Research Article

Nearly Ring Homomorphisms and Nearly Ring Derivations on Non-Archimedean Banach Algebras

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We prove the generalized Hyers-Ulam stability of homomorphisms and derivations on non-Archimedean Banach algebras. Moreover, we prove the superstability of homomorphisms on unital non-Archimedean Banach algebras and we investigate the superstability of derivations in non-Archimedean Banach algebras with bounded approximate identity.

1. Introduction and Preliminaries

In 1897, Hensel [1] has introduced a normed space which does not have the Archimedean property.

During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings, and superstrings [2]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition [3–9].

Let \( \mathbb{K} \) be a field. A non-Archimedean absolute value on \( \mathbb{K} \) is a function \( |\cdot| : \mathbb{K} \to \mathbb{R} \) such that for any \( a, b \in \mathbb{K} \) we have

(i) \( |a| \geq 0 \) and equality holds if and only if \( a = 0 \),

(ii) \( |ab| = |a||b| \),

(iii) \( |a + b| \leq \max\{|a|, |b|\} \).
Condition (iii) is called the strict triangle inequality. By (ii), we have \(|1| = | - 1| = 1\). Thus, by induction, it follows from (iii) that \(|n| \leq 1\) for each integer \(n\). We always assume in addition that \(|\cdot|\) is non trivial, that is, that there is an \(a_0 \in \mathbb{K}\) such that \(|a_0| \neq 0,1\).

Let \(X\) be a linear space over a scalar field \(\mathbb{K}\) with a non-Archimedean nontrivial valuation \(|\cdot|\). A function \(\|\cdot\| : X \to \mathbb{R}\) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) \(\|x\| = 0\) if and only if \(x = 0\);
(NA2) \(\|rx\| = |r|\|x\|\) for all \(r \in \mathbb{K}\) and \(x \in X\);
(NA3) the strong triangle inequality (ultrametric), namely,

\[
\|x + y\| \leq \max\{\|x\|,\|y\|\} \quad (x, y \in X).
\] (1.1)

Then \((X, \| \cdot \|)\) is called a non-Archimedean space.

It follows from (NA3) that

\[
\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m - 1\} \quad (m > l),
\] (1.2)

therefore a sequence \(\{x_m\}\) is Cauchy in \(X\) if and only if \(\{x_{m+1} - x_m\}\) converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra \(\mathcal{A}\) which satisfies \(\|ab\| \leq \|a\|\|b\|\) for all \(a, b \in \mathcal{A}\). For more detailed definitions of non-Archimedean Banach algebras, we can refer to [10].

The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1960 and affirmatively solved by Hyers [12]. Perhaps Aoki was the first author who has generalized the theorem of Hyers (see [13]).

T. M. Rassias [14] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1** (T. M. Rassias). Let \(f : E \to E'\) be a mapping from a normed vector space \(E\) into a Banach space \(E'\) subject to the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)
\] (1.3)

for all \(x, y \in E\), where \(\varepsilon\) and \(p\) are constants with \(\varepsilon > 0\) and \(p < 1\). Then the limit

\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\] (1.4)

exists for all \(x \in E\) and \(L : E \to E'\) is the unique additive mapping which satisfies

\[
\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p
\] (1.5)

for all \(x \in E\). Also, if for each \(x \in E\) the mapping \(f(tx)\) is continuous in \(t \in \mathbb{R}\), then \(L\) is \(\mathbb{R}\)-linear.
Moreover, Bourgin [15] and Găvruţa [16] have considered the stability problem with unbounded Cauchy differences (see also [17–27]).

On the other hand, J. M. Rassias [28–33] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruţa [34]. This stability phenomenon is called the Ulam-Găvruţa-Rassias stability (see also [35]).

**Theorem 1.2 (J. M. Rassias [28]).** Let $X$ be a real normed linear space and $Y$ a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and $f$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q$$

(1.6)

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{2^r - 2}\|x\|^r$$

(1.7)

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L$ is an $\mathbb{R}$-linear mapping.

Very recently, Ravi et al. [36] in the inequality (1.6) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta \left\{ \|x\|^p \|y\|^p + (\|x\|^{2p} + \|y\|^{2q}) \right\}$.

For more details about the results concerning such problems and mixed product-sum stability (J. M.-Rassias Stability) the reader is referred to [37–49].

Khodaei and T. M. Rassias [50] have established the general solution and investigated the Hyers-Ulam-Rassias stability of the following $n$-dimensional additive functional equation:

$$\sum_{k=2}^{n} \left( \sum_{i_1=2}^{k-1} \sum_{i_2=i_1+1}^{k-1} \cdots \sum_{i_k=i_{k-1}+1}^{k-1} \right) f \left( \sum_{i=1}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r} \right)$$

$$+ f \left( \sum_{i=1}^{n} a_i x_i \right) = 2^{n-1} a_1 f(x_1),$$

(1.8)

where $a_1, \ldots, a_n \in \mathbb{Z} - \{0\}$ with $a_1 \neq \pm 1$.

In this paper, we investigate the Hyers-Ulam stability of homomorphisms and derivations associated with functional equation (1.8).
2. Main Results

Before taking up the main subject, for a given \( f : A \rightarrow B \) between vector spaces, we define the difference operator

\[
Df(x_1, \ldots, x_n) := \sum_{k=2}^{n} \left( \sum_{l_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{k+1}=i_{k+1}+1}^{n} \right) f \left( \sum_{i=1}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_i x_i \right) + \frac{f \left( \sum_{i=1}^{n} a_i x_i \right)}{2^{n-1}} - a_1 f(x_1).
\]

(2.1)

\[ \tag{2.1} \]

\[ \text{Theorem 2.1.} \]
Let \( A, B \) be two non-Archimedean Banach algebras and let \( \psi : A^n \rightarrow [0, \infty), \phi : A^2 \rightarrow [0, \infty) \) be functions such that

\[
\lim_{m \to \infty} \frac{1}{|a_1|} \psi \left( a_1^m x_1, \ldots, a_1^m x_n \right) = \lim_{k \to \infty} \frac{1}{k} \phi(kx, y) = 0
\]

(2.2)

for all \( x_1, \ldots, x_n \in A \), and the limit

\[ \tilde{\psi}(x) := \lim_{m \to \infty} \max \left\{ \frac{1}{|a_1|} \psi \left( a_1^\ell x, 0, \ldots, 0 \right) : 0 \leq \ell < m \right\} \]

(2.3)

exists and \( \lim_{k \to \infty} (1/k) \tilde{\psi}(kx) = 0 \) for all \( x \in A \). Suppose that \( f : A \rightarrow B \) is a function satisfying

\[
\|Df(x_1, \ldots, x_n)\| \leq \psi(x_1, \ldots, x_n), \quad \|f(xy) - f(x)f(y)\| \leq \phi(x, y)
\]

(2.4)

for all \( x_1, \ldots, x_n, x, y \in A \). Then there exists a ring homomorphism \( H : A \rightarrow B \) such that

\[
\|f(x) - H(x)\| \leq \frac{1}{|2^{n-1}a_1|} \tilde{\psi}(x)
\]

(2.5)

for all \( x \in A \) and

\[ H(x)(H(y) - f(y)) = (f(x) - H(x))H(y) = 0 \]

(2.6)

for all \( x, y \in A \). Moreover, if

\[
\lim_{j \to \infty} \lim_{m \to \infty} \max \left\{ \frac{1}{|a_1|} \psi \left( a_1^j x, 0, \ldots, 0 \right) : j \leq \ell < m + j \right\} = 0,
\]

(2.7)

then \( H \) is the unique ring homomorphism satisfying (2.5).
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Proof. By [50, Theorem 4.4], there exists an additive function $H : \mathcal{A} \to \mathcal{B}$ which satisfies (2.5). We have

$$H(x) := \lim_{m \to \infty} a_m f \left( \frac{x}{a_m} \right)$$

(2.8)

for all $x \in \mathcal{A}$. Now we show that $H$ is a multiplicative function. It follows from (2.5) that

$$\left\| f(kx) - H(kx) \right\| \leq \frac{1}{2^{n-1} a_1} \bar{\psi}(kx)$$

(2.9)

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. On the other hand $H$ is additive then we have

$$\left\| \frac{1}{k} f(kx) - H(x) \right\| \leq \frac{1}{2^{n-1} a_1 k} \bar{\psi}(kx)$$

(2.10)

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. If $k \to \infty$, then by (2.3), the right hand side of above inequality tends to zero. It follows that

$$H(x) = \lim_{k \to \infty} \frac{1}{k} f(kx)$$

(2.11)

for all $x \in \mathcal{A}$. Applying (2.3), (2.4), and (2.11) we have

$$H(xy) - H(x)f(y) = \lim_{k \to \infty} \frac{1}{k} (f(kxy) - f(kx)f(y)) = 0$$

(2.12)

for all $x, y \in \mathcal{A}$. This means that

$$H(xy) = H(x)f(y)$$

(2.13)

for all $x, y \in \mathcal{A}$. From (2.13) and additivity of $H$ we have

$$H(x)H(y) = H(x) \lim_{k \to \infty} \frac{1}{k} f(ky) = \lim_{k \to \infty} \frac{1}{k} (H(x)f(ky)) = \lim_{k \to \infty} \frac{1}{k} H(x(ky)) = H(xy)$$

(2.14)

for all $x, y \in \mathcal{A}$. In other words, $H$ is multiplicative. It follows from (2.13) and (2.14) that

$$H(x)(H(y) - f(y)) = 0$$

(2.15)

for all $x, y \in \mathcal{A}$. Similarly, we can show that

$$(f(x) - H(x))H(y) = 0$$

(2.16)
for all \( x, y \in \mathcal{A} \). To prove the uniqueness property of \( H \), let \( T : \mathcal{A} \to \mathcal{B} \) be another ring homomorphism which satisfies (2.5). Applying (2.11) and (2.5) we have

\[
\| H(x) - T(x) \| = \lim_{k \to \infty} \frac{1}{k} \| f(kxy) - T(kx) \| \leq \lim_{k \to \infty} \frac{1}{k} |2^{\gamma-1}a_1| \psi(kx) = 0
\]

for all \( x \in \mathcal{A} \) which is the desired conclusion. \( \square \)

Now, we establish the superstability of homomorphisms as follows.

**Corollary 2.2.** Let \( \mathcal{A}, \mathcal{B} \) be two unital non-Archimedean Banach algebras, and let \( \psi : \mathcal{A}^n \to [0, \infty), \phi : \mathcal{A}^2 \to [0, \infty), f : \mathcal{A} \to \mathcal{B} \) be functions with conditions of Theorem 2.1. Suppose that

\[
\lim_{m \to \infty} a_1^m f\left( \frac{1}{a_1^m} \right) = 1_B.
\]

Then the mapping \( f : \mathcal{A} \to \mathcal{B} \) is a ring homomorphism.

**Proof.** It follows from (2.6) and (2.18) that \( f = H \) in Theorem 2.1. Hence, \( f \) is a ring homomorphism. \( \square \)

**Corollary 2.3.** Let \( \eta : [0, \infty) \to [0, \infty) \) be a function satisfying

(i) \( \eta(|a_1|) \leq \eta(|a_1|)\eta(t) \) for all \( t \geq 0 \);

(ii) \( \eta(|a_1|) < |a_1| \);

(iii) \( \lim_{k \to -\infty}(1/k)\eta(k|a_1|) = 0 \).

Suppose that \( \epsilon > 0 \), and let \( f : \mathcal{A} \to \mathcal{B} \) satisfying

\[
\| Df(x_1, \ldots, x_n) \| + \| f(xy) - f(x)f(y) \| \leq \epsilon \min \left\{ \sum_{i=1}^n \eta(\|x_i\|), \eta(\|x\|)\eta(\|y\|) \right\}
\]

for all \( x_1, \ldots, x_n, x, y \in \mathcal{A} \). Then there exists a unique ring homomorphism \( H : \mathcal{A} \to \mathcal{B} \) such that

\[
\| f(x) - H(x) \| \leq \frac{\epsilon}{|2^{\gamma-1}a_1|} \eta(\|x\|)
\]

for all \( x \in \mathcal{A} \).

**Proof.** Defining \( \psi : \mathcal{A}^n \to [0, \infty) \) and \( \phi : \mathcal{A}^2 \to [0, \infty) \) by

\[
\psi(x_1, \ldots, x_n) := \epsilon \sum_{i=1}^n \eta(\|x_i\|), \quad \phi(x, y) := \eta(\|x\|)\eta(\|y\|),
\]

(2.21)
respectively, we have
\[
\lim_{m \to \infty} \frac{1}{|a_1|^m} \psi(a_1^m x_1, \ldots, a_1^m x_n) \leq \lim_{m \to \infty} \left( \frac{\eta(|a_1|)}{|a_1|} \right)^m \psi(x_1, \ldots, x_n) = 0 \tag{2.22}
\]
for all \(x_1, \ldots, x_n \in \mathcal{A}\). Hence
\[
\tilde{\psi}(x) := \lim_{m \to \infty} \max \left\{ \frac{1}{|a_1|^\ell} \psi(a_1^\ell x, 0, \ldots, 0) : 0 \leq \ell < m \right\} = \psi(x, 0, \ldots, 0),
\]
\[
\lim_{j \to \infty} \lim_{m \to \infty} \max \left\{ \frac{1}{|a_1|^j} \psi(a_1^j x, 0, \ldots, 0) : j \leq \ell < m + j \right\} = \lim_{j \to \infty} \frac{1}{|a_1|^j} \psi(a_1^j x, 0, \ldots, 0) = 0 \tag{2.23}
\]
for all \(x \in \mathcal{A}\). On the other hand
\[
\lim_{k \to \infty} \frac{1}{k} \tilde{\phi}(kx, y) = \lim_{k \to \infty} \frac{1}{k} \eta(k \|x\|) \eta(\|y\|) = 0 \tag{2.24}
\]
for all \(x, y \in \mathcal{A}\). The conclusion follows from Theorem 2.1.

**Remark 2.4.** The classical example of the function \(\eta\) is the function \(\eta(t) = t^p\) for all \(t \in [0, \infty)\), where \(p > 1\) with the further assumption that \(|a_1| < 1\).

Now, we prove the stability of derivations non-Archimedean Banach algebras by using Theorem 2.1.

**Theorem 2.5.** Let \(\mathcal{A}\) be a non-Archimedean Banach algebra, and let \(\mathcal{H}\) be a non-Archimedean Banach \(\mathcal{A}\)-module. Let \(\psi : \mathcal{A}^n \to [0, \infty)\), \(\phi : \mathcal{A}^2 \to [0, \infty)\) be a function such that
\[
\lim_{m \to \infty} \frac{1}{|a_1|^m} \psi(a_1^m x_1, \ldots, a_1^m x_n) = \lim_{k \to \infty} \frac{1}{k} \phi(kx, y) = 0 \tag{2.25}
\]
for all \(x_1, \ldots, x_n \in \mathcal{A}\), and the limit
\[
\tilde{\psi}(x) := \lim_{m \to \infty} \max \left\{ \frac{1}{|a_1|^\ell} \psi(a_1^\ell x, 0, \ldots, 0) : 0 \leq \ell < m \right\}
\]
exists and \(\lim_{k \to \infty} (1/k) \tilde{\phi}(kx) = 0 \) for all \(x \in \mathcal{A}\). Suppose that \(f : \mathcal{A} \to \mathcal{H}\) is a function satisfying
\[
\|Df(x_1, \ldots, x_n)\| \leq \psi(x_1, \ldots, x_n), \quad \|f(xy) - f(x)y - xf(y)\| \leq \phi(x, y) \tag{2.27}
\]
We define projection maps $\pi_i$ for all $i$. Then there exists a ring derivation $D : \mathcal{A} \to \mathcal{K}$ such that

$$\|f(x) - D(x)\| \leq \frac{1}{2^{n-1}a_1} \tilde{\varphi}(x)$$

(2.28)

for all $x \in \mathcal{A}$.

**Proof.** It is easy to see that $\mathcal{K} \oplus_1 \mathcal{A}$ is a non-Archimedean Banach algebra equipped with the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1a_2) \quad (a_1, a_2 \in \mathcal{A}, x_1, x_2 \in \mathcal{K})$$

(2.29)

and with the following $\ell_1$-norm:

$$\|(x, a)\| = \|x\| + \|a\| \quad (a \in \mathcal{A}, x \in \mathcal{K}).$$

(2.30)

Let us define the mapping $\varphi_f : \mathcal{A} \to \mathcal{K} \oplus_1 \mathcal{A}$ by $a \mapsto (f(a), a)$. It is easy to see that $\varphi_f : \mathcal{A} \to \mathcal{K} \oplus_1 \mathcal{A}$ satisfies the conditions of Theorem 2.1. By Theorem 2.1, there exists a unique ring homomorphism $H : \mathcal{A} \to \mathcal{K} \oplus_1 \mathcal{A}$ such that

$$\|H(a) - \varphi_f(a)\| \leq \frac{1}{2^{n-1}a_1} \tilde{\varphi}(a) \quad (a \in \mathcal{A}).$$

(2.31)

We define projection maps $\pi_1 : \mathcal{K} \oplus_1 \mathcal{A} \to \mathcal{K}$ and $\pi_2 : \mathcal{K} \oplus_1 \mathcal{A} \to \mathcal{A}$ by $(x, b) \mapsto x$ and $(x, b) \mapsto b$, respectively.

It follows from (2.31) that

$$\|(\pi_2 \circ \varphi_f)(ka) - (\pi_2 \circ H)(ka)\| \leq \|\varphi_f(ka) - H(ka)\| \leq \frac{1}{2^{n-1}a_1} \tilde{\varphi}(ka) \quad (k \in \mathbb{N}, a \in \mathcal{A}).$$

(2.32)

By the additivity of mappings under consideration

$$(\pi_2 \circ \varphi_f)(ka) = k(\pi_2 \circ \varphi_f)(a),$$

$$(\pi_2 \circ \varphi_f)(ka) = \pi_2(f(ka), ka) = ka,$$

(2.33)

whence, by (2.32),

$$\|a - (\pi_2 \circ H)(a)\| \leq \frac{1}{k} \frac{1}{2^{n-1}a_1} \tilde{\varphi}(ka)$$

(2.34)

for all $k \in \mathbb{N}, a \in \mathcal{A}$. By letting $k$ tend to $\infty$ in (2.34), we obtain by (2.25) that

$$(\pi_2 \circ H)(a) = a \quad (a \in \mathcal{A}).$$

(2.35)
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Put $D := \pi_1 \circ H$. Then we have

$$(\pi_1 \circ H)(ab), ab = (\pi_1(H(ab)), \pi_2(H(ab))) = H(ab) = H(a)H(b)$$

$$= (\pi_1(H(a)), \pi_2(H(a)))(\pi_1(H(b)), \pi_2(H(b)))$$

$$= (\pi_1(H(a)), a)(\pi_1(H(b)), b)$$

$$= (a\pi_1(H(b)) + \pi_1(H(a))b, ab)$$

for all $a, b \in \mathcal{A}$. It follows that $D$ is a derivation. On the other hand, by (2.31) we have

$$\|D(a) - f(a)\| = \|\pi_1(H(a)) - \pi_1(\psi_f(a))\| \leq \|H(a) - \psi_f(a)\| \leq \frac{1}{|2^\nu - a_1|} \bar{\psi}(a)$$

for all $a \in \mathcal{A}$.

To prove the uniqueness property of $D$, assume that $D^*$ is another derivation from $\mathcal{A}$ into $\mathcal{K}$ satisfying

$$\|D^*(a) - f(a)\| \leq \frac{1}{|2^\nu - a_1|} \bar{\psi}(a) \quad (a \in \mathcal{A}).$$

Then by (2.25), we have

$$\|D(a) - D^*(a)\| = \lim_{k \to \infty} \frac{1}{k} \|D(ka) - D^*(ka)\| \leq \lim_{k \to \infty} \left( \frac{1}{k} \|D^*(a) - f(a)\| + \frac{1}{k} \|D(a) - f(a)\| \right)$$

$$\leq \lim_{k \to \infty} \frac{2}{k} \frac{1}{|2^\nu - a_1|} \bar{\psi}(ka)$$

$$= 0$$

for all $a \in \mathcal{A}$. This means that $D(a) = D^*(a)$ for all $a \in \mathcal{A}$. \hfill $\square$

**Corollary 2.6.** Let $\eta : [0, \infty) \to [0, \infty)$ be a function satisfying

(i) $\eta(|t|) \leq \eta(|a_1|)\eta(t)$ for all $t \geq 0$;

(ii) $\eta(|a_1|) < |a_1|$;

(iii) $\lim_{k \to \infty} (1/k) \eta(k|a_1|) = 0$.

Suppose that $\varepsilon > 0$, and let $f : \mathcal{A} \to \mathcal{K}$ satisfying

$$\|Df(x_1, \ldots, x_n)\| + \|f(xy) - f(x)y - xf(y)\| \leq \varepsilon \min\left\{ \sum_{i=1}^{n} \eta(|x_i|), \eta(||x||)\eta(||y||) \right\}$$

(2.40)
for all \( x_1, \ldots, x_n, x, y \in \mathcal{A} \). Then there exists a unique ring derivation \( D : \mathcal{A} \to \mathcal{X} \) such that

\[
\|f(x) - D(x)\| \leq \frac{\varepsilon}{\|2^{n-1} a_1\|} \eta(\|x\|) \tag{2.41}
\]

for all \( x \in \mathcal{A} \).

Now, we would like to prove the superstability of derivations on non-Archimedean Banach algebras.

**Theorem 2.7.** Let \( \mathcal{A} \) be a non-Archimedean Banach algebra with bounded approximate identity. Let \( \varphi : \mathcal{A}^n \to [0, \infty) \), \( \phi : \mathcal{A}^2 \to [0, \infty) \), \( f : \mathcal{A} \to \mathcal{A} \) be functions satisfying the conditions of Theorem 2.5. Then \( f : \mathcal{A} \to \mathcal{A} \) is a ring derivation.

**Proof.** In the proof of Theorem 2.5, we can see that

\[
H(b)(H(a) - \varphi_f(a)) = (H(a) - \varphi_f(a))H(b) = 0 \tag{2.42}
\]

for all \( a, b \in \mathcal{A} \).

\[
(f(a) - D(a))b = \pi_1((f(a) - D(a))b, 0)
\]

\[
= \pi_1((f(a) - D(a), 0)(D(b), b))
\]

\[
= \pi_1((\pi_1(H(a) - \varphi_f(a)), 0)(\pi_1(H(b)), b))
\]

\[
= \pi_1((\pi_1(H(a) - \varphi_f(a)), 0)H(b))
\]

\[
= \pi_1((\pi_1(H(a)), a - (\pi_1(\varphi_f(a)), a))H(b))
\]

\[
= \pi_1(0, 0) \quad \text{(by (2.42))}
\]

\[
= 0
\]

for all \( a, b \in \mathcal{A} \). Since \( \mathcal{A} \) has a bounded approximate identity, then by above equation, we have \( f(a) = D(a) \) for all \( a \in \mathcal{A} \). \( f \) is a ring derivation on \( \mathcal{A} \). \( \square \)

**References**


