Research Article

On Regularized Quasi-Semigroups and Evolution Equations

M. Janfada

Department of Mathematics, Sabzevar Tarbiat Moallem University, P.O. Box 397, Sabzevar, Iran

Correspondence should be addressed to M. Janfada, mjanfada@gmail.com

Received 26 November 2009; Accepted 16 April 2010

Academic Editor: Wolfgang Ruess

Copyright © 2010 M. Janfada. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce the notion of regularized quasi-semigroup of bounded linear operators on Banach spaces and its infinitesimal generator, as a generalization of regularized semigroups of operators. After some examples of such quasi-semigroups, the properties of this family of operators will be studied. Also some applications of regularized quasi-semigroups in the abstract evolution equations will be considered. Next some elementary perturbation results on regularized quasi-semigroups will be discussed.

1. Introduction and Preliminaries

The theory of quasi-semigroups of bounded linear operators, as a generalization of strongly continuous semigroups of operators, was introduced in 1991 [1], in a preprint of Barcenas and Leiva. This notion, its elementary properties, exponentially stability, and some of its applications in abstract evolution equations are studied in [2–5]. The dual quasi-semigroups and the controllability of evolution equations are also discussed in [6].

Given a Banach space $X$, we denote by $B(X)$ the space of all bounded linear operators on $X$. A biparametric commutative family $\{R(s, t)\}_{s,t \geq 0} \subseteq B(X)$ is called a quasi-semigroup of operators if for every $s, t, r \geq 0$ and $x \in X$, it satisfies

1. $R(t, 0) = I$, the identity operator on $X$,
2. $R(r, s + t) = R(r + t, s)R(r, t)$,
3. $\lim_{(s, t) \to (s_0, t_0)} \|R(s, t)x - R(s_0, t_0)x\| = 0, x \in X$,
4. $\|R(s, t)\| \leq M(s + t)$, for some continuous increasing mapping $M : [0, \infty) \to [0, \infty)$.

Also regularized semigroups and their connection with abstract Cauchy problems are introduced in [7] and have been studied in [8–12] and many other papers.
We mention that if $C \in B(X)$ is an injective operator, then a one-parameter family $\{T(t)\}_{t \geq 0} \subseteq B(X)$ is called a $C$-semigroup if for any $s, t \geq 0$ it satisfies $T(s + t) = T(s) T(t)$ and $T(0) = C$.

In this paper we are going to introduce regularized quasi-semigroups of operators. In Section 2, some useful examples are discussed and elementary properties of regularized quasi-semigroups are studied. In Section 3 regularized quasi-semigroups are applied to find solutions of the abstract evolution equations. Also perturbations of the generator of regularized quasi-semigroups are also considered in this section. Our results are mainly based on the work of Barcenas and Leiva [1].

### 2. Regularized Quasi-Semigroups

Suppose $X$ is a Banach space and $\{K(s, t)\}_{s, t \geq 0}$ is a two-parameter family of operators in $B(X)$. This family is called commutative if for any $r, s, t, u \geq 0$, 

$$K(r, t)K(s, u) = K(s, u)K(r, t).$$  

**Definition 2.1.** Suppose $C$ is an injective bounded linear operator on Banach space $X$. A commutative two-parameter family $\{K(s, t)\}_{s,t \geq 0}$ in $B(X)$ is called a regularized quasi-semigroups (or $C$-quasi-semigroups) if 

1. $K(t, 0) = C$, for any $t \geq 0$; 
2. $CK(r, t + s) = K(r + t, s)K(r, t)$, $r, t, s \geq 0$; 
3. $\{K(s, t)\}_{s,t \geq 0}$ is strongly continuous, that is, 

$$\lim_{(s, t) \to (s_0, t_0)} \|K(s, t)x - K(s_0, t_0)x\| = 0, \quad x \in X;$$  

4. there exists a continuous and increasing function $M : [0, \infty) \to [0, \infty)$, such that for any $s, t > 0$, $\|K(s, t)\| \leq M(s + t)$.

For a $C$-quasi-semigroups $\{K(s, t)\}_{s,t \geq 0}$ on Banach space $X$, let $D$ be the set of all $x \in X$ for which the following limits exist in the range of $C$:

$$\lim_{t \to 0^+} \frac{K(s, t)x - Cx}{t} = \lim_{t \to 0^+} \frac{K(s - t, t)x - Cx}{t}, \quad s > 0$$

$$\lim_{t \to 0^+} \frac{K(0, t)x - Cx}{t}.$$

Now for $x \in D$ and $s \geq 0$, define 

$$A(s)x = C^{-1} \lim_{t \to 0^+} \frac{K(s, t)x - Cx}{t}.$$  

$\{A(s)\}_{s \geq 0}$ is called the infinitesimal generator of the regularized quasi-semigroup $\{K(s, t)\}_{s,t \geq 0}$. Somewhere we briefly apply generator instead of infinitesimal generator.
Abstract and Applied Analysis

Here are some useful examples of regularized quasi-semigroups.

**Example 2.2.** Let \( \{T_t\}_{t \geq 0} \) be an exponentially bounded strongly continuous C-semigroup on Banach space \( X \), with the generator \( A \). Then

\[
K(s, t) := T_t, \quad s, t \geq 0,
\]

defines a C-quasi-semigroup with the generator \( A(s) = A, s \geq 0 \), and so \( D = D(A) \).

**Example 2.3.** Let \( X = \text{BUC}(\mathbb{R}) \), the space of all bounded uniformly continuous functions on \( \mathbb{R} \) with the supremum-norm. Define \( C, K(s, t) \in B(X) \), by

\[
Cf(x) = e^{-x^2}f(x), \quad K(s, t)f(x) = e^{-x^2}f(t^2 + 2st + x), \quad s, t \geq 0.
\]

One can see that \( \{K(s, t)\}_{s, t \geq 0} \) is a regularized C-quasi-semigroup of operators on \( X \), with the infinitesimal generator \( A(s) = 2sf \) on \( D \), where \( D = \{f \in X : f \in X\} \).

**Example 2.4.** Let \( \{T_t\}_{t \geq 0} \) be a strongly continuous semigroup of operators on Banach space \( X \), with the generator \( A \). If \( C \in B(X) \) is injective and commutes with \( T_t, t \geq 0 \), then

\[
K(s, t) := Ce^{T_0 - T_s}, \quad s, t \geq 0,
\]

is a C-quasi-semigroup with the generator \( A(s) = AT_s \). Thus \( D = D(A) \). In fact, for \( x \in D \), boundedness of \( C \) implies that

\[
CA(s)x = \lim_{t \to 0^+} \frac{Ce^{T_0 - T_s}x - Cx}{t} = C \lim_{t \to 0^+} \frac{e^{T_0 - T_s}x - x}{t} = C \frac{d}{ds} \big|_{s=0} (T_{s+t} - T_s)x = CAT_sx.
\]

Now injectivity of \( C \) implies that \( A(s)x = AT_sx \), and so \( D = D(A) \).

**Example 2.5.** Let \( \{T_t\}_{t \geq 0} \) be a strongly continuous exponentially bounded C-semigroup of operators on Banach space \( X \), with the generator \( A \). For \( s, t \geq 0 \), define

\[
K(s, t) = T(g(s + t) - g(s)), \quad s, t \geq 0,
\]

where \( g(t) = \int_0^t a(s)ds, \) and \( a \in C[0, \infty) \), with \( a(t) > 0 \). We have \( K(s, 0) = T(0) = C \) and the C-semigroup properties of \( \{T(t)\}_{t \geq 0} \) imply that

\[
CK(r, s + t) = CT(g(r + t + s) - g(r))
\]

\[= CT(g(r + t + s) - g(t + r) + g(t + r) - g(r))
\]

\[= T(g(r + t + s) - g(t + r))T(g(t + r) - g(r))
\]

\[= K(r + t, s)K(r, t).
\]
So \( \{K(s, t)\}_{s,t \geq 0} \) is a C-quasi-semigroup (the other properties can be also verified easily). Also \( D = D(A) \) and for \( x \in D \), \( A(s)x = a(s)Ax \).

Some elementary properties of regularized quasi-semigroups can be seen in the following theorem.

**Theorem 2.6.** Suppose \( \{K(s, t)\}_{s,t \geq 0} \) is a C-quasi-semigroup with the generator \( \{A(s)\}_{s \geq 0} \) on Banach space \( X \). Then

(i) for any \( x \in D \) and \( s_0, t_0 \geq 0 \), \( K(s_0, t_0)x \in D \) and

\[
K(s_0, t_0)A(s)x = A(s)K(s_0, t_0)x; \tag{2.11}
\]

(ii) for each \( x_0 \in D \),

\[
\frac{\partial}{\partial t}K(r, t)Cx_0 = A(r + t)K(r, t)Cx_0 = K(r, t)A(r + t)Cx_0; \tag{2.12}
\]

(iii) if \( A(s) \) is locally integrable, then for each \( x_0 \in D \) and \( r \geq 0 \),

\[
K(r, t)x_0 = Cx_0 + \int_0^t A(r + s)K(r, s)x_0 ds, \quad t \geq 0; \tag{2.13}
\]

(iv) let \( f : [0, \infty) \rightarrow X \) be a continuous function; then for every \( t \in [0, \infty) \),

\[
\lim_{h \to 0} \frac{1}{t} \int_t^{t+h} K(s, u)f(u)du = K(t)f(t); \tag{2.14}
\]

(v) Let \( C' \) \( \in B(X) \) be injective and for any \( s, t \geq 0 \), \( C'K(s, t) = K(s, t)C' \). Then \( R(s, t) := C'K(s, t) \) is a \( CC' \)-quasi-semigroup with the generator \( \{A(s)\}_{s \geq 0} \). Then \( R(s, t) := C'K(s, t) \) is a \( C'K \)-quasi-semigroup with the generator \( \{A(s)\}_{s \geq 0} \).

(vi) Suppose \( \{R(s, t)\}_{s,t \geq 0} \) is a quasi-semigroup of operators on Banach space \( X \) with the generator \( \{A(s)\}_{s \geq 0} \), and \( C \in B(X) \) commutes with every \( R(s, t), s, t \geq 0 \). Then \( K(s, t) := CR(s, t) \) is a \( C \)-quasi-semigroup of operators on \( X \) with the generator \( \{A(s)\}_{s \geq 0} \).

**Proof.** First we note that from the commutativity of \( \{K(s, t)\}_{s,t \geq 0} \):

\[
CK(s, t) = K(s, t)C \quad s, t \geq 0. \tag{2.15}
\]

Also \( x \in D \) implies that

\[
\lim_{t \to 0} \frac{K(s, t)x - Cx}{t} = CA(s)x \quad s \geq 0. \tag{2.16}
\]
Abstract and Applied Analysis

Thus from continuity of \( K(s_0, t_0) \), we have

\[
\lim_{t \to 0^+} \frac{K(s, t)K(s_0, t_0)x - CK(s_0, t_0)x}{t} = K(s_0, t_0) \lim_{t \to 0^+} \frac{K(s, t)x - Cx}{t} = K(s_0, t_0)CA(s)x
\]

(2.17)

Thus \( K(s_0, t_0)x \in D \) and \( A(s)K(s_0, t_0) = K(s_0, t_0)A(s)x \).

To prove (ii), consider the quotient

\[
\frac{K(r, t + s)Cx_0 - K(r, t)Cx_0}{s} = \frac{K(r + t, s)K(r, t)x_0 - K(r, t)Cx_0}{s} = K(r, t)\frac{K(r + t, s)x_0 - Cx_0}{s}
\]

(2.18)

which tends to \( K(r, t)CA(r + t)x_0 \) as \( s \to 0^+ \).

Also for \( s < 0 \),

\[
\frac{K(r, t + s)Cx_0 - K(r, t)Cx_0}{s} = \frac{K(r, t)Cx_0 - K(r, t + s)Cx_0}{-s} = \frac{K(r + t + s, -s)K(r, t + s)x_0 - K(r, t + s)Cx_0}{-s} = K(r, t + s)\frac{1}{-s}(K(r + t + s, -s)x_0 - K(r + t, -s)x_0 + K(r + t, -s)x_0 - Cx_0).
\]

(2.19)

Now the strongly continuity of \( \{K(s, t)\}_{s,t \geq 0} \) implies that

\[
\lim_{s \to 0^+} K(r + t + s, -s)x_0 - K(r + t, -s)x_0 = 0.
\]

(2.20)

Thus

\[
\lim_{s \to 0^+} \frac{K(r + t + s, -s)x_0 - Cx_0}{-s} = CA(r + t)x_0.
\]

(2.21)

Hence by the strongly continuity of \( K(s, t) \),

\[
\lim_{s \to 0^+} \frac{K(r, t + s)Cx_0 - K(r, t)Cx_0}{s} = K(r, t)CA(r + t)x_0.
\]

(2.22)

Thus \((\partial/\partial t)K(r, t)Cx_0 = K(r, t)CA(r + t)x_0 \). The second equality holds by (i).
Now integrating of this equation, we have

\[ K(r,t)Cx_0 - Cx_0 = C \int_0^t K(r,s)A(r+s)x_0 ds. \]  

(2.23)

Hence injectivity of \( C \) implies (iii).

(iv) is trivial from continuity of \( f \) and strongly continuity of \( \{K(s,t)\}_{s,t \geq 0} \). In (v), obviously \( \{R(s,t)\}_{s,t \geq 0} \) is a \( C' \)-quasi-semigroup. For \( x \in D \), we have

\[ \frac{R(s,t)x - CC'x}{t} = C, \frac{K(s,t)x - Cx}{t}, \]  

(2.24)

which tends to \( C'CA(s) \), as \( t \to 0^+ \). This proves (v).

(vi) can be seen easily. \( \square \)

3. Evolution Equations and Regularized Quasi-Semigroups

Suppose \( C \) is an injective bounded linear operator on Banach space \( X \) and \( r > 0 \). In this section, we study the solutions of the following abstract evolution equation using the regularized quasi-semigroups:

\[ \dot{x}(t) = A(t + r)x(t), \quad t > 0, \]

\[ x(0) = C^2x_0, \quad x_0 \in X. \]  

(3.1)

One can see [13, 14] for a comprehensive studying of abstract evolution equations.

**Theorem 3.1.** Let \( \{A(s)\}_{s \geq 0} \) be the infinitesimal generator of a \( C \)-quasi-semigroups \( \{K(s,t)\}_{s,t \geq 0} \) on Banach space \( X \), with domain \( D \). Then for each \( x_0 \in D \) and \( r \geq 0 \), the initial value problem (3.1) admits a unique solution.

**Proof.** Let \( x(t) = K(r,t)Cx_0 \). By Theorem 2.6(ii), \( x(t) \) is a solution of (3.1).

Now we show that this solution is unique. Suppose \( y(s) \) is another solution of (3.1). Trivially \( y(s) \in D \). Let \( t > 0 \). For \( s \in [0, t] \) and \( x \in X \), define

\[ F(s)x = K(r+s,t-s)Cx, \quad G(s) = F(s)Cy(s). \]  

(3.2)

From \( C \)-quasi-semigroup properties, for small enough \( h > 0 \), we have

\[ K(r+s,t-s)C = K(r+s+t-s-(t-s-h),t-s-h)K(r+s,t-s-(t-s-h)) \]

\[ = K(r+s+h,t-s-h)K(r+s,h). \]  

(3.3)
Therefore, from this, the fact that
\[
\frac{F(s + h)x - F(s)x}{h} = \frac{K(r + s + h, t - s - h)Cx - K(r + s + h, t - s - h)K(r + s, h)x}{h}
\]
\[
= -K(r + s + h, t - s - h) \left[ \frac{K(r + s, h)x - Cx}{h} \right]
\]
\[
\rightarrow -K(r + s, t - s)CA(r + s)x, \quad \text{as } h \rightarrow 0.
\] (3.4)

This means that
\[
\dot{F}(s)x = -K(r + s, t - s)CA(r + s)x.
\] (3.5)

Therefore, from this, the fact that \(y(s)\) satisfies (3.1), and \(CF(s) = F(s)C\), we obtain that
\[
\dot{G}(s) = \dot{F}(s)Cy(s) + F(s)C\dot{y}(s) = -K(r + s, t - s)CA(r + s)Cy(s) + K(r + s, t - s)C^2\dot{y}(s)
\]
\[
= -K(r + s, t - s)CA(r + s)Cy(s) + K(r + s, t - s)C^2A(r + s)y(s) = 0.
\] (3.6)

Hence for every \(s \in (0, t)\), \(\dot{G}(s) = 0\). Consequently, \(G(s)\) is a constant function on \([0, t]\). In particular, \(G(0) = G(t)\). So from \(y(0) = Cx_0\), we have
\[
G(0) = F(0)Cy(0) = K(r, t)C^2x_0 = G(t) = F(t)Cy(t) = K(r + t, 0)C^2y(t) = C^3y(t).
\] (3.7)

Hence \(C^2K(r, t)x_0 = C^3y(t)\). Now injectivity of \(C\) implies that \(y(t) = K(r, t)Cx_0\), which proves the uniqueness of the solution.

Now with the above notation, we consider the inhomogeneous evolution equation
\[
\dot{x}(t) = A(r + t)x(t) + C^2f(t), \quad 0 < t \leq T,
\]
\[
x(0) = C^2x_0, \quad x_0 \in D.
\] (3.8)

The following theorem guarantees the existence and uniqueness of solutions of (3.8) with some sufficient conditions on \(f\).

**Theorem 3.2.** Let \(K(s, t)\) be a \(C\)-quasi-semigroup on Banach space \(X\), with the generator \(\{ A(s) \}_{s \geq 0}\) whose domain is \(D\). If \(f : [0, T] \rightarrow D\) is a continuous function, each operator \(A(s)\) is closed, and
\[
C \int_0^t K(r + s, t - s) f(s)ds \in D, \quad 0 < t \leq T,
\] (3.9)

then the initial value equation (3.8) admits a unique solution
\[
x(t) = K(r, t)Cx_0 + \int_0^t K(r + s, t - s)Cf(s)ds.
\] (3.10)
Proof. For the existence of the solution, it is enough to show that \( x(t) \) in (3.10) is continuously differentiable and satisfies (3.8).

Trivially \( x(0) = Cx_0 \). We know that \( y(t) = K(r,t)Cx_0 \) is a solution of (3.1) by Theorem 3.1. Define

\[
g(t) = \int_0^t K(r + s, t - s) Cf(s) ds, \tag{3.11}
\]

which is in \( D \) by our hypothesis. We have

\[
\frac{g(t + h) - g(t)}{h} = \frac{1}{h} \left[ \int_0^t K(r + s, t + h - s) Cf(s) ds - \int_0^t K(r + s, t - s) Cf(s) ds \right]
\]

\[
= \frac{1}{h} \left[ \int_0^t K(r + s, t + h - s) Cf(s) ds - \int_0^t K(r + s, t - s) Cf(s) ds \right] + \int_t^{t+h} K(r + s, t + h - s) Cf(s) ds \tag{3.12}
\]

On the other hand, the \( C \)-quasi-semigroup properties imply that

\[
K(r + s, t + h - s) Cf(s) = K(r + s + t + h - s - h, h) K(r + s, t + h - s - h) f(s)
\]

\[
= K(r + t, h) K(r + s, t - s) f(s). \tag{3.13}
\]

So

\[
\frac{g(t + h) - g(t)}{h} = \frac{1}{h} \left[ \int_0^t K(r + t, h) K(r + t, t - s) f(s) ds \right.
\]

\[
- \int_0^t K(r + s, t - s) Cf(s) ds + \int_t^{t+h} K(r + s, t + h - s) Cf(s) ds \right] + \int_t^{t+h} K(r + s, t + h - s) Cf(s) ds. \tag{3.14}
\]

Since the range of \( f \) is in \( D \), passing to the limit when \( h \to 0 \), and using Theorem 2.6(v), we have

\[
g(t) = \int_0^t K(r + s, t - s) CA(r + t) f(s) ds + K(r + t, t - t) Cf(t)
\]

\[
= \int_0^t K(r + s, t - s) CA(r + t) f(s) ds + C^2 f(t). \tag{3.15}
\]
Therefore, \( \dot{g}(t) \) exists. Also by our hypothesis \( A(r + t) \) is closed, and \[ \int_0^t K(r + s, t - s)Cf(s)ds \in D, \] thus

\[
\int_0^t K(r + s, t - s)CA(r + t)f(s)ds = A(r + t) \int_0^t K(r + s, t - s)Cf(s)ds. \tag{3.16}
\]

Consequently,

\[
\dot{g}(t) = A(r + t)g(t) + C^2f(t), \quad t \geq 0. \tag{3.17}
\]

Hence

\[
x(t) = \frac{\partial}{\partial t}K(r, t)Cx_0 + A(r + t) \int_0^t K(r + s, t - s)Cf(s)ds + C^2f(t)
\]

\[
= A(r + t) \left( K(r, t)Cx_0 + \int_0^t K(r + s, t - s)Cf(s)ds \right) + C^2f(t) \tag{3.18}
\]

\[
= A(r + t)x(t) + C^2f(t).
\]

This completes the proof. \( \square \)

We conclude this section with two simple perturbation theorems and some examples, as applications of our discussion.

**Theorem 3.3.** (a) Suppose \( B \) is the infinitesimal generator of a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) and \( \{A(s)\}_{s \geq 0} \) with domain \( D \) is the generator of a regularized \( C \)-quasi-semigroup \( \{K(s, t)\}_{s, t \geq 0} \), which commutes with \( \{T(t)\}_{t \geq 0} \). Then \( \{A(s) + B\}_{s \geq 0} \) with domain \( D \cap D(B) \) is the infinitesimal generator of a regularized \( C \)-quasi-semigroup.

(b) Suppose \( B \) is the infinitesimal generator of an exponentially bounded \( C \)-semigroup \( \{T(t)\}_{t \geq 0} \) and \( \{A(s)\}_{s \geq 0} \) with domain \( D \) is the generator of a quasi-semigroup (resp., regularized \( C \)-quasi-semigroup) \( \{K(s, t)\}_{s, t \geq 0} \), which commutes with \( \{T(t)\}_{t \geq 0} \). Then \( \{A(s) + B\}_{s \geq 0} \) with domain \( D \cap D(B) \) is the infinitesimal generator of a \( C \)-regularized quasi-semigroup (resp., regularized \( C'^{C} \)-quasi-semigroup).

**Proof.** In (a) and (b), define

\[
R(s, t) = T(t)K(s, t). \tag{3.19}
\]

One can see that \( \{R(s, t)\}_{s \geq 0} \) is a \( C \)-regularized quasi-semigroup (in (b), resp., regularized \( C'^{C} \)-quasi-semigroup). We just prove that \( \{A(s) + B\}_{s \geq 0} \) is its generator. In (a), let \( \{B(s)\}_{s \geq 0} \) be the infinitesimal generator of \( \{R(s, t)\}_{s \geq 0} \) and \( x \in D \cap D(B) \). Hence

\[
\lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad \lim_{t \to 0^+} \frac{K(s, t)x - Cx}{t} \tag{3.20}
\]
exist in \( X \) and the range of \( C \), respectively. Now the fact that \( C \) commutes with \( T(t) \) and strongly continuity of \( T(t) \) implies that

\[
\lim_{t \to 0^+} T(t) \frac{K(s,t)x - Cx}{t} = \lim_{t \to 0^+} T(t) \frac{K(s,t)x - Cx}{t} = \lim_{t \to 0^+} T(t) \frac{K(s,t)x - Cx}{t} + C \lim_{t \to 0^+} T(t)x - x
gives
\[
(3.21)
\]
exists in the range of \( C \). So

\[
\lim_{t \to 0^+} R(s,t)x - Cx = \lim_{t \to 0^+} T(t)K(s,t)x - Cx = \lim_{t \to 0^+} T(t) \frac{K(s,t)x - Cx}{t} + C \lim_{t \to 0^+} T(t)x - x
\]
exists in the range of \( C \) and

\[
CB(s)x = \lim_{t \to 0^+} R(s,t)x - Cx = CA(s)x + CBx.
\]

By injectivity of \( C \), \( B(s)x = A(s)x + Bx \).

The proof of the other parts is similar. \( \square \)

**Theorem 3.4.** Let \( K(s,t) \) be a \( C \)-quasi-semigroup of operator on Banach space \( X \) with the generator \( \{A(s)\} \) on domain \( D \). If \( B \in B(X) \) commutes with \( K(s,t) \), \( s,t \geq 0 \) and \( B^2 = B \), then \( \{BA(s)\}_{s \geq 0} \) is the infinitesimal generator of \( C \)-regularized quasi-semigroup

\[
R(s,t) = B(K(s,t) - C) + C.
\]

**Proof.** The \( C \)-quasi-semigroup properties of \( \{R(s,t)\}_{s,t \geq 0} \) can be easily verified. We just prove that its generator is \( \{BA(s)\}_{s \geq 0} \). Let \( x \in D \); we have

\[
\frac{R(s,t)x - Cx}{t} = \frac{B(K(s,t) - C)x + Cx - Cx}{t} = B \frac{K(s,t)x - Cx}{t}
\]
which tends to \( BA(s)x \), as \( t \to 0 \). This completes the proof. \( \square \)

**Example 3.5.** Let \( r > 0 \). Consider the following initial value problem:

\[
\frac{\partial}{\partial t} x(t,\varepsilon) = 2(r + t) \frac{\partial}{\partial \varepsilon} x(t,\varepsilon) + \varepsilon x(t,\varepsilon), \quad \varepsilon, t \geq 0.
\]

\[
x(0,\varepsilon) = e^{-\varepsilon t} x_0(\varepsilon),
\]

Let \( X = \text{BUC}(\mathbb{R}) \), with the supremum-norm. Define \( C \in B(X) \) by \( Cx(\varepsilon) = e^{-\varepsilon t} x(\varepsilon) \), \( x(\cdot) \in X \). Also define \( B : D(B) \to X \) by \( Bx(\varepsilon) = \varepsilon x(\varepsilon) \), where \( D(B) = \{ x \in X : Bx \in X \} \). It is well known that \( B \) is the infinitesimal generator of \( C \)-regularized semigroup \( T(t) \), defined by \( T(t)x(\varepsilon) = e^{-\varepsilon t} x(\varepsilon) \). Now with \( D = \{ x \in X : x \in X \} \), if \( A(s) : D \to X \) is defined by \( A(s)x = 2s\hat{x} \), then by Example 2.3, \( \{A(s)\}_{s \geq 0} \) is the infinitesimal generator of the regularized
Abstract and Applied Analysis

C²-quasi-semigroup \( K(s,t)x(\varepsilon) = e^{-\varepsilon^2 x(t^2 + 2st + \varepsilon)}. \) Using Theorem 3.3 and the fact that \( T(t)K(s,r) = K(s,r)T(t), \) \( s, t, t \geq 0, \) we obtain that \( \{ A(s) + B \} \) is the infinitesimal generator of regularized C²-quasi-semigroup \( R(s,t) = T(t)K(s,t). \) Also using these operators, (3.26) can be written as

\[
\dot{x}(t) = (A(r + t) + B)x(t),
\]

\[
x(0) = C^4 x_0. \tag{3.27}
\]

Thus by Theorem 3.1 for any \( x_0 \in D \cap D(B), \) (3.26) has the unique solution

\[
x(t, \varepsilon) = R(r, t)C^2 x_0(\varepsilon) = e^{-4\varepsilon^2 + rt} x_0 \left( t^2 + 2rt + \varepsilon \right). \tag{3.28}
\]

**Example 3.6.** For a given sequence \( (p_n)_{n \in \mathbb{N}} \) of complex numbers with nonzero elements and \( (y_n)_{n \in \mathbb{N}}, \) consider the following equation:

\[
\frac{d}{dt} x_n(t) = e^{int} x_n(t) + p_n x_n(t), \tag{3.29}
\]

\[
x_n(0) = p_n^2 y_n, \quad n \in \mathbb{N}.
\]

Let \( X \) be the space \( c_0, \) the set of all complex sequence with zero limit at infinity. For a bounded sequence \( p = (p_n)_{n \in \mathbb{N}} \) define \( A : D(A) : \rightarrow X \) and \( M_p \) on \( X \) by

\[
A(x_n)_{n \in \mathbb{N}} = \left( e^{int} x_n \right)_{n \in \mathbb{N}}, \quad M_p(x_n)_{n \in \mathbb{N}} = (p_n x_n). \tag{3.30}
\]

One can easily see that \( D(A) = \{ (x_n)_{n \in \mathbb{N}} \in c_0 : (e^{int} x_n)_{n \in \mathbb{N}} \in c_0 \} \) and \( M_p \) is a bounded linear operator which is injective. It is well known that \( A \) is the infinitesimal generator of strongly continuous semigroup

\[
T(t)(x_n)_{n \in \mathbb{N}} = \left( e^{int} x_n \right)_{n \in \mathbb{N}}. \tag{3.31}
\]

Thus by Example 2.4, \( \{ A(t) \}_{t \geq 0}, \) defined by

\[
A(t)(x_n)_{n \in \mathbb{N}} := AT(t)(x_n)_{n \in \mathbb{N}} = \left( e^{int(1+t)} x_n \right)_{n \in \mathbb{N}}, \tag{3.32}
\]

is the infinitesimal generator of the \( M_p \)-quasi-semigroup

\[
K(s,t) = M_p \left( e^{T(s+t) - T(s)} \right). \tag{3.33}
\]
Using these operators, one can rewrite (3.29) as

\[\dot{x}(t) = (A(t) + M_{\rho})x(t),\]

\[x(0) = M_{\rho}^2 y_0,\]  \hspace{1cm} (3.34)

where \(x_0 = (y_n)_{n \in \mathbb{N}}\). Trivially \(T(t)\) commutes with \(K(r,s)\), for any \(r,s,t \geq 0\). Now using Theorem 3.3 we obtain that \(\{A(t) + M_{\rho}\}_{t \geq 0}\) is the infinitesimal generator of of \(M_{\rho}\)-quasi-semigroup

\[R(s,t) = T(t)K(s,t).\]  \hspace{1cm} (3.35)

Also from Theorem 3.1, with \(r = 0\), for any \(y \in D(A)\), (3.34) has a unique solution

\[x(t) = R(0,t)M_{\rho}y = T(t)K(0,t)M_{\rho}^2 x_0.\]  \hspace{1cm} (3.36)

But from definition of \(K(s,t)\), for a given \((x_n)_{n \in \mathbb{N}} \in c_0,\)

\[K(0,t)(x_n)_{n \in \mathbb{N}} = e^{T(t)-1} = e^{-1} \sum_{k=0}^{\infty} \frac{T^k(t)}{k!} (x_n)_{n \in \mathbb{N}} = e^{-1} \sum_{k=0}^{\infty} \left( \frac{e^{iknt}x_n}{k!} \right)_{n \in \mathbb{N}} = e^{-1} \left( \sum_{k=0}^{\infty} \frac{e^{iknt}x_n}{k!} \right)_{n \in \mathbb{N}}.\]  \hspace{1cm} (3.37)

So the solution of (3.34) is

\[x(t) = R(0,t)M_{\rho}y = \left( \sum_{k=0}^{\infty} \frac{e^{ik(t+1)}-1}{k!} P_n^k y_n \right)_{n \in \mathbb{N}},\]  \hspace{1cm} (3.38)

or equivalently the solution of (3.29) is

\[x_n(t) = \sum_{k=0}^{\infty} \frac{e^{ik(t+1)}-1}{k!} P_n^k y_n, \quad n \in \mathbb{N}.\]  \hspace{1cm} (3.39)

**References**


