A New Approximation Method for Common Fixed Points of a Finite Family of Generalized Asymptotically Quasinonexpansive Mappings in Banach Spaces

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We introduce a new iterative scheme to approximate a common fixed point for a finite family of generalized asymptotically quasinonexpansive mappings. Several strong and weak convergence theorems of the proposed iteration in Banach spaces are established. The main results obtained in this paper generalize and refine many known results in the current literature.

1. Introduction

Let $C$ be a convex subset of a Banach space $X$, and let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of self-mappings of $C$. Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$
\begin{align*}
  x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\
  y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\
  y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\
  &\vdots \\
  y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\
  y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n},
\end{align*}
$$

(1.1)
where $y_{0n} = x_n$ for all $n$. The iterative process (1.1) for a finite family of mappings introduced by Khan et al. [1], and the iterative process is the generalized form of the modified Mann (one-step) iterative process by Schu [2], the modified Ishikawa (two-step) iterative process by Tan and Xu [3], and the three-step iterative process by Xu and Noor [4].

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [1, 5–19] are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [12–16, 20, 21].


Motivated by Khan et al. [1], we introduce a new iterative scheme for finding a common fixed point of a finite family of generalized asymptotically quasinonexpansive mappings as follows:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$
\begin{align*}
x_{n+1} &= (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_n^{k}y_{(k-1)n}, \\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_n^{k-1}y_{(k-2)n}, \\
y_{(k-2)n} &= (1 - \alpha_{(k-2)n})y_{(k-3)n} + \alpha_{(k-2)n}T_n^{k-2}y_{(k-3)n}, \\
&\vdots \\
y_{2n} &= (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_n^{n}y_{1n}, \\
y_{1n} &= (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_n^{1}y_{0n},
\end{align*}
$$

(1.2)

where $y_{0n} = x_n$ for all $n$.

The aim of this paper is to obtain strong and weak convergence results for the iterative process (1.2) of a finite family of generalized asymptotically quasinonexpansive mappings in Banach spaces.

2. Preliminaries

In this section, we give some definitions and lemmas used in the main results.
Let $C$ be a nonempty subset of a real Banach space $X$, and let $T$ be a self-mapping of $C$. The fixed point set of $T$ is denoted by $F(T) = \{x \in C: Tx = x\}$.

Then let $T$ is called

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;

(ii) quasinonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$;

(iii) asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} r_n = 0$ and $\|T^nx - T^n'y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(iv) asymptotically quasinonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} r_n = 0$ and $\|T^nx - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(v) generalized quasinonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \to 0$ as $n \to \infty$ such that $\|T^nx - p\| \leq \|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(vi) generalized asymptotically quasinonexpansive [19] if $F(T) \neq \emptyset$ and there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that $\|T^nx - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(vii) uniformly L-Lipschitzian if there exists constant $L > 0$ such that $\|T^nx - T^n'y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(viii) $(L - \gamma)$ uniform Lipschitz if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^nx - T^n'y\| \leq L\|x - y\|^\gamma$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(ix) semicompact if for a sequence $\{x_n\}$ in $C$ with $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

From the definition of these mappings, it can be seen that

(i) a quasinonexpansive mapping is generalized quasinonexpansive;

(ii) an asymptotically quasinonexpansive mapping is generalized asymptotically quasinonexpansive;

(iii) a generalized quasinonexpansive mapping is generalized asymptotically quasinonexpansive;

(iv) a uniformly L-Lipschitzian mapping is $(L - 1)$ uniform Lipschitz.

The map $T : C \to X$ is said to be demiclosed at $0$ if for each sequence $\{x_n\}$ in $C$ converging weakly to $x \in C$ and $Tx_n$ converging strongly to $0$, we get $Tx = 0$.

A Banach space $X$ is said to have Opial's property if for each sequence $\{x_n\}$ converging weakly to $x \in C$ and $x \neq y$, we have the condition

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$  \hspace{1cm} (2.1)

Condition (A'). Let $C$ be a subset of a normed space $X$. A family of self-mappings $\{T_i : i = 1, 2, \ldots, k\}$ of $C$ is said to have Condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T_ix\| \geq f(d(x, F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F = \bigcap_{i=1}^k F(T_i)\}$. 
Lemma 2.1 (cf. [17, Lemma 2.2]). Let the sequences \(\{a_n\}, \{\delta_n\}\) and \(\{c_n\}\) of real numbers satisfy:

\[
a_{n+1} \leq (1 + \delta_n)a_n + c_n, \quad \text{where } a_n \geq 0, \quad \delta_n \geq 0, \quad c_n \geq 0 \quad \forall n = 1, 2, 3, \ldots
\]

and \(\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} c_n < \infty\).

(i) \(\lim_{n \to \infty} a_n\) exists;

(ii) if \(\lim \inf_{n \to \infty} a_n = 0\), then \(\lim_{n \to \infty} a_n = 0\).

Lemma 2.2 (see [2, Lemma 1.3]). Let \(X\) be a uniformly convex Banach space. Assume that \(0 < b \leq t_n \leq c < 1, \ n = 1, 2, 3, \ldots\). Let the sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) be such that \(\lim \sup_{n \to \infty} \|x_n\| \leq a\), \(\lim \sup_{n \to \infty} \|y_n\| \leq a\) and \(\lim_{n \to \infty} \|t_n x_n + (1-t_n)y_n\| = a\), where \(a \geq 0\). Then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\).

3. Convergence in Banach Spaces

The aim of this section is to establish the strong convergence of the iterative scheme (1.2) to converge to a common fixed point of a finite family of asymptotically quasinonexpansive mappings in a Banach space under some appropriate conditions.

Lemma 3.1. Let \(C\) be a nonempty closed convex subset of a real Banach space \(X\), and \(\{T_i : i = 1, 2, \ldots, k\}\) be a finite family of generalized asymptotically quasinonexpansive self-mappings of \(C\), that is, \(\|T_i^n x - p_i\| \leq (1 + r_i^n) \|x - p_i\| + s_i\), for all \(x \in C\) with the sequence \(\{r_i^n\}, \{s_i\} \subset [0, \infty)\) and \(p_i \in F(T_i), i = 1, 2, \ldots, k\). Suppose that \(F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset, x_1 \in C\), and the iterative sequence \(\{x_n\}\) is defined by (1.2). Let \(r_n = \max_{1 \leq i \leq k} \{r_i^n\}\) and \(s_n = \max_{1 \leq i \leq k} \{s_i\}\). Then for \(p \in F\), we get the following:

(i) \(\|x_n - T_i^n x_n\| \leq (2 + r_n) \|x_n - p\| + s_n\), for all \(i = 1, 2, \ldots, k\);

(ii) \(\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n) \|y_{(i-1)n} - p\| + s_n\), for all \(i = 1, 2, \ldots, k\);

(iii) \(\|y_{kn} - p\| \leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}\);

(iv) \(\|x_{n+1} - p\| \leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}\);

(v) \(\|x_{n+1} - p\| \leq (1 + \delta_n) \|x_n - p\| + c_n\), for all \(n \in \mathbb{N}\), where \(c_n = s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}\) and \(\delta_n = (\frac{1}{k}) r_n + (\frac{1}{k}) r_n^2 + \cdots + (\frac{1}{k}) r_n^k\).

(vi) If \(\sum_{n=1}^{\infty} r_n < \infty\) and \(\sum_{n=1}^{\infty} s_n < \infty\), then \(\lim_{n \to \infty} \|x_n - p\|\) exists.

Proof. Let \(p \in F\).

(i) For \(i = 1, 2, 3, \ldots, k\), we have

\[
\|x_n - T_i^n x_n\| \leq \|x_n - p\| + \|T_i^n x_n - p\|
\leq \|x_n - p\| + (1 + r_n) \|x_n - p\| + s_{in},
\]

where \(s_{in} = s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}\) and \(\delta_{in} = (\frac{1}{k}) r_n + (\frac{1}{k}) r_n^2 + \cdots + (\frac{1}{k}) r_n^k\).

3. Convergence in Banach Spaces

The aim of this section is to establish the strong convergence of the iterative scheme (1.2) to converge to a common fixed point of a finite family of asymptotically quasinonexpansive mappings in a Banach space under some appropriate conditions.
(ii) Similarly to part (i), we have
\[
\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n) \|y_{(i-1)n} - p\| + s_n, \quad \forall i = 1, 2, \ldots, k. \tag{3.2}
\]

(iii) By part (i) and \(0 \leq \alpha_n \leq 1\), we have
\[
\|y_{1n} - p\| = \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_{1n}^n x_n - p)\|
\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_{1n}^n x_n - p\|
\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}(1 + r_{1n})\|x_n - p\| + \alpha_{1n}s_{1n}
\leq (1 + r_n)\|x_n - p\| + s_n,
\]

\[
\|y_{2n} - p\| = \|(1 - \alpha_{2n})(y_{1n} - p) + \alpha_{2n}(T_{2n}^n y_{1n} - p)\|
\leq (1 - \alpha_{2n})\|y_{1n} - p\| + \alpha_{2n}\|T_{2n}^n y_{1n} - p\|
\leq (1 - \alpha_{2n})\|y_{1n} - p\| + \alpha_{2n}(1 + r_{2n})\|y_{1n} - p\| + \alpha_{2n}s_{2n}
\leq (1 + r_n)\|y_{1n} - p\| + s_n
\leq (1 + r_n)(1 + r_n)\|x_n - p\| + s_n(1 + r_n) + s_n
\leq (1 + r_n)^2\|x_n - p\| + s_n((1 + r_n) + 1), \tag{3.3}
\]

\[
\|y_{3n} - p\| = \|(1 - \alpha_{3n})(y_{2n} - p) + \alpha_{3n}(T_{3n}^n y_{2n} - p)\|
\leq (1 - \alpha_{3n})\|y_{2n} - p\| + \alpha_{3n}\|T_{3n}^n y_{2n} - p\|
\leq (1 - \alpha_{3n})\|y_{2n} - p\| + \alpha_{3n}(1 + r_{3n})\|y_{2n} - p\| + \alpha_{3n}s_{3n}
\leq (1 + r_n)\|y_{2n} - p\| + s_n
\leq (1 + r_n)^3\|x_n - p\| + s_n\left((1 + r_n)^2 + (1 + r_n) + 1\right),
\]

\[
\vdots
\]

\[
\|y_{kn} - p\| \leq (1 + r_n)^k\|x_n - p\| + s_n\sum_{i=1}^k (1 + r_n)^{(k-i)}.
\]

(iv) By part (ii) and part (iii), we get
\[
\|x_{n+1} - p\| = \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_{kn}^n y_{(k-1)n} - p)\|
\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_{kn}^n y_{(k-1)n} - p\|
\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}s_{kn}
\leq (1 + r_n)\|y_{(k-1)n} - p\| + s_n
\leq (1 + r_n)(1 + r_n)^{k-1}\|x_n - p\| + (1 + r_n)s_n\sum_{i=1}^{k-1} (1 + r_n)^{(k-1-i)} + s_n
\]

\[
= (1 + r_n)^k\|x_n - p\| + s_n\sum_{i=1}^k (1 + r_n)^{(k-i)}.
\]
(v) Put \( \delta_n = (\frac{1}{k}) r_n + (\frac{k}{2}) r_n^2 + \cdots + (\frac{k}{k}) r_n^k \) and \( c_n = s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)} \). Then (v) is directly obtained by (iv).

(vi) By (v), we have \( \|x_{n+1} - p\| \leq (1 + \delta_n) \|x_n - p\| + c_n \) for all \( n \in \mathbb{N} \), where \( c_n = s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)} \) and \( \delta_n = (\frac{1}{k}) r_n + (\frac{k}{2}) r_n^2 + \cdots + (\frac{k}{k}) r_n^k \). From \( \sum_{n=1}^{\infty} r_n < \infty \) and \( \sum_{n=1}^{\infty} c_n < \infty \), it follows that \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} c_n < \infty \). By Lemma 2.1, we get \( \lim_{n \to \infty} \|x_n - p\| \) exists.

\[ \square \]

**Theorem 3.2.** Let \( C \) be a nonempty closed convex subset of a real Banach space \( X \), and \( \{T_i : i = 1, 2, \ldots, k\} \) be a finite family of generalized asymptotically quasinonexpansive self-mappings of \( C \), that is, \( \|T_i x - p_i\| \leq (1 + r_n) \|x - p_i\| + s_{i,n}, \) for all \( x \in C \), and \( p_i \in F(T_i), i = 1, 2, \ldots, k. \) Suppose that \( F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset \) is closed, \( x_1 \in C \) and the iterative sequence \( \{x_n\} \) is defined by (1.2). Assume that \( \sum_{n=1}^{\infty} r_n < \infty \) and \( \sum_{n=1}^{\infty} s_n < \infty \), where \( r_n = \max_{1 \leq i \leq k} \{r_{in}\} \) and \( s_n = \max_{1 \leq i \leq k} \{s_{i,n}\}. \) Then \( \{x_n\} \) converges strongly to a common fixed point of the family of mappings if and only if
\[ \liminf_{n \to \infty} d(x_n, F) = 0, \text{ where } d(x, F) = \inf_{p \in F} \|x - p\|. \]

**Proof.** The necessity is obvious and then we prove only the sufficiency. Let \( p \in F. \) Since \( 1 + t \leq e^t \) for \( t \geq 0, \) we obtain \( (1 + t)^k \leq e^{kt}, \) for \( k = 1, 2, \ldots. \) Thus by Lemma 3.1(iv) and (v), for positive integers \( m \) and \( n, \) we have
\[
\|x_{n+m} - p\| \leq (1 + r_{(n+m-1)}^{n}) \|x_{(n+m-1)} - p\| + c_{(n+m-1)}
\]
\[
\leq \exp\left\{ k \sum_{i=n}^{n+m-1} r_i \right\} \|x_n - p\| + \sum_{i=n}^{n+m-1} c_i
\]
\[
\leq \exp\left\{ k \sum_{i=1}^{\infty} r_i \right\} \|x_n - p\| + \sum_{i=n}^{\infty} c_i
\]
\[
= M \|x_n - p\| + \sum_{i=n}^{\infty} c_i
\]
where \( M = \exp\{k \sum_{i=1}^{\infty} r_i\}. \)

By Lemma 3.1(v), we have
\[
\|x_{n+1} - p\| \leq (1 + \delta_n) \|x_n - p\| + c_n, \quad \forall p \in F,
\]
where \( \delta_n = (\frac{1}{k}) r_n + (\frac{k}{2}) r_n^2 + \cdots + (\frac{k}{k}) r_n^k \) and \( c_n = s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}. \)

It follows that
\[
d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F).
\]

From the given condition \( \liminf_{n \to \infty} d(x_n, F) = 0 \) and Lemma 2.1, we get
\[
\lim_{n \to \infty} d(x_n, F) = 0.
\]
Next, we show that \{x_n\} is a Cauchy sequence in C. By (3.8) and \(\sum_{n=1}^{\infty} c_n < \infty\), we get that for any \(\epsilon > 0\), there exists a positive integer \(n_0\) such that, for all \(n \geq n_0\),

\[
d(x_n, F) < \frac{\epsilon}{3M}, \quad \sum_{n=n_0}^{\infty} c_n < \frac{\epsilon}{3}.
\]  

(3.9)

From \(d(x_n, F) < \epsilon/3M\), there exists \(p_0 \in F\) such that

\[
\|x_{n_0} - p_0\| < \frac{\epsilon}{3M}.
\]  

(3.10)

For any positive integer \(m\), by (3.5), (3.9), and (3.10), we have

\[
\|x_{n_0+m} - x_{n_0}\| \leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\|
\]

\[
\leq M\|x_{n_0} - p_0\| + \sum_{i=n_0}^{\infty} c_i + \|x_{n_0} - p_0\|
\]

\[
< M\frac{\epsilon}{3M} + \frac{\epsilon}{3} + M\frac{\epsilon}{3M} = \epsilon.
\]  

(3.11)

Thus, \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete, \(x_n \to q \in X\). Actually, \(q \in C\) because \(\{x_n\} \subset C\) and \(C\) is a closed subset of \(X\). Next we show that \(q \in F\). Since \(F = \bigcap_{i=1}^{k} F(T_i)\) is closed, by the continuity of \(d(x, F)\) with \(d(x_n, F) \to 0\) and \(x_n \to q\) as \(n \to \infty\), we get \(d(q, F) = 0\) and then \(q \in F\). Therefore, the proof is complete. \(\square\)

Since any asymptotically quasinonexpansive mapping is generalized asymptotically quasinonexpansive, the next corollary is obtained immediately from Theorem 3.2.

**Corollary 3.3** (see [5, Theorem 3.2]). Let \(C\) be a nonempty closed convex subset of a real Banach space \(X\), and \(\{T_i : i = 1, 2, \ldots, k\}\) be a finite family of asymptotically quasinonexpansive self-mappings of \(C\), that is, \(\|T_i^nx - p_i\| \leq (1 + r_{in})\|x - p_i\|\) for all \(x \in C\) and \(p_i \in F(T_i)\), \(i = 1, 2, \ldots, k\). \(x_1 \in C\) and the iterative sequence \(\{x_n\}\) be defined by (1.2). Then \(\{x_n\}\) converges strongly to a common fixed point of the family of mappings if and only if \(\lim\inf_{n \to \infty} d(x_n, F) = 0\), where \(d(x, F) = \inf_{p \in F} \|x - p\|\).

**4. Convergence in Uniformly Convex Banach Spaces**

In this section, strong and weak convergence results for the iterative process (1.2) on uniformly convex Banach spaces are proved without using the condition \(\lim\inf_{n \to \infty} d(x_n, F) = 0\) appearing in Section 3.

**Theorem 4.1.** Let \(C\) be a nonempty closed convex subset of an uniformly convex real Banach space \(X\). Let \(\{T_i : i = 1, 2, \ldots, k\}\) be a finite family of uniformly \((L - \gamma)\) Lipschitzian and generalized asymptotically quasinonexpansive self-mappings of \(C\), that is, \(\|T_i^nx - T_i^ny\| \leq L\|x - y\|^{\sigma}\) and \(\|T_i^nx - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}\), for all \(x, y \in C\) and \(p_i \in F(T_i)\), \(i = 1, 2, \ldots, k\). Suppose that \(\{T_i : i = 1, 2, \ldots, k\}\) satisfies condition \((A^\sigma)\) and \(F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset\). Let \(x_1 \in C\) and the iterative sequence \(\{x_n\}\) be defined by (1.2) with \(\alpha_{in} > 0, \alpha_{in} \in [a, b]\), where \(0 < a < b < 1\). Assume that \(\sum_{i=1}^{\infty} r_i < \infty\),
\[
\sum_{n=1}^{\infty} s_n < \infty \text{ where } r_n = \max_{1 \leq i \leq k} \{ r_{in} \} \text{ and } s_n = \max_{1 \leq i \leq k} \{ s_{in} \}. \text{ Then } \{ x_n \} \text{ converges strongly to a common fixed point of the family of mappings.}
\]

**Proof.** Let \( p \in F \). By Lemma 3.1(vi), we get that \( \lim_{n \to \infty} \| x_n - p \| \) exists. Then there is a real number \( c \geq 0 \) such that

\[
\lim_{n \to \infty} \| x_n - p \| = c. \tag{4.1}
\]

By Lemma 3.1(iii), we have

\[
\| y_{kn} - p \| \leq (1 + r_n)^k \| x_n - p \| + s_n \sum_{i=1}^{k} (1 + r_n)^{(k-i)}, \quad \forall p \in F. \tag{4.2}
\]

By taking \( \lim \sup \) on both sides of the above inequality, we get

\[
\limsup_{n \to \infty} \| y_{in} - p \| \leq c, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{4.3}
\]

Since \( \| T_i^n \cdot y_{(i-1)n} - p \| \leq (1 + r_n) \| y_{(i-1)n} - p \| + s_n \) and (4.3), we obtain

\[
\limsup_{n \to \infty} \| T_i^n \cdot y_{(i-1)n} - p \| \leq c, \quad \text{for } i = 1, 2, \ldots, k. \tag{4.4}
\]

Since \( \lim_{n \to \infty} \| x_{n+1} - p \| = c \), we have

\[
\lim_{n \to \infty} \left\| (1 - \alpha_{kn}) \left( y_{(k-1)n} - p \right) + \alpha_{kn} \left( T_k^n \cdot y_{(k-1)n} - p \right) \right\| = c. \tag{4.5}
\]

Using (4.1), (4.4), and Lemma 2.2, we conclude that

\[
\lim_{n \to \infty} \| y_{(k-1)n} - T_k^n \cdot y_{(k-1)n} \| = 0. \tag{4.6}
\]

We assume that

\[
\lim_{n \to \infty} \left\| y_{(i-1)n} - T_i^n \cdot y_{(i-1)n} \right\| = 0, \quad \text{for some } 2 \leq j \leq k. \tag{4.7}
\]

It follows from (4.5) and (4.7) that

\[
c \leq \liminf_{n \to \infty} \| y_{(j-1)n} - p \|, \quad \text{for } 2 \leq j \leq k. \tag{4.8}
\]

By Lemma 3.1(iv), (1.2), and (4.8), we get

\[
\lim_{n \to \infty} \left\| (1 - \alpha_{(j-1)n}) \left( y_{(j-2)n} - p \right) + \alpha_{(j-1)n} \left( T_{j-1}^n \cdot y_{(j-2)n} - p \right) \right\| = \lim_{n \to \infty} \| y_{(j-1)n} - p \| = c. \tag{4.9}
\]
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Using (4.3), (4.4) and Lemma 2.2, we conclude that

$$\lim_{n \to \infty} \| y_{(j-2)n} - T_{j-1}^n y_{(j-2)n} \| = 0. \quad (4.10)$$

Therefore, by mathematical induction, we obtain

$$\lim_{n \to \infty} \| y_{(i-1)n} - T_i^n y_{(i-1)n} \| = 0, \quad \text{for } i = 1, 2, \ldots, k. \quad (4.11)$$

From (1.2), we have

$$\| y_{in} - y_{(i-1)n} \| = \alpha_{in} \| T_i^n y_{(i-1)n} - y_{(i-1)n} \|, \quad \text{for } i = 1, 2, \ldots, k - 1. \quad (4.12)$$

By (4.11), we obtain that

$$\| y_{in} - y_{(i-1)n} \| \to 0 \quad \text{as } n \to \infty, \quad \text{for } i = 1, 2, \ldots, k - 1. \quad (4.13)$$

From

$$\| x_n - y_{in} \| \leq \| x_n - y_{1n} \| + \| y_{1n} - y_{2n} \| + \cdots + \| y_{(i-1)n} - y_{in} \|, \quad (4.14)$$

for $i = 1, 2, \ldots, k - 1$, it follows by (4.13) that

$$\| x_n - y_{in} \| \to 0 \quad \text{as } n \to \infty, \quad \text{for } i = 1, 2, \ldots, k - 1. \quad (4.15)$$

From (4.11), when $i = 1$, we get $\lim_{n \to \infty} x_n - T_1^n x_n = 0$. For $2 \leq i \leq k$, we have

$$\| x_n - T_i^n x_n \| \leq \| x_n - y_{(i-1)n} \| + \| y_{(i-1)n} - T_i^n y_{(i-1)n} \| + \| T_i^n y_{(i-1)n} - T_i^n x_n \|
\leq \| x_n - y_{(i-1)n} \| + \| y_{(i-1)n} - T_i^n y_{(i-1)n} \| + |L| \| y_{(i-1)n} - x_n \|^\eta. \quad (4.16)$$

From (4.11) and (4.15), we conclude that

$$\lim_{n \to \infty} y_{in} = 0, \quad \text{for } i = 1, 2, \ldots, k, \quad (4.17)$$

where $y_{in} = \| x_n - T_i^n x_n \|$. From (1.2), we have

$$\| x_{n+1} - x_n \| \leq (1 - \alpha_{kn}) \| y_{(k-1)n} - x_n \| + \alpha_{kn} \| T_k^n y_{(k-1)n} - x_n \|
\leq (1 - \alpha_{kn}) \| y_{(k-1)n} - x_n \| + \alpha_{kn} \left( \| T_k^n y_{(k-1)n} - y_{(k-1)n} \| + \| y_{(k-1)n} - x_n \| \right) \quad (4.18)
\leq \| y_{(k-1)n} - x_n \| + \alpha_{kn} \| T_k^n y_{(k-1)n} - y_{(k-1)n} \|. \quad (4.18)$$
From (4.11) and (4.15),
\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \] (4.19)

For \( i = 1, 2, \ldots, k \), we have
\[
\|x_{n+1} - T_i x_{n+1}\| \leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i x_{n+1} - T_i^n x_{n+1}\|
\leq \gamma_i(n+1) + L \|x_{n+1} - T_i^n x_{n+1}\|^\gamma_i
\leq \gamma_i(n+1) + L (\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|)^\gamma_i
\leq \gamma_i(n+1) + L (\|x_{n+1} - x_n\| + \gamma_i n + L \|x_n - x_{n+1}\|)^\gamma_i. \] (4.20)

Using (4.17) and (4.19), we obtain
\[ \lim_{n \to \infty} \|x_{n+1} - T_i x_{n+1}\| = 0, \quad \text{for } i = 1, 2, \ldots, k. \] (4.21)

Therefore, by using condition (A′), there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, \infty) \) such that
\[ \lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} \|x_n - T_j x_n\| = 0, \] (4.22)

for some \( 1 \leq j \leq k \), that is
\[ \lim_{n \to \infty} d(x_n, F) = 0. \] (4.23)

By Theorem 3.2, we conclude that \( \{x_n\} \) converges strongly to a point \( p \in F \). \( \Box \)

**Lemma 4.2.** Let \( C \) be a nonempty closed convex subset of an uniformly convex real Banach space \( X \), and \( \{T_i : i = 1, 2, \ldots, k\} \) be a family of \((L - \gamma_i)\) uniform Lipschitz and generalized asymptotically quasinonexpansive self-mappings of \( C \), that is, \( \|T_i^n x - T_i^n y\| \leq L \|x - y\|^{\gamma_i} \) and \( \|T_i^n x - p_i\| \leq (1 + r_n) \|x - p_i\| + s_{in}, \) for all \( x, y \in C \) and \( p_i \in F(T_i), i = 1, 2, \ldots, k \). Suppose that \( F = \bigcap_{i=1}^k F(T_i) \neq \emptyset \). Let \( x_1 \in C \) and the iterative sequence \( \{x_n\} \) be defined by (1.2) with \( \{\alpha_i\}_{i=1}^n \subset [a, b], \) where \( 0 < a < b < 1 \). Assume that \( \sum_{n=1}^\infty r_n < \infty, \sum_{n=1}^\infty s_n < \infty \) where \( r_n = \max_{1 \leq j \leq k} \{r_{jn}\} \) and \( s_n = \max_{1 \leq j \leq k} \{s_{jn}\} \). Then,

(i) \( \lim_{n \to \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0, \) for all \( i = 1, 2, \ldots, k \);

(ii) \( \lim_{n \to \infty} \|x_n - T_i x_n\| = 0, \) for all \( i = 1, 2, \ldots, k \).

**Proof.** (i) Let \( p \in F \). By Lemma 3.1(vi), we obtain that \( \lim_{n \to \infty} \|x_n - p\| \) exists and we then suppose that
\[ \lim_{n \to \infty} \|x_n - p\| = c. \] (4.24)
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By (4.24) and Lemma 3.1(iii), we have

\[
\limsup_{n \to \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \ldots, k - 1.
\]  
(4.25)

By (1.2), we have

\[
\|x_{n+1} - p\| \leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - p\|
\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n) \|y_{(k-1)n} - p\| + \alpha_{kn} s_{kn}
\leq (1 + r_n) \|y_{(k-1)n} - p\| + s_n
= (1 + r_n) [(1 - \alpha_{(k-1)n}) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} \|T_{k-1}^n y_{(k-2)n} - p\|] + s_n
\leq (1 + r_n) [(1 - \alpha_{(k-1)n}) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} (1 + r_n) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} s_n] + s_n
\leq (1 + r_n)^2 \|y_{(k-2)n} - p\| + s_n (1 + r_n) + s_n
\vdots
\leq (1 + r_n)^{k-i} \|y_{in} - p\| + s_n \sum_{i=1}^{k-1} (1 + r_n)^{k-i},
\]  
(4.26)

for \( i = 1, 2, \ldots, k - 1 \). It follows that

\[
c \leq \liminf_{n \to \infty} \|y_{in} - p\|, \quad \text{for } i = 1, 2, \ldots, k - 1.
\]  
(4.27)

From (4.25) and (4.27), we obtain

\[
\lim_{n \to \infty} \|y_{in} - p\| = c, \quad \text{for } i = 1, 2, \ldots, k - 1,
\]  
(4.28)

and then

\[
\lim_{n \to \infty} \| (1 - \alpha_{in}) (y_{(i-1)n} - p) + \alpha_{in} (T_i^n y_{(i-1)n} - p) \| = c,
\]  
(4.29)

for \( i = 1, 2, \ldots, k - 1 \).

Since \( \|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n) \|y_{(i-1)n} - p\| + s_n \), for \( i = 1, 2, \ldots, k - 1 \), we have

\[
\limsup_{n \to \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \ldots, k - 1.
\]  
(4.30)
From (4.25), (4.29), (4.30) and Lemma 2.2, we obtain
\[
\lim_{n \to \infty} \| T_i^n y_{(i-1)n} - y_{(i-1)n} \| = 0, \quad \text{for } i = 1, 2, \ldots, k - 1. \tag{4.31}
\]

Now we want to show that (4.31) is also true for \( i = k \).

Since \( \| T_k^n y_{(k-1)n} - p \| \leq (1 + r_n) \| y_{(k-1)n} - p \| + s_n \), \( \lim_{n \to \infty} r_n = 0 \) and \( \lim_{n \to \infty} s_n = 0 \), it follows by (4.28),
\[
\limsup_{n \to \infty} \| T_k^n y_{(k-1)n} - p \| \leq c. \tag{4.32}
\]

We also have
\[
\lim_{n \to \infty} \| (1 - \alpha_{kn}) (y_{(k-1)n} - p) + \alpha_{kn} (T_k^n y_{(k-1)n} - p) \| = \lim_{n \to \infty} \| x_{n+1} - p \| = c. \tag{4.33}
\]

Hence, by (4.25), (4.32), and Lemma 2.2, we obtain
\[
\lim_{n \to \infty} \| y_{(k-1)n} - T_k^n y_{(k-1)n} \| = 0. \tag{4.34}
\]

Then, (4.31) and (4.34) give us
\[
\lim_{n \to \infty} \| T_i^n y_{(i-1)n} - y_{(i-1)n} \| = 0, \quad \text{for } i = 1, 2, \ldots, k. \tag{4.35}
\]

From
\[
\| x_n - T_i^n y_{(i-1)n} \| \leq \| x_n - y_{(i-1)n} \| + \| y_{(i-1)n} - T_i^n y_{(i-1)n} \|, \tag{4.36}
\]

it implies by (4.15) and (4.35) that
\[
\lim_{n \to \infty} \| x_n - T_i^n y_{(i-1)n} \| = 0, \tag{4.37}
\]

for \( i = 1, 2, 3, \ldots, k \).

(ii) From part (i), for \( i = 1 \), we have
\[
\lim_{n \to \infty} \| T_1^n x_n - x_n \| = 0. \tag{4.38}
\]

For \( i = 2, 3, 4, \ldots, k \), we get
\[
\| T_i^n x_n - x_n \| \leq \| T_i^n x_n - T_i^n y_{(i-1)n} \| + \| T_i^n y_{(i-1)n} - x_n \| \leq L \| x_n - y_{(i-1)n} \|^j + \| T_i^n y_{(i-1)n} - x_n \|. \tag{4.39}
\]
By part (i) and (4.15),
\[ \lim_{n \to \infty} \|T^i_n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k. \] (4.40)

For \(1 \leq i \leq k\), we obtain
\[
\|x_n - T_i x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}_i x_{n+1}\|
+ \|T^{n+1}_i x_{n+1} - T^{n+1}_i x_n\| + \|T^{n+1}_i x_n - T_i x_n\|
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}_i x_{n+1}\|
+ L \|x_{n+1} - x_n\|^\gamma + L \|T^n x_n - x_n\|^\gamma.
\] (4.41)

From (4.19) and (4.40), we then have
\[ \lim_{n \to \infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k. \] (4.42)

\textbf{Theorem 4.3.} Under the hypotheses of Lemma 4.2, assume that \(T^m_j\) is semicompact for some positive integers \(m\) and \(1 \leq j \leq k\). Then \(\{x_n\}\) converges strongly to a common fixed point of the family \(\{T_i : i = 1, 2, \ldots, k\}\).

\textbf{Proof.} Suppose that \(T^m_j\) is semicompact for some positive integers \(m \geq 1\) and \(1 \leq j \leq k\). We have
\[
\|T^m_j x_n - x_n\| \leq \|T^m_j x_n - T^{m-1}_j x_n\| + \|T^{m-1}_j x_n - T^{m-2}_j x_n\|
+ \cdots + \|T^2_j x_n - T_j x_n\| + \|T_j x_n - x_n\|
\leq (m - 1) L \|T_j x_n - x_n\|^\gamma + \|T_j x_n - x_n\|.
\] (4.43)

Then, by Lemma 4.2(ii), we get \(\|T^m_j x_n - x_n\| \to 0\) as \(n \to \infty\). Since \(\{x_n\}\) is bounded and \(T^m_j\) is semicompact, there exists a subsequence \(\{x_{n_l}\}\) of \(\{x_n\}\) such that \(x_{n_l} \to q \in C\) as \(l \to \infty\).

By continuity of \(T_i\) and Lemma 4.2(ii), we obtain
\[ \|q - T_j q\| = \lim_{l \to \infty} \|x_{n_l} - T_j x_{n_l}\| = 0, \quad \forall j = 1, 2, \ldots, k. \] (4.44)

Therefore, \(q \in F\) and then Theorem 3.2 implies that \(\{x_n\}\) converges strongly to a common fixed point \(q\) of the family \(\{T_i : i = 1, 2, \ldots, k\}\). \(\square\)

We note that in practical Theorem 4.3 is very useful in the case that one of \(T_i\), \(i = 1, 2, 3, \ldots, k\), is semicompact.

\textbf{Theorem 4.4.} Let \(C\) be a nonempty closed convex subset of an uniformly convex real Banach space \(X\) satisfying the Opial property, and \(\{T_i : i = 1, 2, \ldots, k\}\) be a family of \((L - \gamma_i)\) uniform Lipschitz and
generalized asymptotically quasinonexpansive self-mappings of $C$, that is, $\|T^n_i x - T^n_i y\| \leq L \|x - y\|^p$ and $\|T^n_i x - p_i\| \leq (1+r_n)\|x - p_i\| + \sin, \text{for all } x, y \in C \text{ and } p_i \in F(T_i), i = 1, 2, \ldots, k. \text{ Suppose that } F = \bigcap_{i=1}^k F(T_i) \neq \emptyset. \text{ Let } x_n \in C \text{ and the iterative sequence } \{x_n\} \text{ be defined by (1.2) with } \{r_n\} \subset [a, b], \text{ where } 0 < a < b < 1. \text{ Assume that } \sum_{n=1}^\infty r_n < \infty, \text{ where } r_n = \max_{1 \leq i \leq k} \{r_n\}. \text{ If } I - T_i, i = 1, 2, \ldots, k, \text{ is demiclosed at } 0, \text{ then } \{x_n\} \text{ converges weakly to a common fixed point of the family of mappings.}

Proof. Let $p \in F$. By Lemma 3.1(v), we get $\lim_{n \to \infty} \|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [1] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$. 

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References


