Research Article

On the Nevanlinna’s Theory for Vector-Valued Mappings

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The purpose of this paper is to establish the first and second fundamental theorems for an \( E \)-valued meromorphic mapping from a generic domain \( D \subset \mathbb{C} \) to an infinite dimensional complex Banach space \( E \) with a Schauder basis. It is a continuation of the work of C. Hu and Q. Hu. For \( f(z) \) defined in the disk, we will prove Chuang’s inequality, which is to compare the relationship between \( T(r,f) \) and \( T(r,f') \). Consequently, we obtain that the order and the lower order of \( f(z) \) and its derivative \( f'(z) \) are the same.

1. Introduction

In 1980s, Ziegler [1] established Nevanlinna’s theory for the vector-valued meromorphic functions in finite dimensional spaces. After Ziegler some works in finite dimensional spaces were done in 1990s [2–4]. In 2006, C. Hu and Q. Hu [5] considered the case of infinite dimensional spaces and they investigated the \( E \)-valued meromorphic mappings defined in the disk \( C_r = \{ z : |z| < r \} \). In this article, by using Green function technique, we will consider this theory defined in generic domain \( D \subset \mathbb{C} \) (see Section 2). In Section 3, motivated by the work of [6–8], we will prove Chuang’s inequality, which is to compare the relationship between \( T(r,f) \) and \( T(r,f') \). Consequently, we obtain that the order and the lower order of \( f(z) \) and its derivative \( f'(z) \) are the same. This is an extension of an important result for meromorphic functions.

2. First and Second Fundamental Theorem in Generic Domains

Let \( (E, \| \cdot \|) \) be a complex Banach space with a Schauder basis \( \{ e_i \} \) and the norm \( \| \cdot \| \). Thus an \( E \)-valued meromorphic mapping \( f(z) \) defined in a domain \( D \subset \mathbb{C} \) can be written as \( f(z) = \)}
Theorem 2.3. A vector-valued mapping is called holomorphic (meromorphic) if all \( f_i(z) \) are holomorphic (meromorphic). The \( j \)th derivative \( (j=1,2,\ldots) \) and the integration of \( f(z) \) are defined by

\[
f^{(j)}(z) = \left( f^{(j)}_1(z), f^{(j)}_2(z), \ldots, f^{(j)}_k(z), \ldots \right),
\]

\[
\int f(\zeta) d\zeta = \left( \int f_1(\zeta) d\zeta, \int f_2(\zeta) d\zeta, \ldots, \int f_k(\zeta) d\zeta, \ldots \right),
\]

respectively. We assume that \( f^{(0)}(z) = f(z) \). A point \( z_0 \in D \) is called a “pole” or “\( \hat{\infty} \)-point” of \( f(z) = (f_1(z), \ldots, f_k(z), \ldots) \) if \( z_0 \) is a pole of at least one of the component functions \( f_k(z) \) \( (k = 1, 2, \ldots) \). We define \( \| f(z_0) \| = +\infty \) when \( z_0 \) is a pole. A point \( z_0 \in D \) is called “zero” of \( f(z) \) if all the component functions \( f_k(z) \) \( (k = 1, 2, \ldots) \) have zeros at \( z_0 \).

Remark 2.1. The integrals are well defined because the set of singularities making \( \hat{\infty} - \hat{\infty} \) meaningless is zero measurable.

In order to make our statement clear, we first recall some knowledge of Green functions.

Definition 2.2. Let \( D \) be a domain surrounded by finitely many piecewise analytic curves. Then for any \( a \in D \), there exists a Green function, denoted by \( G_D(z,a) \), for \( D \) with singularity at \( a \in D \) which is uniquely determined by the following:

1. \( G_D(z,a) \) is harmonic in \( D \setminus \{a\} \);
2. in a neighborhood of \( a \), \( G_D(z,a) = -\log |z - a| + w(z,a) \) for some function \( w(z,a) \) harmonic in \( D \);
3. \( G_D(z,a) \equiv 0 \), on the boundary of \( D \).

By \( \partial D \) we denote the boundary of \( D \) and \( \vec{n} \) the inner normal of \( \partial D \) with respect to \( D \). Using Green function we can establish the following general Poisson formula for the \( E \)-valued meromorphic mapping, which is similar with [5, Lemma 2.2] (see [9, Theorem 2.1], or [10, Theorem 2.1.1]). We do not give the details here.

Theorem 2.3. Let \( f : \overline{D}(\subset \mathbb{C}) \rightarrow E \) be an \( E \)-valued meromorphic mapping, which does not reduce to the constant zero element \( 0 \in E \). Then

\[
\log \| f(z) \| = \frac{1}{2\pi} \int_{\partial D} \log \| f(\zeta) \| \frac{\partial G_D(\zeta,z)}{\partial H} d\zeta - \sum_{a_m \in D} G_D(a_m,z)
\]

\[
+ \sum_{b_n \in D} G_D(b_n,z) - \frac{1}{2\pi} \int_D G_D(\zeta,z) \Delta \log \| f(\zeta) \| dx \land dy,
\]

where \( \zeta = x + iy \), \( \{a_m\} \) are the zeros of \( f(z) \) and \( \{b_n\} \) are the poles of \( f(z) \) according to their multiplicities.
Remark 2.4. A simple inspection to the \( \mathbb{C}^2 \)-valued case shows that \( \log \| f(z) \| \) is not harmonic for a holomorphic (or meromorphic) \( E \)-valued function. Therefore we have an additional term in formula (2.2).

Following Theorem 2.3, we introduce some notations.

\[
N(D, a, f) = \sum_{b_n \in D} G_D(b_n, a),
\]
\[
m(D, a, f) = \frac{1}{2\pi} \int_{\partial D} \log^+ \| f(\zeta) \| \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} \, ds,
\]
\[
V(D, a, f) = \frac{1}{2\pi} \int_{D} G(\zeta, a) \Delta \log \| f(\zeta) \| \, dx \wedge dy,
\]

where \( a \) is a point in \( D \) and \( \{ b_n \} \) are the poles of \( f(z) \) in \( D \) appearing according to their multiplicities, \( \log^+ x = \log \max\{x, 1\} \). Define

\[
T(D, a, f) = m(D, a, f) + N(D, a, f).
\]

\( T(D, a, f) \) is called the Nevanlinna characteristic function of \( f(z) \) with the center \( a \in D \).

Next, we give the first (FFT) and the second (SFT) fundamental theorems for \( f(z) \).

**Theorem 2.5** (FFT). Let \( f(z) \) be an \( E \)-valued meromorphic mapping on \( \overline{D} \). Then for a fixed vector \( b \in E \) and for any \( a \in D \) such that \( f(a) \neq b \), one has

\[
T\left(D, a, \frac{1}{f - b}\right) = T(D, a, f) - V(D, a, f - b) - \log \| f(a) - b \| + \varepsilon(b, D),
\]

where

\[
|\varepsilon(b, D)| \leq \begin{cases} 
\log^+ \| b \| + \log 2, & b \neq 0, \\
0, & b = 0.
\end{cases}
\]

**Proof.** We can rewrite Theorem 2.3 as follows:

\[
T(D, a, f) = T\left(D, a, \frac{1}{f}\right) + V(D, a, f) + \log \| f(a) \|.
\]

Applying this formula to the function \( f(z) - b \), we can prove the theorem. \( \square \)
Theorem 2.6 (SFT). Let \( f(z) \) be an \( E \)-valued meromorphic mapping on \( \overline{D} \), let \( a^{[j]} (j = 1, 2, \ldots, q) \in E \cup \{ \infty \} \) be \( q \) distinct vectors, and let \( f(a) \neq a^{[j]} \). Then,

\[
(q - 2)T(D, a, f) \leq \sum_{j=1}^{q} \left[ N(D, a, f = a^{[j]}) + V(D, a, f - a^{[j]}) \right] - V(D, a, f^*) - N_1(D, a, f) + S(D, a, f),
\]

where

\[
S(D, a, f) = \frac{1}{2\pi} \int_{\partial D} \log^{+} \left[ \sum_{j=1}^{q} \frac{\| f'(\zeta) \|}{\| f(\zeta) - a^{[j]} \|} \right] \frac{\partial G_D(\zeta, a)}{\partial \vec{n}} ds - \log \| f'(a) \| + q \log^+ \frac{2q}{\delta} + \sum_{i=1}^{q} \log \| f(a) - a^{[i]} \|,
\]

\[
N_1(D, a, f) = 2N(D, a, f) - N(D, a, f^*) + N(D, a, \frac{1}{f^*}),
\]

\[
\delta = \min_{i \neq j} \| a^{[i]} - a^{[j]} \| > 0.
\]

Furthermore, one has the following form:

\[
(q - 2)T(D, a, f) \leq \sum_{j=1}^{q} \left[ N(D, a, \frac{1}{f - a^{[j]}}) + V(D, a, f - a^{[j]}) \right] - V(D, a, f^*) + S(D, a, f). \tag{2.10}
\]

Proof. Set

\[
F(\zeta) = \sum_{j=1}^{q} \frac{1}{\| f(\zeta) - a^{[j]} \|}.
\]

According to the property of the logarithm function, we get

\[
\frac{1}{2\pi} \int_{\partial D} \log^{+} F(\zeta) \frac{\partial G_D(\zeta, a)}{\partial \vec{n}} ds \leq m(D, a, \frac{1}{f^*}) + \frac{1}{2\pi} \int_{\partial D} \log^{+} [F(\zeta) \| f'(\zeta) \|] \frac{\partial G_D(\zeta, a)}{\partial \vec{n}} ds. \tag{2.12}
\]

Denote \( \delta = \min_{i \neq j} \| a^{[i]} - a^{[j]} \| \), and fix \( \mu \in \{ 1, 2, \ldots, q \} \). Then we obtain

\[
\| f(z) - a^{[\mu]} \| \geq \| a^{[\mu]} - a^{[j]} \| - \| a^{[\mu]} - f(z) \| > \frac{3\delta}{4} \tag{2.13}
\]
for $\mu \neq v$ by

$$
\|f(z) - a^{|\mu|}\| < \frac{\delta}{2q} \leq \frac{\delta}{4}.
$$

(2.14)

So either the set of points on $\partial D$ which is determined by (2.14) is empty or any two of some sets for different $\mu$ have intersection. In any case, on $\partial D$ we have

$$
\frac{1}{2\pi} \int_{\partial D} \log^+ F(\zeta) \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds \geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int_{\|f - a^{|\mu|}\|<\delta/2q} \log^+ F(\zeta) \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds
$$

$$
\geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int_{\|f - a^{|\mu|}\|\geq\delta/2q} \log^+ \frac{1}{\|f(\zeta) - a^{|\mu|}\|} \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds.
$$

(2.15)

Since

$$
\frac{1}{2\pi} \int_{\|f - a^{|\mu|}\|<\delta/2q} \log^+ \frac{1}{\|f(\zeta) - a^{|\mu|}\|} \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds
$$

$$
= m(D, a, a^{|\mu|}) - \frac{1}{2\pi} \int_{\|f - a^{|\mu|}\|>\delta/2q} \log^+ \frac{1}{\|f(\zeta) - a^{|\mu|}\|} \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds
$$

$$
\geq m(D, a, a^{|\mu|}) - \log^+ \frac{2q}{\delta},
$$

it follows that

$$
\frac{1}{2\pi} \int_{\partial D} \log^+ F(\zeta) \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds \geq \sum_{\mu=1}^{q} m(D, a, a^{|\mu|}) - q \log^+ \frac{2q}{\delta}.
$$

(2.17)

From (2.12), we get

$$
m(D, a, \frac{1}{f'}) \geq \sum_{\mu=1}^{q} m(D, a, a^{|\mu|}) - q \log^+ \frac{2q}{\delta} - \frac{1}{2\pi} \int_{\partial D} \log^+[F(\zeta)\|f'(\zeta)\|] \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds.
$$

(2.18)

Since $f(z)$ is nonconstant vector, $f'(z)$ does not reduce to the constant zero element $0$. Applying FFT to $f'(z)$, we can obtain

$$
T(D, a, f') = N(D, a, \frac{1}{f'}) + m(D, a, \frac{1}{f'}) + V(D, a, f') + \log \|f'(a)\|.
$$

(2.19)
Using this formula, we have

\[
T(D, a, f') \geq \sum_{\mu=1}^{q} m(D, a, a^{[\mu]}) - q \log \frac{2q}{\delta} \\
- \frac{1}{2\pi} \int_{\partial D} \log^+ \left| F(\xi) \| f' (\xi) \| \right| \frac{\partial G_D (\xi, a)}{\partial n} d\sigma \\
+ N(D, a, \frac{1}{f'}) + V(D, a, f') + \log \| f' (a) \|.
\] (2.20)

On the other hand, we have

\[
T(D, a, f') = m(D, a, f') + N(D, a, f') \\
\leq m(D, a, f) + N(D, a, f') + \frac{1}{2\pi} \int_{\partial D} \log \left| f' (\xi) \right| \frac{\partial G_D (\xi, a)}{\partial n} d\sigma.
\] (2.21)

The two inequalities above give

\[
\sum_{\mu=1}^{q} m(D, a, a^{[\mu]}) + V(D, a, f') \\
\leq m(D, a, f) + N(D, a, f') - \frac{1}{2\pi} \int_{\partial D} \log \left| f' (\xi) \right| \frac{\partial G_D (\xi, a)}{\partial n} d\sigma + \frac{1}{2\pi} \int_{\partial D} \log \left| f' (\xi) \right| \frac{\partial G_D (\xi, a)}{\partial n} d\sigma
\]

\[- \log \left| f' (a) \right| + q \log \frac{2q}{\delta}.
\] (2.22)

That is to say,

\[
\sum_{\mu=1}^{q} m(D, a, a^{[\mu]}) + V(D, a, f') \\
\leq m(D, a, f) + N(D, a, f') - \frac{1}{2\pi} \int_{\partial D} \log \left| f' (\xi) \right| \frac{\partial G_D (\xi, a)}{\partial n} d\sigma + S(D, a, f).
\] (2.23)

Adding \( \sum_{\mu=1}^{q} N(D, a, f = a^{[\mu]} ) \) to the above inequality and applying FFT, we can formulate

\[
(q - 1) T(D, a, f) < N(D, a, f) + \sum_{j=1}^{q} \left[ N(D, a, f = a^{[j]}) + V(D, a, f - a^{[j]}) \right] \\
- N_1(D, a, f) + S(D, a, f),
\] (2.24)
If \( f \) where \( S(D, a, f) = \frac{1}{2\pi} \int_{\partial D} \log^+ \left[ \sum_{j=0}^{q} \frac{\| f'(\zeta) \|}{\| f(\zeta) - a^{[j]} \|} \right] \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds \)

\((2.25)\)

\[- \log \| f'(a) \| + q \log^+ \frac{2q}{\delta} + \sum_{i=0}^{q} \log \| f(a) - a^{[i]} \|, \quad a^{[0]} = 0.\]

Since \( N(D, a, f) \leq T(D, a, f) \), (2.24) can be written as

\((q - 2)T(D, a, f) \leq \sum_{j=1}^{q} \left[ N(D, a, f = a^{[j]}) + V(D, a, f - a^{[j]}) \right]\)

\(- N_1(D, a, f) + S(D, a, f). \)

If \( \{a^{[j]}\} \) contains \( \infty \), (2.26) also holds. Let \( a^{[q+1]} = \infty \), and substitute \( q \) with \( q + 1 \); then we have (2.26), where \( a^{[q]} = \infty \), and

\( S(D, a, f) = \frac{1}{2\pi} \int_{\partial D} \log^+ \left[ \sum_{j=0}^{q-1} \frac{\| f'(\zeta) \|}{\| f(\zeta) - a^{[j]} \|} \right] \frac{\partial G_D(\zeta, a)}{\partial \bar{n}} ds \)

\[- \log \| f'(a) \| + q \log^+ \frac{2q}{\delta} + \sum_{i=0}^{q-1} \log \| f(a) - a^{[i]} \|. \]

Next we establish Hiong King-Lai’s inequality for \( f(z) \).

**Theorem 2.7.** Let \( f(z) \) be an \( E \)-valued meromorphic mapping on \( \overline{D} \), \( l \in D \), let \( a, b, c \in E \) be three finite vectors, and let \( b \neq 0, c \neq 0, b \neq c, f^{(k)}(l) \neq 0, b, c \). Then one has

\( T(D, l, f) < N(D, l, f = a) + N(D, l, f^{(k)} = b) + N(D, l, f^{(k)} = c) \)

\(+ V(D, l, f^{(k)}) + V(D, l, f^{(k)} - b) + V(D, l, f^{(k)} - c) - N(D, l, \frac{1}{f^{(k+1)}}) \)

\(+ S(D, l, f^{(k)}). \)

\((2.28)\)

**Proof.** First, we have

\[ \frac{1}{2\pi} \int_{\partial D} \log^+ \left\| \frac{1}{f'(\zeta) - a} \right\| \frac{\partial G_D(\zeta, l)}{\partial \bar{n}} ds \]

\[ \leq \frac{1}{2\pi} \int_{\partial D} \log^+ \left\| \frac{1}{f^{(k)}(\zeta)} \right\| \frac{\partial G_D(\zeta, l)}{\partial \bar{n}} ds + \frac{1}{2\pi} \int_{\partial D} \log^+ \left\| \frac{f^{(k)}(\zeta)}{f(\zeta) - a} \right\| \frac{\partial G_D(\zeta, l)}{\partial \bar{n}} ds. \]

\((2.29)\)
Applying FFT to \( f(z) \) and \( f^{(k)}(z) \), respectively, we have

\[
\frac{1}{2\pi} \int_{\partial D} \log^+ \left\| \frac{1}{f(\zeta) - a} \frac{\partial G_D(\zeta, l)}{\partial \eta} \right\|_2 \, ds
= T(D, l, f) - N(D, l, f = a) - V(D, l, f - a) - \log \|f(l) - a\| + \varepsilon(a, D),
\]

\[
\frac{1}{2\pi} \int_{\partial D} \log^+ \left\| \frac{1}{f^{(k)}(\zeta)} \frac{\partial G_D(\zeta, l)}{\partial \eta} \right\|_2 \, ds
= T(D, l, f^{(k)}) - N\left(D, l, \frac{1}{f^{(k)}}\right) - V\left(D, l, f^{(k)}\right) - \log \|f^{(k)}(l)\|.
\]  

Thus we have

\[
T(D, l, f) \leq T\left(D, l, f^{(k)}\right) + N(D, l, f = a) + V(D, l, f - a) - N\left(D, l, \frac{1}{f^{(k)}}\right) - V\left(D, l, f^{(k)}\right) - \log \|f(l) - a\| - \varepsilon(a, D).
\]  

Applying SFT to \( f^{(k)} \) with \( 0, b, c \), we have

\[
T\left(D, l, f^{(k)}\right) \leq N\left(D, l, \frac{1}{f^{(k)}}\right) + N\left(D, l, f^{(k)} = b\right) + N\left(D, l, f^{(k)} = c\right) - N\left(D, l, f^{(k+1)}\right) + V\left(D, l, f^{(k)}\right) + V\left(D, l, f^{(k) - b}\right) + V\left(D, l, f^{(k) - c}\right) - V\left(D, l, f^{(k+1)}\right) + S\left(D, l, f^{(k)}\right). \]

Combining (2.31) with (2.32), we have

\[
T(D, l, f) \leq N(D, l, f = a) + N\left(D, l, f^{(k)} = b\right) + N\left(D, l, f^{(k)} = c\right) - N\left(D, l, f^{(k+1)}\right) + V\left(D, l, f - a\right) + V\left(D, l, f^{(k)} - b\right) + V\left(D, l, f^{(k)} - c\right) - V\left(D, l, f^{(k+1)}\right) + S\left(D, l, f^{(k)}\right). \]
3. The Vector-Valued Mapping and Its Derivative

In this section, we will discuss the value distribution theory of \( f(z) \) defined in the disk \( C_r = \{ z : |z| < r \} \). We will prove Chuang’s inequality. According to (2.3), we have the following terms:

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \| f(re^{it}) \| d\theta,
\]
\[
N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,
\]
\[
V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\zeta} \right| \Delta \log \| f(\zeta) \| dx \wedge dy,
\]
\[
\zeta = x + iy,
\]
\[
T(r, f) = m(r, f) + N(r, f),
\]

where \( n(r, f) \) denotes the number of poles of \( f(z) \) in \( \{ z : |z| < r \} \). The order and the lower order of an \( E \)-valued meromorphic mapping \( f(z) \) are defined by

\[
\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},
\]
\[
\mu(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

The following lemma is well known.

**Lemma 3.1** (see [11, Boutroux-Cartan Theorem]). Let \( \{ a_j \}_{j=1}^n \) be \( n \) complex numbers. Then the set of the point \( z \) satisfying

\[
\prod_{j=1}^n |z - a_j| < h^n
\]

can be contained in several disks, denoted by \( \gamma \); the total sum of its radius does not exceed \( 2eh \).

The next lemma is a special case of Theorem 2.3.
Lemma 3.2 (see [5]). Let \( f : C \rightarrow E \) be an \( E \)-valued meromorphic mapping, which does not reduce to the constant zero element \( 0 \in E \). Then, for a \( z \in C \), one has

\[
\log \|f(z)\| = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| \frac{r^2 - t^2}{r^2 - 2rt \cos(\theta - \phi) + t^2} \, d\phi
\]

\[
- \sum_{z_j(0) \in C} \log \left| \frac{r^2 - z_j(0)z}{r(z - z_j(0))} \right| + \sum_{z_j(\infty) \in C} \log \left| \frac{r^2 - z_j(\infty)z}{r(z - z_j(\infty))} \right|.
\]

Here \( z_j(0) \) and \( z_j(\infty) \) are all the zeros and poles counting their multiplies of \( f \) in \( D \).

In order to obtain the relationship between \( T(r, f) \) and \( T(r, f') \), we should first establish the following two lemmas.

Lemma 3.3. Let \( f : \mathbb{C} \rightarrow E \) be a nonzero \( E \)-valued meromorphic mapping, and \( f(0) \neq \infty \). If \( R \) and \( R' \) are two positive numbers, and \( R < R' \), then there exists a \( \theta_0 \in [0, 2\pi) \), such that for any \( 0 \leq r \leq R' \) one has

\[
\log^+ \|f(re^{i\theta_0})\| \leq \frac{R' + R}{R - R'} m(R', f) + n(R', f) \log 4 + N(R', f).
\]

Proof. For \( z = re^{i\theta}, 0 \leq r \leq R \). By Lemma 3.2 we have

\[
\log \|f(z)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| \frac{R^2 - r^2}{R^2 - 2R'r \cos(\theta - \phi) + r^2} \, d\phi
\]

\[
+ \sum_{j=1}^n \log \left| \frac{R^2 - b_j z}{R'(z - b_j)} \right|,
\]

where \( \{b_j\}_{j=1}^n \) are the poles of \( f(z) \) in \( |z| \leq R' \). Then

\[
\log^+ \|f(z)\| \leq \frac{R' + R}{R - r} m(R', f) + \sum_{j=1}^n \log \frac{2R'}{|z - b_j|}
\]

\[
\leq \frac{R' + R}{R' - r} m(R', f) + \log \frac{(2R')^n}{\prod_{j=1}^n |z - b_j|}.
\]

Writing \( b_j = |b_j|e^{i\phi_j} \), we have

\[
|re^{i\theta} - |b_j|e^{i\phi_j}| \geq |b_j| \sin(\theta - \phi_j)|.
\]
Thus
\[ \prod_{j=1}^{n} |z - b_j| \geq \left( \prod_{j=1}^{n} |b_j| \right) \left( \prod_{j=1}^{n} |\sin(\theta - \phi_j)| \right). \quad (3.9) \]

However,
\[ \int_{0}^{\pi} \log \left| \prod_{j=1}^{n} \sin(\theta - \phi_j) \right| d\theta = n \int_{0}^{\pi} \log |\sin \theta| d\theta = -n\pi \log 2. \quad (3.10) \]

Hence there exists a real number \( \theta_0 \) such that
\[ \left| \prod_{j=1}^{n} \sin(\theta_0 - \phi_j) \right| > \frac{1}{2^n}. \quad (3.11) \]

Combining (3.7) and (3.9) with (3.11), we have
\[ \log^{+} \|f(re^{i\theta})\| \leq \frac{R' + R}{R' - R} m(R', f) + n \log 4 + \sum_{j=1}^{n} \log \frac{R'}{|b_j|} \]
\[ \leq \frac{R' + R}{R' - R} m(R', f) + n \log 4 + N(R', f). \quad (3.12) \]

**Lemma 3.4.** Let \( f : \mathbb{C} \to E \) be a nonzero \( E \)-valued meromorphic mapping, and let \( R < R' < R'' \) be three positive numbers. Then there exists a positive number \( R \leq \rho \leq R' \), and for \( |z| = \rho \), one has
\[ \log^{+} \|f(z)\| \leq \frac{R'' + R'}{R'' - R'} m(R'', f) + n(R'', f) \log \frac{8eR''}{R'' - R'}. \quad (3.13) \]

**Proof.** Let \( \{b_j\}_{j=1}^{n} \) be the poles of \( f(z) \) in \( |z| \leq R'' \). By Boutroux-Cartan Theorem, we have
\[ \prod_{j=1}^{n} |z - b_j| \geq \left( \frac{R' - R}{4e} \right)^{n}. \quad (3.14) \]

except for some points contained in a pack of disks whose radius does not exceed \((R' - R)/2\). Then there exists a circle \( \{z : |z| = \rho\} \) such that \( R \leq \rho \leq R' \) and \( \{|z| = \rho\} \cap (\gamma) = \emptyset \).
Thus (3.14) holds on \( \{ |z| = \rho \} \). For any \( z \in \{ z : |z| = \rho \} \), we have

\[
\log^+ \| f(re^{i\theta}) \| \leq \frac{R^n + \rho}{R^n - \rho} m(R^n, f) + \sum_{j=1}^{n} \log \left| \frac{R^n 2 - b_j z}{R^n (z - b_j)} \right|
\]

(3.15)

Now we are in the position to establish the following Chuang’s inequality.

**Theorem 3.5.** Let \( f : C \to E \) be a nonzero \( E \)-valued meromorphic mapping and \( f(0) \neq \infty \). Then for \( \tau > 1 \) and \( 0 < r < R \), one has

\[
T(r, f) < C_\tau T(\tau r, f') + \log^+ \tau r + 4 + \log^+ \| f(0) \|,
\]

(3.16)

where \( C_\tau \) is a positive constant.

**Proof.** Take a \( \sigma \) such that \( \sigma^3 = \tau \) and denote \( r_1 = \sigma r, r_2 = \sigma r_1, r_3 = \sigma r_2 \). Applying Lemma 3.3 to \( f'(z) \), we can find a real number \( \theta_0 \) such that \( 0 \leq t \leq r_1 \), and we have

\[
\log^+ \| f'(re^{i\theta}) \| \leq \frac{r_2 + r_1}{r_2 - r_1} m(r_2, f') + n(r_2, f') \log 4 + N(r_2, f').
\]

(3.17)

In view of Lemma 3.4, for a fixed \( \rho \in [r, r_1] \) we have

\[
\log^+ \| f'(z) \| \leq \frac{r_2 + r_1}{r_2 - r_1} m(r_2, f') + n(r_2, f') \log \frac{8er_2}{r_1 - r},
\]

(3.18)

on \( \{ z : |z| = \rho \} \).

From the origin along the segment \( \arg z = \theta_0 \) to \( \rho e^{i\theta_0} \), and along \( \{ z : |z| = \rho \} \) turn a rotation to \( \rho e^{i\theta_0} \). We denote this curve by \( L \), and its length is \( (2\pi + 1)\rho \).

We notice that \( \varphi(z) = \| f(z) \| \) is continuous on \( L \). As in [5], \( E_n \) is an \( n \)-dimensional projective space of \( E \) with a basis \( \{ e_i \}_{i=1}^n \). The projection operator \( P_n : E \to E_n \) is a realization of \( E_n \) associated to the basis and \( P_n(f(z)) = (f_1(z), f_2(z), \ldots, f_n(z)) \). We have \( P_n(f'(z)) = (P_n(f(z)))' = \sum_{i=1}^n f_i'(z)e_i \) and \( P_n(f(z)) = P_n(f(0)) + \sum_{i=1}^n (j_i f_i(\zeta)d\zeta)e_i \). Therefore, since \( E_n \) is
finite dimensional, there exists $K > 0$ (appearing in the inequality $\| \cdot \|_1 \leq K \| \cdot \|_2$, where $\| \cdot \|_1$ and $\| \cdot \|_2$ are any two norms on $E_n$) such that

$$
\| P_n(f(z)) \| \leq \| P_n(f(0)) \| + \left\| \frac{1}{K} \left( \sum_{i=1}^{n} \left( \int_{0}^{z} f_i'(\xi) d\xi \right) e_i \right) \right\| \leq \| P_n(f(0)) \| + \frac{1}{K} \left( \sum_{i=1}^{n} \| f_i'(z) \|_{\infty}^{2} \right)^{1/2} \leq \| P_n(f(0)) \| + \frac{1}{K} M_n(2\pi + 1)\rho,
$$

where $M_n = \max_{z \in L} \| P_n(f'(z)) \|$. Thus, we have

$$
\| P_n(f(z)) \| \leq \| f(0) \| + M_n(2\pi + 1)\rho + O(1), \quad |z| = \rho.
$$

In virtue of [6–8], every meromorphic mapping $f(z)$ with values in a Banach space $E$ with a Schauder basis and the projections $P_n(f)$ are convergent in its natural topology; that is, they converge uniformly to $f$ in any compact subset $W$ of $\mathbb{C} \setminus P_f$ ($P_f$ being the set of poles of the $f$ in $\mathbb{C}$). Thus for $n$ large enough, we have

$$
\| P_n(f(z)) \| = \| f(z) \| + O(1), \quad \text{for any } z \in W \subseteq \mathbb{C} \setminus P_f.
$$

A similar argument to $f'(z)$ implies that for $n$ large enough

$$
\| P_n(f'(z)) \| = \| f'(z) \| + O(1), \quad M_n \leq M + O(1) \quad \text{for any } z \in W \subseteq \mathbb{C} \setminus P_f,
$$

where $M = \max_{z \in L} \| f'(z) \|$. Combining (3.20), (3.21), and (3.22) and the fact that the compact set $\{ z : |z| = \rho \} \subseteq L \subseteq \mathbb{C} \setminus P_f$, we get

$$
\| f(z) \| \leq \| f(0) \| + M(2\pi + 1)\rho + O(1).
$$

Then

$$
\log^+ \| f(z) \| \leq \log^+ \| f(0) \| + \log^+ M + \log^+ \rho + \log 8\pi + O(1).
$$
In virtue of (3.13) and (3.17), we have

$$\log^+ M \leq \frac{r_2 + r_1}{r_2 - r_1} m(r_2, f') + n(r_2, f') \log \frac{8r_2}{r_1 - r} + N(r_2, f')$$

$$\leq \left\{ \log \frac{8r_2/(r_1 - r)}{\log (r_3/r_2)} + \frac{r_2 + r_1}{r_2 - r_1} \right\} T(r_3, f') = C_3^r T(r_3, f').$$

Therefore,

$$m(\rho, f) < C_3^r T(r_3, f') + \log^+ (\tau r) + 4 + \log^+ \|f(0)\|.$$  \hspace{1cm} (3.26)

Thus we have

$$T(r, f) \leq T(\rho, f) < (C_3^r + 1)T(r_3, f') + \log^+ (\tau r) + 4 + \log^+ \|f(0)\|$$

$$= C_3 T(\tau r, f') + \log^+ (\tau r) + 4 + \log^+ \|f(0)\|.$$ \hspace{1cm} (3.27)

The following result says that we can also control the $T(r, f')$ by $T(r, f)$.

**Theorem 3.6.** Let $f(z)(z \in \mathbb{C})$ be a nonconstant $E$-valued meromorphic mapping. Then one has

$$T(r, f') \leq 2T(r, f) + O(\log r + \log^+ T(r, f)).$$  \hspace{1cm} (3.28)

**Proof.** One has

$$T(r, f') = m(r, f') + N(r, f')$$

$$\leq m(r, f) + N(r, f') + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi$$

$$= m(r, f) + N(r, f) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi$$

$$\leq 2T(r, f) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi$$

$$= 2T(r, f) + O(\log r + \log^+ T(r, f)).$$ \hspace{1cm} (3.29)

From Theorems 3.5 and 3.6, we have the following.

**Corollary 3.7.** For a nonconstant $E$-valued meromorphic mapping $f(z)(z \in \mathbb{C})$, one has $\lambda(f) = \lambda(f'), \mu(f) = \mu(f')$. 
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