d(x, \partial \Omega) \to 0 \)), such that for every solution \( u(x) \) for problem (P), one has \( u(x) \geq \Phi_2(x) \).

**Proof.** We will prove this theorem in the following two cases.

(i) \( \beta_1(R) > 0 \).

(ii) \( \beta_1(R) \leq 0 \).

**Case 1** (\( \beta_1(R) > 0 \)). Let \( z_1 \) be a radial solution of

\[
-\Delta_p(x) z_1(x) = -\mu, \quad \text{in} \ \Omega_1 = B(0, \sigma), \quad z_1 = 0, \quad \text{on} \ \partial \Omega_1, \tag{3.17}
\]

where \( \mu > 2(\max_{x \in [0, R_0]} p(r) + 1)^{(p-1)/p+1} \) is a positive constant. We denote \( z_1(x) = z_1(r) = z_1(|x|) \). Then, \( z_1 \) satisfies

\[
-\left( r^{N-1} |z_1'|^{p(r)-2} z_1' \right)' = -r^{N-1} \mu, \quad z_1(\sigma) = 0, \quad z_1'(0) = 0,
\]

\[
z_1' = \left[ \frac{r \mu}{N} \right]^{1/(p(r)-1)} , z_1 = - \int_r^\sigma \left[ \frac{r \mu}{N} \right]^{1/(p(r)-1)} dr. \tag{3.18}
\]

Denote \( h_b(r, \lambda) \) on \([\sigma, R_0] \) as

\[
h_b(r, \lambda) = \int_r^{R_0} \left\{ \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t - \sigma}{R_0 - \sigma} \left[ \frac{b(1 - \theta)(R - R_0)^{-\theta}}{s((R - R_0)^{1-\theta} + \lambda)} \left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda}}{(R - R_0)^{1-\theta} + \lambda} \right)^{(1/\alpha)-1} \right] \right\}^{p(R_0) - 1} dt,
\]

\[
+ \frac{(\sigma)^{N-1}}{t^{N-1}} \frac{R_0 - t}{R_0 - \sigma} \left[ \frac{\sigma \mu}{N} \right]^{1/(p(r)-1)} dt. \tag{3.19}
\]
It is easy to see that

\[-h'_b(\sigma, \lambda) = z'_1(\sigma) = \left| \frac{\sigma \mu}{N} \right|^{1/(p(\sigma)-1)}, \]

\[-h'_b(R_0, \lambda) = \frac{b(1 - \theta)(R - R_0)^{-\theta}}{s} \left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda}}{(R - R_0)^{1-\theta} + \lambda} \right)^{(1/s) - 1}. \quad (3.20)\]

Define the function \( v(r, b, \lambda) \) on \([0, R]\) as

\[ v(r, b, \lambda) = \begin{cases} 
\left( \frac{b \ln \frac{1}{(R - r)^{1-\theta} + \lambda}}{(R - r)^{1-\theta} + \lambda} \right)^{1/s} - k^*, & R_0 \leq r < R, \\
\left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda}}{(R - R_0)^{1-\theta} + \lambda} \right)^{1/s} - k^* - h_b(r, \lambda), & \sigma < r < R_0, \\
- \int_r^\sigma \left( \frac{r \mu}{N} \right)^{1/(p(r)-1)} \ln 1 d\lambda + \left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda}}{(R - R_0)^{1-\theta} + \lambda} \right)^{1/s} - k^* - h_b(\sigma, \lambda), & r \leq \sigma,
\end{cases} \]

(3.21)

where \( \theta \in (\beta_1(R) / p(R), 1) \), \( b \in (0, (1/\delta) \inf_{x \geq R} p(x) ) \) are constants, \( R_0 \in (\sigma, R), R - R_0 \) is small enough, parameter \( \lambda \in [0, (R - R_0)^{1-\theta}/2] \), and

\[ k^* = M + \left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta}}}{(R - R_0)^{1-\theta}} \right)^{1/s}, \quad (3.22) \]

where \( M \) satisfies

\[ (\sigma)^{N-1} \frac{1}{R_0 - \sigma} \geq r^{N-1} p(r) e^{f(r, \psi)}, \quad \forall y \leq -M, \; \forall r \in [0, R_0]. \quad (3.23) \]

Obviously, for any positive constant \( b \), \( v(r, b, \lambda) \in C^1[0, R] \).

By computation, when \( r \in (R_0, R) \), we have

\[ v' = v'(r, b, \lambda) = \frac{b^{1/s}}{s} \left( \ln \frac{1}{(R - r)^{1-\theta} + \lambda} \right)^{1/s-1} \left( \frac{1 - \theta)(R - r)^{-\theta}}{(R - r)^{1-\theta} + \lambda} \right) ^{(1/s) - 1} \left( \frac{(R - r)^{-\theta} p(r^{p(r)-1})}{(R - r)^{1-\theta} + \lambda} \right)^{1/(p(r)-1)}, \]

\[ |v'|^{p(r)-2} v' = \left[ \frac{(1 - \theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R - r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \left( \frac{(R - r)^{-\theta} p(r^{p(r)-1})}{(R - r)^{1-\theta} + \lambda} \right)^{1/(p(r)-1)}. \]
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\[
\left( r^{N-1} |v'|^{(p(r)-2)v'} \right)' = r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \\
\times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)-1}}{(R-r)^{1-\theta} + \lambda}^{p(r)-1} (\theta + \Lambda(r)),
\]

where

\[
\Lambda(r) = \frac{\left\{ r^{N-1} \left[ (1-\theta)b^{1/s} / s \right]^{p(r)-1} \right\}'}{(p(r)-1)r^{N-1} \left[ (1-\theta)b^{1/s} / s \right]^{p(r)-1} (R-r) + \frac{(1/s-1)(1-\theta)}{\left( \ln \left( 1/((R-r)^{1-\theta} + \lambda) \right) \right) \left((R-r)^{1-\theta} + \lambda \right)}} \\
\times (R-r)^{1-\theta} + \frac{(1/s-1)p'(r)}{(p(r)-1)} (R-r) \ln \left[ \frac{1}{(R-r)^{1-\theta} + \lambda} \right] + \frac{\theta p'(r)}{(p(r)-1)} (R-r) \ln \left[ \frac{1}{(R-r)^{1-\theta} + \lambda} \right] \\
+ \frac{(1-\theta)}{(R-r)^{1-\theta} + \lambda} (R-r)^{1-\theta} + \frac{-p'(r)}{p(r)-1} (R-r) \ln \left[ (R-r)^{1-\theta} + \lambda \right].
\]

(3.25)

By computation, when \( R - R_0 \) is small enough, for \( \lambda \in [0,(R-R_0)^{1-\theta}/2] \) uniformly, we have

\[
\left( r^{N-1} |v'|^{(p(r)-2)v'} \right)'
\geq r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \\
\times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)-1}}{(R-r)^{1-\theta} + \lambda}^{p(r)-1} (\theta + \frac{1}{2})
\geq \frac{\theta}{2} r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \\
\times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)-1}}{(R-r)^{1-\theta} + \lambda}^{p(r)-1} (R-r)^{1-\theta}
\geq \frac{\theta}{2} r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \\
\times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)}}{(R-r)^{1-\theta} + \lambda}^{p(r)} \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{(R-r)^{1-\theta} + \lambda}^{p(r)}
\geq r^{N-1} \rho_1 (R-r)^{-\delta_1} e^{\delta \nu}
\geq r^{N-1} \rho(r) e^{f(\sigma,\nu)}, \quad \forall r \in (R_0, R).
\]

(3.26)
Then, for $\lambda \in [0, (R - R_0)^{1-\theta}/2]$ uniformly, we have

$$
\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geq r^{N-1}\rho(r)e^{f(r,v)}, \quad \forall r \in (R_0, R).
$$

(3.27)

When $R - R_0$ is small enough, for all $r \in (\sigma, R_0)$, since $v \leq -M$, it is easy to see that

$$
\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geq \left(r^{N-1}|h'|^{p(r)-2}h'\right)',
$$

(3.28)

where $h = b(1 - \theta)(R - R_0)^{-\theta}\left(\frac{b}{(R - R_0)^{1-\theta}} + \lambda\right)^{1/s-1}$. Combining (3.27), (3.29), and (3.30), when $R - R_0$ is large enough, for any $\lambda \in [0, (R - R_0)^{1-\theta}/2]$, one can see that $v(r, \sigma, \lambda)$ is a subsolution for (P).

Define the function $v_m(r, b + \epsilon)$ on $B(0, R)$ as

$$
v_m(r, b + \epsilon) = v_m\left(r, b + \epsilon, \frac{1}{m}\right),
$$

(3.31)

where $\epsilon$ is a small enough positive constant such that $(b + \epsilon) < (1/\delta)\inf_{|x| \geq R_0} p(x)$.

For any $m = 1, 2, \ldots$, we can see that $v_m(r, b + \epsilon) \in C^1([0, R])$ is a subsolution for (P) on $B(R_0, R)$. According to the comparison principle, we get that $v_m(r, b + \epsilon) \leq u(x)$ for any $x \in B(0, R)$. For any $x \in B(0, R) \setminus B(0, R_0)$, we have $v_m(|x|, b + \epsilon) \leq v_{m+1}(|x|, b + \epsilon)$. Thus

$$
u(x) \geq \lim_{m \to +\infty} v_m(|x|, b + \epsilon), \quad \forall x \in B(0, R) \setminus B(0, R_0).
$$

(3.32)

When $d(x, \delta \Omega)$ is small enough, we have $\lim_{m \to +\infty} v_m(|x|, b + \epsilon) > v(|x|, b, 0)$. 

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According to the comparison principle, we get that $v(|x|, b, 0) \leq u(x), \forall x \in B(0, R)$; then $\Phi_2(x) = \Phi_2(|x|) = v(|x|, b, 0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x)$ is a radial subsolution for (P).

**Case 2** ($\beta_1(R) \leq 0$). Let $\mu > 2(\max_{r \in [0, R]} p(r) + 1)^{2(p' - 1)/(p - 1)}$ be a positive constant. Denote $\varpi_b(r, \lambda)$ on $[\sigma, R_0]$ as

$$
\varpi_b(r, \lambda) = \int_r^{R_0} \left\{ \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t - \sigma}{R_0 - \sigma} \frac{b}{s(R + \lambda - R_0)} \left( b \ln \left( R + \lambda - R_0 \right)^{-1} \right)^{(p(R_0) - 1)/s - 1} \right\}^{(p(R_0) - 1)/s} dt.
$$

(3.33)

It is easy to see that

$$
-\varpi_b'(\sigma, \lambda) = z_1'(\sigma) = \frac{\sigma \mu}{N} \left( \frac{1}{p(\sigma)} - 1 \right), \quad -\varpi_b'(R_0, \lambda) = \frac{b}{s(R + \lambda - R_0)} \left( b \ln \left( R + \lambda - R_0 \right)^{-1} \right)^{(p(R_0) - 1)/s - 1}.
$$

(3.34)

Define the function $\eta(r, b, \lambda)$ on $B(0, R)$ as

$$
\eta(r, b, \lambda) = \begin{cases} 
\left( b \ln \left( R + \lambda - r \right)^{-1} \right)^{1/s} - k^*, & R_0 \leq r < R, \\
\left( b \ln \left( R + \lambda - R_0 \right)^{-1} \right)^{1/s} - k^* - \varpi_b(r, \lambda), & \sigma < r < R_0, \\
-\int_{\sigma}^{r} \frac{\sigma \mu}{N} \left( \frac{1}{p(r)} - 1 \right) dr + \left( b \ln \left( R + \lambda - R_0 \right)^{-1} \right)^{1/s} - k^* - \varpi_b(\sigma, \lambda), & r \leq \sigma,
\end{cases}
$$

(3.35)

where $b \in (0, (1/\delta) \inf_{x \in [R_0, R]} \{ p(x) - \beta_1(x) \})$ is a constant, $R_0 \in (\sigma, R)$, $R - R_0$ is small enough, parameter $\lambda \in [0, (R - R_0)/2]$, and

$$
k^* = M + \left( b \ln \frac{1}{R - R_0} \right)^{1/s},
$$

(3.36)

where $M$ is defined in (3.23).

Obviously, for any positive constant $b$, $\eta(r, b, \lambda) \in C^1[0, R)$. 

Similar to the proof of Case 1, when $R - R_0$ is small enough, we have

\[
\left( r^{N-1} |\eta'|^{p(r)-2} \eta \right)' \geq r^{N-1} \left( \frac{\rho^{1/s}}{s} \right)^{p(r)-1} \left( \rho(r) - 1 \right)(R + \lambda - r)^{-p(r)} \left( \ln (R + \lambda - r)^{-1} \right)^{(1/s-1)(p(r)-1)} \left( 1 - \frac{1}{2} \right) \\
\geq r^{N-1} \rho(r) e^{f(r,\eta)}, \quad \forall r \in (R_0, R).
\]  

(3.37)

When $R - R_0$ is small enough, for all $r \in (\sigma, R_0)$, from the definition of $k^*$, it is easy to see that

\[
\left( r^{N-1} |\eta'|^{p(r)-2} \eta \right)' \geq (\sigma)^{N-1} \frac{1}{R_0 - \sigma} \geq r^{N-1} \rho(r) e^{f(r,\eta)}.
\]

(3.38)

Obviously

\[
\left( r^{N-1} |\eta'|^{p(r)-2} \eta \right)' = r^{N-1} \mu \geq r^{N-1} \rho(r) e^{f(r,\eta)}, \quad \forall r \in (0, \sigma).
\]

(3.39)

Combining (3.37), (3.38), and (3.39), when $R - R_0$ is large enough, for any $\lambda \in [0, (R - R_0)/2]$, one can see that $\eta(r, a, \lambda)$ is a subsolution for (P).

Define the function $\eta_m(r, b + \varepsilon)$ on $B(0, R)$ as

\[
\eta_m(r, b + \varepsilon) = \eta \left( r, b + \varepsilon, \frac{1}{m} \right),
\]

(3.40)

where $\varepsilon$ is a small enough positive constant such that $(b + \varepsilon) < (1/\delta) \inf_{|x| \geq R_0} p(x)$.

We can see that $\eta_m(r, b + \varepsilon) \in C^1[0, R]$ is a subsolution for (P) for any $m = 1, 2, \ldots$. According to the comparison principle, we get that $\eta_m(r, b + \varepsilon) \leq u(x)$ for any $x \in B(0, R)$. For any $x \in B(0, R) \setminus B(0, R_0)$, we have $\eta_m(\|x\|, b + \varepsilon) \leq \eta_{m+1}(\|x\|, b + \varepsilon)$. Then,

\[
u(x) \geq \lim_{m \to +\infty} \eta_m(\|x\|, b + \varepsilon), \quad \forall x \in B(0, R) \setminus B(0, R_0).
\]

(3.41)

When $d(x, \partial \Omega)$ is small enough, we have

\[
\lim_{m \to +\infty} \eta_m(\|x\|, b + \varepsilon) > \eta(\|x\|, b, 0).
\]

(3.42)

According to the comparison principle, we get that $\eta(\|x\|, b, 0) \leq u(x)$, $\forall x \in B(0, R)$; then $\Phi_2(x) = \Phi_2(|x|) = \eta(|x|, b, 0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x) = \Phi_2(|x|)$ is a radial subsolution for (P).
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Theorem 3.3. If \( f(r, u) \) satisfies

\[
\lim_{u \to +\infty} \frac{f(r, u)}{u^s} = \delta \quad \text{as } u \to +\infty \text{ for } r \in [\sigma, R) \text{ uniformly},
\]

where \( \sigma \) is defined in (H4), \( \delta \) and \( s \) are positive constants, \( \rho(r) = \rho_0(R-r)^{-\beta(r)} \), where \( \beta(R) < p(R) \), then each solution \( u(x) \) for (P) satisfies

\[
\lim_{|x| \to R} \frac{u(x)}{\left( (p(R)/\delta) \left( \ln 1/(R - |x|)^{1-\theta} \right) \right)^{1/s}} = 1, \quad \text{where } \theta = \frac{\beta(R)}{p(R)}.
\]

Proof. It is easy to be seen from Theorems 3.1 and 3.2 \( \square \)

4. The Existence of Boundary Blow-Up Solutions

Theorem 4.1. If \( \inf_{x \in \Omega} p(x) > N \) and \( f(r, u) \) satisfies

\[
f(r, u) \geq au^s \quad \text{as } u \to +\infty \text{ for } r \in [\sigma, R) \text{ uniformly},
\]

where \( \sigma \) is defined in (H4), \( a \) and \( s \) are positive constants, then (P) possesses a boundary blow-up solution.

Proof. In order to deal with the existence of boundary blow-up solutions, let us consider the problem

\[
-\Delta_{p(x)} u + \rho(r)e^{f(x,u)} = 0, \quad \text{in } \Omega_0,
\]

\[
u(x) = c, \quad \text{for } x \in \partial \Omega_0,
\]

where \( c \) is a positive constant and \( \Omega_0 \Subset \Omega \) is a radial subdomain of \( \Omega \). Since \( \inf_{x \in \Omega} p(x) > N \), then \( W^{1,p(x)}(\Omega_0) \to C^\alpha(\overline{\Omega_0}) \), where \( \alpha \in (0, 1) \). The relative functional of (4.2) is

\[
\varphi = \int_{\Omega_0} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_{\Omega_0} F(x, u) dx,
\]

where \( F(x, u) = \int_0^u e^{f(x,t)} dt \). Since \( \varphi \) is coercive in \( X := c + W^{1,p(x)}(\Omega_0) \), then \( \varphi \) possesses a nontrivial minimum point \( u \). So, problem (4.2) possesses a weak solution \( u \).

Since \( au^s \leq f(r, u) \leq C_1 + C_2|u|^{f(x)} \), from Theorems 3.1 and 3.2, we get that (P) possesses a supersolution \( g^*(x) \) and a subsolution \( g_*(x) \), which satisfy \( g^*(x) \geq g_*(x) \), when \( d(x, \partial \Omega) \) (the distance from \( x \) to \( \partial \Omega \)) is small enough. According to the comparison principle, we get that \( g^*(x) \geq g_*(x) \) for any \( x \in \Omega \).
Denote $D_j = \{ x \mid |x| < 1 - 1/(j + 1)R \} \ (j = 1, 2, \ldots)$. Let us consider the problem

$$
-\Delta_{p(x)} u_j + \rho(x) e^{f(x,u_j)} = 0, \quad \text{in} \ D_j,
$$

$$
u_j(x) = g_*(x), \quad \text{for} \ x \in \partial D_j,
$$

and the relative functional is

$$
\varphi = \int_{D_j} \frac{1}{p(x)} |\nabla u_j(x)|^{p(x)} \, dx + \int_{D_j} \rho(x) F(x, u_j) \, dx.
$$

Let $g_*(x) = g_*(x)|_{D_j}$. Since the functional $\varphi$ is coercive in $X_j = g_*(x) + W^{1,p(x)}_0(D_j)$, then $\varphi$ has a nontrivial minimum point $u_j$. Therefore, problem (4.4) has a weak solution $u_j$.

According to the comparison principle, we get that $g_*(x) \leq u_j(x)$ for any $x \in D_j \ (j = 1, 2, \ldots)$. Since $u_j(x) = g_*(x)$ for any $x \in \partial D_j$, then $u_j(x) \leq u_{j+1}(x)$ for any $x \in \partial D_j$ \ ($j = 1, 2, \ldots$). According to the comparison principle, we get that $u_j(x) \leq u_{j+1}(x)$ for any $x \in D_j \ (j = 1, 2, \ldots)$.

Since $g_*(x)$ is a supersolution and $g_*(x) \geq g_*(x)$ for any $x \in \Omega$, so we have $u_j(x) = g_*(x)$ for any $x \in \partial D_j \ (j = 1, 2, \ldots)$. According to the comparison principle, we get that $u_j(x) \geq g_*(x)$ for any $x \in \partial D_j \ (j = 1, 2, \ldots)$.

Since $g_*(x)$ and $g_*(x)$ are locally bounded, from Lemma 2.4, each weak solution of (4.4) is a $C^{1,\alpha}$ function. The $C^{1,\alpha}$ interior regularity result implies that the sequences $\{u_j\}$ and $\{\nabla u_j\}$ are equicontinuous in $D_2$, and hence we can choose a subsequence, which we denoted by $\{u_j^\prime\}$, such that $u_j^\prime \to u_1$ and $\nabla u_j^\prime \to \nabla u_1$ uniformly on $D_1$ for some $w_1 \in C(D_1)$ and $\nabla w_1 \in (C^{\alpha}(D_1))^N$. In fact, $w_1 = \nabla w_1$ on $D_1$, and from the interior $C^{1,\alpha}$ estimate, we conclude that $\nabla w_1 \in (C^{\alpha}(D_1))^N$ for some $0 < \alpha < 1$. Thus, $w_1 \in W^{1,p(x)}(D_1) \cap C^{1,\alpha}(D_1)$. From the $C^{1,\alpha}$ interior regularity result, we see that $|\nabla u_j|^{p(x)-1} |\nabla \varphi| \leq C |\nabla \varphi|$ on $D_1$, and since the function $\xi \to \|\xi\|^{p^2-2} \xi$ is continuous on $\mathbb{R}^N$, it follows that $|\nabla u_j^\prime(x)|^{p(x)-2} \nabla u_j^\prime(x) \cdot \nabla \varphi(x) \to |\nabla w_1(x)|^{p(x)-2} \nabla w_1(x) \cdot \nabla \varphi(x)$ for $x \in D_1$. Thus, by the dominated convergence theorem, we have

$$
\int_{D_1} |\nabla u_j^\prime(x)|^{p(x)-2} \nabla u_j^\prime(x) \cdot \nabla \varphi(x) \, dx \to \int_{D_1} |\nabla w_1(x)|^{p(x)-2} \nabla w_1(x) \cdot \nabla \varphi(x) \, dx, \quad \forall \varphi \in W^{1,p(x)}_0(D_1).
$$

Furthermore, since $0 \leq f(u_j^\prime) \leq f(u_{j+1}^\prime)$ and $f(u_j^\prime(x)) \to f(w_1(x))$ for each $x \in D_1$, by the monotone convergence theorem, we obtain

$$
\int_{D_1} \rho e^{f(u_j^\prime)} q \, dx \to \int_{D_1} \rho e^{f(w_1)} q \, dx, \quad \forall q \in W^{1,p(x)}_0(D_1).
$$

Therefore, it follows that

$$
\int_{D_1} |\nabla w_1(x)|^{p(x)-2} \nabla w_1(x) \cdot \nabla q(x) \, dx + \int_{D_1} \rho e^{f(w_1)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_1),
$$

and hence $w_1$ is a weak solution for $-\Delta_{p(x)} w_1 + \rho e^{f(w_1)} = 0$ on $D_1$. 

Thus, there exists a subsequence of \( \{ u_j \} \) which we denote it by \( \{ u_j^1 \} \), such that \( u_j^1 \to w_1 \) in \( D_1 \) (as \( j \to \infty \)), where \( w_1 \in W^{1,p(x)}(D_1) \cap C^{1,a_1}(D_1) \) and satisfies

\[
\int_{D_1} |\nabla w_1|^{p(x)-2} \nabla w_1 \nabla q \, dx + \int_{D_1} \rho(x)e^{f(x,w_1)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_1). \quad (4.9)
\]

Similarly, we can prove that there exists a subsequence of \( \{ u_j^1 \} \) which we denote by \( \{ u_j^2 \} \), such that \( u_j^2 \to w_2 \) in \( D_2 \) (as \( j \to \infty \)), where \( w_2 \in W^{1,p(x)}(D_2) \cap C^{1,a_1}(D_2) \) satisfies \( w_1 = w_2|_{D_1} \) and

\[
\int_{D_2} |\nabla w_2|^{p(x)-2} \nabla w_2 \nabla q \, dx + \int_{D_2} \rho(x)e^{f(x,w_2)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_2). \quad (4.10)
\]

Repeating the above steps, we can get a subsequence of \( \{ u_j^i \} \) which we denote by \( \{ u_j^{i+1} \} \) (i = 1, 2, \ldots) which satisfies the following:

1. For any fixed \( i \), \( \{ u_j^{i+1} \} \) is a subsequence of \( \{ u_j^i \} \).
2. For any fixed \( i \), \( u_j^{i+1} \to w_{i+1} \) in \( D_{i+1} \) (as \( j \to \infty \)), where \( w_{i+1} \in W^{1,p(x)}(D_{i+1}) \cap C^{1,a_1}(D_{i+1}) \) satisfies \( w_i = w_{i+1}|_{D_i} \).
3. For any fixed \( i \), \( w_i \) satisfies

\[
\int_{D_i} |\nabla w_i|^{p(x)-2} \nabla w_i \nabla q \, dx + \int_{D_i} \rho(x)e^{f(x,w_i)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_i). \quad (4.11)
\]

Thus, we can conclude that

(i) \( \{ u_j^i \} \) is a subsequence of \( \{ u_j \} \),

(ii) there exists a function \( w \in W^{1,p(x)}_{\text{loc}}(\Omega) \cap C^{1,a}_{\text{loc}}(\Omega) \) such that \( w_i = w|_{D_i} \), and for any \( x \in \Omega \), there exists a constant \( j_x \) such that when \( j \geq j_x \), \( u_j^i(x) \) is defined at \( x \), and \( \lim_{j \to \infty} u_j^i(x) = w(x) \),

(iii)

\[
\int_{\Omega} |\nabla w|^{p(x)-2} \nabla w \nabla q \, dx + \int_{\Omega} \rho(x)e^{f(x,w)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_{0,\text{loc}}(\Omega). \quad (4.12)
\]

Obviously, \( w \) is a boundary blow-up solution for (P).

This completes the proof. \( \Box \)
In Theorem 4.1, when \( \inf_{x \in \Omega} p(x) > N \), the existence of solutions for (P) is given. In the following, we will consider the existence of solutions for (P) in the general case \( 1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < \infty \). We need to do some preparation. Let us consider

\[
\left( r^{N-1} |u'|^p(r-2) u' \right)' = r^{N-1} \rho(r)e^{f(r,u)}, \quad r \in (0, R_1),
\]

\[
u'(0) = 0, \quad u(R_1) = d,
\]

where \( R_1 \in (0, R) \) and \( d \) is a constant.

**Lemma 4.2.** If \( \Phi_2(R_1) \leq d \leq \Phi_1(R_1) \), where \( \Phi_1 \) and \( \Phi_2 \) are defined in Theorems 3.13.2, respectively, then (4.13) has a solution \( u \) satisfying

\[
\Phi_2(r) \leq u(r) \leq \Phi_1(r), \quad \forall r \in [0, R_1].
\]

**Proof.** Denote

\[
h(r, u) = \begin{cases} 
\frac{e^{f(r,\Phi_1(r))} + \arctan(u(r) - \Phi_1(r))}{u(r)} & u(r) > \Phi_1(r), \\
e^{f(r,u)}, & \Phi_2(r) \leq u(r) \leq \Phi_1(r), \\
\frac{e^{f(r,\Phi_2(r))} + \arctan(u(r) - \Phi_2(r))}{u(r)} & u(r) < \Phi_2(r).
\end{cases}
\]

Let \( \rho_E(t) = \rho(|t|) \), and \( h_E(t, u) = h(|t|, u) \), for all \( t \in [-R_1, R_1] \). Let us consider the even solutions of the following

\[
\left( |t|^{N-1} |u'|^p(|t|)^{-2} u' \right)' = |t|^{N-1} \rho_E(t) |t|^{N-1} \rho_E(t) h_E(t, u), \quad t \in (-R_1, R_1),
\]

\[
u(-R_1) = d, \quad u(R_1) = d.
\]

It is easy to see that \( u \) is an even solution for (4.15) if and only if \( u \) is even and

\[
u = d - \int_{R_1}^{R_1} \left[ |t|^{1-N} \int_0^t |s|^{N-1} \rho(s) h(s, u(s)) ds \right]^{1/(p-1)} dt, \quad \forall r \in [0, R_1].
\]

Denote \( \Psi(u, \mu) = \mu d - \mu \int_{R_1}^{R_1} \left[ |t|^{1-N} \int_0^t |s|^{N-1} \rho(s) h(s, u(s)) ds \right]^{1/(p-1)} dt \). Similar to the proof of Lemma 2.3 of [18], for any \( \mu \in [0, 1] \), it is easy to see that \( \Psi(u, \mu) \) is compact continuous and bounded from \( C^1[0, R_1] \) to \( C^1[0, R_1] \), where \( C^1[0, R_1] = \{ u \in C^1[0, R_1] \mid u \text{ is even} \} \). Thus, \( u = \Psi(u, 1) \) has a solution \( u \) in \( C^1[0, R_1] \) and satisfies \( u'(0) = \lim_{r \to 0^+} u'(r) = 0 \). Then, \( u(|t|) \) is an even solution for (4.15).

Denote \( \Phi_1(t) = \Phi_1(|t|), \Phi_2(t) = \Phi_2(|t|) \). From the definitions of \( \Phi_1 \) and \( \Phi_2 \), we can see that \( \Phi_1'(0) = 0 = \Phi_2'(0) \); therefore, \( \Phi_1(t) \) and \( \Phi_2(t) \) are supersolution and subsolution for (4.15), respectively.
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Since \( \Phi_2(R_i) \leq u(R_i) \leq \Phi_1(R_i) \) and \( h_E(t, \cdot) \) is increasing, from the comparison principle, we have

\[
\Phi_2E(t) \leq u(t) \leq \Phi_1E(t), \quad \forall t \in [-R_i, R_i].
\]  

(4.16)

It means that \( u \) is a solution for (4.13) and \( u \) satisfies

\[
\Phi_2(r) \leq u(r) \leq \Phi_1(r), \quad \forall r \in [0, R_i].
\]  

(4.17)

Thus \( u \) is a radial solution for (P). This completes the proof. \( \square \)

Theorem 4.3. If \( f(r, u) \) satisfies

\[
f(r, u) \geq au^s \quad \text{(as} \ u \rightarrow +\infty) \text{for} \ r \in [\sigma, R] \text{uniformly},
\]  

(4.18)

where \( \sigma \) is defined in (H_4) and \( a \) and \( s \) are positive constants, then (P) possesses a boundary blow-up solution.

Proof. From Lemma 4.2, we have that (4.4) has a weak solution \( u_j(x) = u_j(|x|) = u_j(r) \). Similar to the proof of Theorem 4.1, we can obtain the existence of solutions for (P). \( \square \)

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References


