Research Article

Multiple Positive Solutions of a Second Order Nonlinear Semipositone \( m \)-Point Boundary Value Problem on Time Scales

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In this paper, we study a general second-order \( m \)-point boundary value problem for nonlinear singular dynamic equation on time scales

\[ u^{ΔΔ}(t) + a(t)u^{Δ}(t) + b(t)u(t) + λq(t)f(t, u(t)) = 0, \quad t \in (0, 1)_T, \]
\[ u(ρ(0)) = 0, \quad u(σ(1)) = \sum_{i=1}^{m-2} α_i u(η_i), \]  \( \text{for all } i = 1, 2, \ldots, m-2; \)

where \( α_i \geq 0, \ 0 < η_i < η_{i+1} < 1; \) for all \( i = 1, 2, \ldots, m-2; \) \( f, q, a \) and \( b \) satisfy

\( \text{(C1)} \) \( q \in L \) is continuously and nonnegative function and there exists \( t_0 \in (ρ(0), σ(1)) \) s.t. \( q(t_0) > 0, \ q(t) \) may be singular at \( t = ρ(0), σ(1); \)

\( \text{(C2)} \) \( a \in C([0, 1], [0, +∞)), \ b \in C([0, 1], (−∞, 0]). \)

In the past few years, the boundary value problems of dynamic equations on time scales have been studied by many authors (see [1–15] and references therein). Recently,
multiple-point boundary value problems on time scale have been studied, for instance, see [1–9].

In 2008, Lin and Du [2] studied the \( m \)-point boundary value problem for second-order dynamic equations on time scales:

\[
\begin{align*}
\Delta^m u(t) + f(t,u) &= 0, \\
\Delta^2 u(0) &= 0,
\end{align*}
\]

where \( T \) is a time scale. This paper deals with the existence of multiple positive solutions for second-order dynamic equations on time scales. By using Green’s function and the Leggett-Williams fixed point theorem in an appropriate cone, the existence of at least three positive solutions of the problem is obtained.

In 2009, Topal and Yantir [1] studied the general second-order nonlinear \( m \)-point boundary value problems (1.1) with no singularities and the case. The authors deal with the determining the value of \( \lambda \); the existences of multiple positive solutions of (1.1) are obtained by using the Krasnosel’skii and Legget-Williams fixed point theorems.

Motivated by the abovementioned results, we continue to study the general second-order nonlinear \( m \)-point boundary value problem (1.1), but the nonlinear term may be singularity and semipositone.

In this paper, the nonlinear term \( f \) of (1.1) is suit to and semipositone and the superlinear case, we will prove our two existence results for problem (1.1) by using Krasnosel’skii fixed point theorem. This paper is organized as follows. In Section 2, starting with some preliminary lemmas, we state the Krasnosel’skii fixed point theorem. In Section 3, we give the main result which state the sufficient conditions for the \( m \)-point boundary value problem (1.1) to have existence of positive solutions.

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results. From Lemmas 2.1 and 2.3 in [1], we have the following lemma.

**Lemma 2.1** (see [1]). *Assuming that (C2) holds. Then the equations*

\[
\begin{align*}
\phi_1^\Delta(t) + a(t)\phi_1^\Delta(t) + b(t)\phi_1(t) &= 0, \\
\phi_1(\rho(0)) &= 0, \\
\phi_1(\sigma(1)) &= 1,
\end{align*}
\]

\[
\begin{align*}
\phi_2^\Delta(t) + a(t)\phi_2^\Delta(t) + b(t)\phi_2(t) &= 0, \\
\phi_2(\rho(0)) &= 1, \\
\phi_2(\sigma(1)) &= 0
\end{align*}
\]

*have unique solutions \( \phi_1 \) and \( \phi_2 \), respectively, and*

(a) \( \phi_1 \) is strictly increasing on \([\rho(0), \sigma(1)]\),

(b) \( \phi_2 \) is strictly decreasing on \([\rho(0), \sigma(1)]\).
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For the rest of the paper we need the following assumption:

(C3) \(0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) < 1\).

**Lemma 2.2** (see [1]). Assuming that (C2) and (C3) hold. Let \(y \in C[\rho(0), \sigma(1)]\). Then boundary value problem

\[
x^\Delta v(t) + a(t)x^\Delta(t) + b(t)x(t) + y(t) = 0, \quad t \in (0, 1)_v,
\]

\[
x(\rho(0)) = 0, \quad x(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i)
\]

is equivalent to integral equation

\[
x(t) = \int_{\rho(0)}^{\sigma(1)} H(t, s)p(s)y(s)\nabla s + A\phi_1(t),
\]

where

\[
p(t) = e_\alpha(\rho(t), \rho(0)), \quad A = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s,
\]

\[
H(t, s) = \frac{1}{\phi_1^\Delta(\rho(0))} \begin{cases} 
\phi_1(s)\phi_2(t), & s \leq t, \\
\phi_1(t)\phi_2(s), & t \leq s.
\end{cases}
\]

**Proof.** First we show that the unique solution of (2.3) can be represented by (2.4). From Lemma 2.1, we know that the homogenous part of (2.3) has two linearly independent solution \(\phi_1\) and \(\phi_2\) since

\[
\begin{vmatrix}
\phi_1(\rho(0)) & \phi_1^\Delta(\rho(0)) \\
\phi_2(\rho(0)) & \phi_2^\Delta(\rho(0))
\end{vmatrix} = -\phi_1^\Delta(\rho(0)) \neq 0.
\]

Now by the method of variations of constants, we can obtain the unique solution of (2.3) which can be represented by (2.4) where \(A\) and \(H\) are as in (2.5) and (2.6), respectively. Next we check the function defined in (2.4) is the solution of the boundary value problem (2.3). For this purpose we first show that (2.4) satisfies (2.3). From the definition of Green’s function (2.6), we get

\[
x(t) = \frac{1}{\phi_1^\Delta(\rho(0))} \left( \int_{\rho(0)}^{t} \phi_1(s)\phi_2(t)p(s)y(s)\nabla s + \int_{t}^{\sigma(1)} \phi_1(t)\phi_2(s)p(s)y(s)\nabla s \right) + A\phi_1(t).
\]
Hence, the derivatives $x^\Delta$ and $x^{\Delta\nabla}$ are as follows:

$$
x^\Delta(t) = \frac{1}{\phi_1^\Delta(\rho(0))} \left( \phi_2^\Delta(t) \int_{\rho(0)}^t \phi_1(s)p(s)y(s)\nabla s + \phi_1^\Delta(t) \int_t^{\sigma(1)} \phi_2(s)p(s)y(s)\nabla s \right) + A\phi_1^\Delta(t),
$$

$$
x^{\Delta\nabla}(t) = \frac{1}{\phi_1^\Delta(\rho(0))} \left( \phi_2^{\Delta\nabla}(t) \int_{\rho(0)}^{\rho(t)} \phi_1(s)p(s)y(s)\nabla s + \phi_1^\Delta(t)\phi_1(t)p(t)y(t) 
\quad + \phi_1^\Delta(t) \int_{\rho(t)}^{\sigma(1)} \phi_2(s)p(s)y(s)\nabla s + \phi_1^\Delta(t)\phi_2(t)p(t)y(t) \right) + A\phi_1^{\Delta\nabla}(t). \tag{2.9}
$$

Replacing the derivatives in (2.3), we deduce that

$$
x^{\Delta\nabla}(t) + a(t)x^\Delta(t) + b(t)x(t)
= A \left( \phi_1^{\Delta\nabla}(t) + a(t)\phi_1^\Delta(t) + b(t)\phi_1(t) \right)
+ \left( \frac{1}{\phi_1^\Delta(\rho(0))} \int_{\rho(0)}^{\rho(t)} \phi_1(s)p(s)y(s)\nabla s \right) \left( \phi_1^{\Delta\nabla}(t) + a(t)\phi_1^\Delta(t) + b(t)\phi_1(t) \right)
+ \left( \frac{1}{\phi_1^\Delta(\rho(0))} \int_t^{\sigma(1)} \phi_2(s)p(s)y(s)\nabla s + \phi_1^{\Delta\nabla}(t) \int_{\rho(t)}^{\sigma(1)} \phi_2(s)p(s)y(s)\nabla s \right)
+ \frac{1}{\phi_1^\Delta(\rho(0))} \left( \phi_2^\Delta(t)\phi_1(t) - \phi_1^\Delta(t)\phi_2(t) \right)p(t)y(t)
= \frac{1}{\phi_1^\Delta(\rho(0))} \left( \phi_2^{\Delta\nabla}(t)(\rho(t) - t)\phi_1(t)p(t)y(t) - \phi_1^{\Delta\nabla}(t)\phi_2(t)p(t)y(t) \right)
+ \phi_2^\Delta(t)\phi_1(t)p(t)y(t) - \phi_1^\Delta(t)\phi_2(t)p(t)y(t)
= \frac{1}{\phi_1^\Delta(\rho(0))} p(t)y(t) \left( \phi_2^\Delta(t)\phi_1(t) - \phi_1^\Delta(t)\phi_2(t) \right)
+ \frac{1}{\phi_1^\Delta(\rho(0))} p(t)y(t)(\rho(t) - t) \left( \phi_2^{\Delta\nabla}(t)\phi_1(t) - \phi_1^{\Delta\nabla}(t)\phi_2(t) \right)
= \frac{1}{\phi_1^\Delta(\rho(0))} \left( \phi_2^\Delta(t)\phi_1(t) - \phi_1^\Delta(t)\phi_2(t) \right) + (\rho(t) - t) \left( \phi_2^\Delta(t)\phi_1(t) - \phi_1^\Delta(t)\phi_2(t) \right)^\nabla
= \frac{1}{\phi_1^\Delta(\rho(0))} p(t)y(t) \left( \phi_2^\Delta(\rho(t))\phi_1(\rho(t)) - \phi_1^\Delta(\rho(t))\phi_2(\rho(t)) \right).
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\[ \begin{align*}
&= \frac{1}{\phi_1^2(\rho(0))}p(t)y(t)e_{\Theta_0}(\rho(t), \rho(0))\left(-\phi_1^2(\rho(0))\right) \\
&= -y(t).
\end{align*} \]

(2.10)

Therefore the function defined in (2.4) satisfies (2.3). Further we obtain that the boundary value conditions are satisfied by (2.4). The first condition follows from (2.5) and (2.6) and Lemma 2.1. Now we verify the second boundary condition. Since

\[ H(\sigma(1), s) = \frac{1}{\phi_1^2(\rho(0))} \phi_1(s)\phi_2(\sigma(1)) = 0, \]

we obtain that

\[ x(\sigma(1)) = \int_{\rho(0)}^{\sigma(1)} H(\sigma(1), s)p(s)y(s)\nabla s + A\phi_1(\sigma(1)) = A. \]  

(2.12)

On the other hand, by using (2.5), we find that

\[ \begin{align*}
\sum_{i=1}^{m-2} \alpha_i x(\eta_i) &= \sum_{i=1}^{m-2} \alpha_i \left( \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s + A\phi_1(\eta_i) \right) \\
&= \sum_{i=1}^{m-2} \alpha_i \left( \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s \right. \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s \left. \right) \\
&= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s = A.
\end{align*} \]

(2.13)

Combining the two equations above finishes the proof.

\[ \square \]

**Lemma 2.3.** Green’s function \( H(t, s) \) has the following properties:

\[ H(t, s) \leq H(t, t), \quad \frac{\phi_1^2(\rho(0))}{\|\phi_1\|\|\phi_2\|} H(t, t) H(s, s) \leq H(t, s) \leq H(s, s), \quad H(t, t) \leq \phi_1(t) \frac{\|\phi_2\|}{\phi_1^2(\rho(0))}. \]

(2.14)

**Lemma 2.4.** Assume that (C2) and (C3) hold. Let \( u \) be a solution of boundary value problem (1.1) if and only if \( u \) is a solution of the following integral equation:

\[ u(t) = \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s)f(s, u(s))\nabla s, \]

(2.15)
where

\[ G(t, s) = H(t, s) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, s) \phi_1(t). \]  

(2.16)

The proofs of the Lemmas 2.3 and 2.4 can be obtained easily by Lemmas 2.1 and 2.2.

**Lemma 2.5.** Green’s function \( G(t, s) \) defined by (2.16) has the following properties:

\[ C_2 \phi_1(t) H(s, s) \leq G(t, s) \leq C_1 H(s, s), \quad G(t, s) \leq C_3 \phi_1(t), \]

(2.17)

where

\[ C_1 = 1 + \frac{\| \phi_1 \|}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i, \]

\[ C_2 = \frac{\phi_1^\Delta (\rho(0))}{\| \phi_1 \| \| \phi_2 \|} \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, \eta_i), \]

(2.18)

\[ C_3 = \frac{\| \phi_2 \|}{\phi_1^\Delta (\rho(0))} + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, \eta_i). \]

**Proof.** From Lemma 2.3, we have

\[ G(t, s) \leq H(s, s) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(s, s) \| \phi_1 \| \leq C_1 H(s, s), \]

\[ G(t, s) \leq \frac{\| \phi_2 \|}{\phi_1^\Delta (\rho(0))} \phi_1^\Delta(t) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, \eta_i) \phi_1(t) \leq C_3 \phi_1(t), \]

\[ G(t, s) \geq \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, s) \phi_1(t) \]

\[ \geq \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \phi_1^\Delta (\rho(0)) \frac{\| \phi_1 \| \| \phi_2 \|}{\phi_1^\Delta (\rho(0))} H(\eta_i, \eta_i) H(s, s) \phi_1(t) \]

\[ \geq C_2 \phi_1(t) H(s, s). \]

The proof is complete.

The following theorems will play major role in our next analysis.
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3. Main Results

We make the following assumptions:

(\text{H}_1) \ f(t, u) \in C([\rho(0), \sigma(1)] \times [0, +\infty), \mathbb{R}), \text{ moreover there exists a function} \ g(t) \in L^1([\rho(0), \sigma(1)], (0, +\infty)) \text{ such that} \ f(t, u) \geq -g(t), \text{ for any} \ t \in (\rho(0), \sigma(1)), u \in [0, +\infty).

(\text{H}_1^*) f(t, u) \in C((\rho(0), \sigma(1)) \times [0, +\infty), \mathbb{R}), \text{ may be singular at} \ t = \rho(0), \sigma(1), \text{ moreover there exists a function} \ g(t) \in L^1((\rho(0), \sigma(1)), (0, +\infty)) \text{ such that} \ f(t, u) \geq -g(t), \text{ for any} \ t \in (\rho(0), \sigma(1)), u \in [0, +\infty).

(\text{H}_2) f(t, 0) > 0, \text{ for} \ t \in [\rho(0), \sigma(1)].

(\text{H}_3) \text{ There exists} \ [\theta_1, \theta_2] \in (\rho(0), \sigma(1)) \text{ such that} \lim_{u \to +\infty} \min_{t \in [\theta_1, \theta_2]} (f(t, u)/u) = +\infty.

(\text{H}_4) \int_{\rho(0)}^{\sigma(1)} H(s, s)p(s)q(s)f(s, z)\nabla s < +\infty \text{ for any} \ z \in [0, m], \ m > 0 \text{ is any constant.}

In fact, we only consider the boundary value problem

\[ x^\Delta(t) + a(t)x^\Delta(t) + b(t)x(t) + \lambda q(t)(f(t, [x(t) - v(t)])^\Delta + g(t)) = 0, \quad \lambda > 0, \]

\[ x(\rho(0)) = 0, \quad x(\sigma(1)) = \sum_{i=1}^{m-2} a_i x(\eta_i), \]  

(3.1)

where

\[ y(t)^* = \begin{cases} 
  y(t), & y(t) \geq 0, \\
  0, & y(t) < 0, 
\end{cases} \]

(3.2)

and \( v(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s)g(s)\nabla s, \) which is the solution of the boundary value problem

\[ v^\Delta(t) + a(t)v^\Delta(t) + b(t)v(t) + \lambda q(t)g(t) = 0, \]

\[ v(\rho(0)) = 0, \quad v(\sigma(1)) = \sum_{i=1}^{m-2} a_i v(\eta_i). \]  

(3.3)

From Lemma 2.1, it is easy to verify that \( v(t) \leq \lambda C_0 \psi(t) \) and \( C_0 = C_3 \int_{\rho(0)}^{\sigma(1)} q(s)g(s)\nabla s. \)
We will show that there exists a solution \( x \) for boundary value problem (3.1) with \( x(t) \geq v(t), t \in [\rho(0), \sigma(1)] \). If this is true, then \( u(t) = x(t) - v(t) \) is a nonnegative solution (positive on \((\rho(0), \sigma(1))\)) of boundary value problem (3.1). Since for any \( t \in (\rho(0), \sigma(1)) \), from

\[
(u(t) + v(t))\Delta^\gamma + a(t)(u(t) + v(t))\Delta + b(t)(u(t) + v(t)) = -\lambda q(t) [f(t, u) + g(t)],
\]

we have

\[
u\Delta^\gamma(t) + a(t)u\Delta(t) + b(t)u(t) = -\lambda q(t)f(t, u).
\]

As a result, we will concentrate our study on boundary value problem (3.1).

We note that \( x(t) \) is a solution of (3.1) if and only if

\[
x(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) (f(t, [x(t) - v(t)]^*) + g(t))\nabla s. \tag{3.6}
\]

For our constructions, we will consider the Banach space \( X = C[\rho(0), \sigma(1)] \) equipped with standard norm \( \|x\| = \max_{0 \leq s \leq 1} |x(t)|, \ x \in X \). We define a cone \( P \) by

\[
P = \left\{ x \in X \mid x(t) \geq \frac{C_2}{C_1} \phi_1(t)\|x\|, \ t \in [\rho(0), \sigma(1)] \right\}, \tag{3.7}
\]

where \( \phi_1 \) is defined by Lemma 2.1 (namely, \( \phi_1 \) is solution (2.1)). Define an integral operator \( T : P \to X \) by

\[
Tx(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) (f(t, [x(t) - v(t)]^*) + g(t))\nabla s. \tag{3.8}
\]

Notice, from (3.8) and Lemma 2.5, we have \( Tx(t) \geq 0 \) on \([0, 1]\) for \( x \in P \) and

\[
Tx(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) (f(t, [x(t) - v(t)]^*) + g(t))\nabla s \leq C_1 \lambda \int_{\rho(0)}^{\sigma(1)} H(s, s)p(s)q(s) (f(t, [x(t) - v(t)]^*) + g(t))\nabla s, \tag{3.9}
\]

then \( \|Tx\| \leq C_1 \lambda \int_{\rho(0)}^{\sigma(1)} H(s, s)p(s)q(s) (f(t, [x(t) - v(t)]^*) + g(t))\nabla s. \)
On the other hand, we have
\[ Tx(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) \left( f(t, \sigma(t) - \nu(t))^* + g(t) \right) \nabla s \]
\[ \geq \lambda \int_{\rho(0)}^{\sigma(1)} C_2 \phi_1(t) H(s, s)p(s)q(s) \left( f(t, \sigma(t) - \nu(t))^* + g(t) \right) \nabla s \]
\[ \geq \frac{C_2}{C_1} \phi_1(t) \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s)q(s) \left( f(t, \sigma(t) - \nu(t))^* + g(t) \right) \nabla s \]
\[ \geq \frac{C_2}{C_1} \phi_1(t) \|Tx\|. \tag{3.10} \]

Thus, \( T(P) \subset P \). In addition, standard arguments show that \( T(P) \subset P \) and \( T \) is a compact, and completely continuous.

**Theorem 3.1.** Suppose that \( (H_1)-(H_2) \) hold. Then there exists a constant \( \lambda > 0 \) such that, for any \( 0 < \lambda \leq \lambda \), boundary value problem \( (1.1) \) has at least one positive solution.

**Proof.** Fix \( \delta \in (0, 1) \). From \( (H_2) \), let \( 0 < \varepsilon < 1 \) be such that
\[ f(t, z) \geq \delta f(t, 0), \quad \text{for } \rho(0) \leq t \leq \sigma(1), \quad 0 \leq z \leq \varepsilon. \tag{3.11} \]

Suppose that
\[ 0 < \lambda < \frac{\varepsilon}{2c\overline{f}(\varepsilon)} := \bar{\lambda} \tag{3.12} \]

where \( \overline{f}(\varepsilon) = \max_{\rho(0) \leq t \leq 1, 0 \leq z \leq \varepsilon} \{ f(t, z) + g(t) \} \) and \( c = C_1 \int_{\rho(0)}^{\sigma(1)} H(s, s)p(s)q(s) \nabla s \). Since
\[ \lim_{z \to 0} \frac{\overline{f}(z)}{z} = +\infty, \tag{3.13} \]
\[ \frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{2c\lambda}, \]

there exists a \( R_0 \in (0, \varepsilon) \) such that
\[ \frac{\overline{f}(R_0)}{R_0} = \frac{1}{2c\lambda}. \tag{3.14} \]
Let \( x \in P \) and \( \nu \in (0, 1) \) be such that \( x = \nu T(x) \), we claim that \( \|x\| \neq R_0 \). In fact

\[
\|T x(t)\| \leq \nu \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s)q(s) [f(s, [x(s) - \nu(s)])^* + g(s)] \nabla s
\]

\[
\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s)q(s) [f(s, [x(s) - \nu(s)])^* + g(s)] \nabla s
\]

\[
\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s)q(s) \max_{0 \leq s \leq 1, 0 \leq z \leq R_0} [f(s, z) + g(s)] \nabla s
\]

\[
\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s)q(s) \bar{f}(R_0) \nabla s
\]

\[
\leq \lambda c \bar{f}(R_0),
\]

that is,

\[
\frac{\bar{f}(R_0)}{R_0} \geq \frac{1}{c \lambda} > \frac{1}{2c \lambda} = \frac{\bar{f}(R_0)}{R_0}
\]

which implies that \( \|x\| \neq R_0 \). Let \( U = \{ x \in P : \|x\| < R_0 \} \). By nonlinear alternative of Leray-Schauder type theorem, \( T \) has a fixed point \( x \in \overline{U} \). Moreover, combing (3.8), (3.28), and \( R_0 < \varepsilon \), we obtain that

\[
x(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) [f(s, [x(s) - \nu(s)])^* + g(s)] \nabla s
\]

\[
\geq \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) [f(s, 0) + g(s)] \nabla s
\]

\[
\geq \lambda \left[ \delta \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s)f(s, 0) \nabla s + \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s)g(s) \nabla s \right]
\]

\[
\geq \lambda \int_{\rho(0)}^{\sigma(1)} G(t, s)p(s)q(s) g(s) \nabla s
\]

\[
= \nu(t) \quad \text{for} \ t \in (\rho(0), \sigma(1)).
\]

Let \( u(t) = x(t) - \nu(t) > 0 \). Then (1.1) has a positive solution \( u \) and \( \|u\| \leq \|x\| \leq R_0 < 1 \). This completes the proof of Theorem 3.1.
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**Theorem 3.2.** Suppose that $(H^*_1)$ and $(H_3)-(H_4)$ hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$, boundary value problem (1.1) has at least one positive solution.

**Proof.** Let $\Omega_1 = \{x \in P : \|x\| < R_1\}$, where $R_1 = \max\{1, r\}$ and $r = (C_1C_3/C_2)\int_{\rho(0)}^{\alpha(1)} q(s)g(s)\nabla s$. Choose

$$
\lambda^* = \min \left\{ 1, R_1 \left[ C_1 \int_{\rho(0)}^{\alpha(1)} H(s,s)p(s)q(s) \left[ \max_{0 \leq z \leq R_1} f(s,z)+g(s) \right] \nabla s \right]^{-1} \right\}. \quad (3.18)
$$

Then for any $x \in P \cap \partial\Omega_1$, then $\|x\| = R_1$ and $x(s) - v(s) \leq x(s) \leq \|x\|$, we have

$$
\|Tx(t)\| \leq \lambda \int_{\rho(0)}^{\alpha(1)} C_1 H(s,s)p(s)q(s) \left[ f(s, [x(s) - v(s)]^+) + g(s) \right] \nabla s
\leq \lambda \int_{\rho(0)}^{\alpha(1)} C_1 H(s,s)p(s)q(s) \left[ f(s, [x(s) - v(s)]^+) + g(s) \right] \nabla s
\leq \lambda C_1 \int_{\rho(0)}^{\alpha(1)} H(s,s)p(s)q(s) \left[ \max_{0 \leq z \leq R_1} f(s,z)+g(s) \right] \nabla s
\leq R_1 = \|x\|. \quad (3.19)
$$

This implies that

$$
\|Tx\| \leq \|x\|, \quad x \in P \cap \partial\Omega_1. \quad (3.20)
$$

On the other hand, choose a constant $N > 0$ such that

$$
\frac{\lambda C_1^2 N}{2C_1} \int_{\theta_1}^{\beta_2} H(s,s)\phi_1(s)p(s)q(s)\nabla s \min_{\theta_1 \leq t \leq \beta_2} \phi_1(t) \geq 1. \quad (3.21)
$$

By assumption $(H_3)$, for any $t \in [\theta_1, \theta_2]$, there exists a constant $B > 0$ such that

$$
\frac{f(t,z)}{z} > N, \quad \text{namely, } f(t,z) > Nz, \quad \text{for } z > B. \quad (3.22)
$$
Choose $R_2 = \max \{ R_1 + 1, 2\lambda r, 2C_1(B + 1)/C_2\min_{\theta_1 \leq \theta_2} \phi_1(t) \}$, and let $\Omega_2 = \{ x \in P : \| x \| < R_2 \}$, then for any $x \in P \cap \partial \Omega_2$, we have

$$x(t) - v(t) = x(t) - \lambda \int_{p(0)}^{\alpha_1} G(t,s)p(s)q(s)g(s) \nabla s$$

$$\geq x(t) - \lambda C_3 \phi_1(t) \int_{p(0)}^{\alpha_1} p(s)q(s)g(s) \nabla s$$

$$\geq x(t) - \frac{C_1 x(t)}{C_2 \| x \|} \lambda C_3 \int_{p(0)}^{\alpha_1} p(s)q(s)g(s) \nabla s$$

$$\geq x(t) - \frac{x(t)}{R_2 - \lambda r}$$

$$\geq \left(1 - \frac{\lambda r}{R_2}\right) x(t)$$

$$\geq \frac{1}{2} x(t) \geq 0, \quad t \in [0, 1].$$

Then,

$$\min_{\theta_1 \leq \theta_2} \{ x(t) - v(t) \} \geq \min_{\theta_1 \leq \theta_2} \left\{ \frac{1}{2} x(t) \right\} \geq \min_{\theta_1 \leq \theta_2} \left\{ \frac{C_2}{2C_1} \phi_1(t) \| x \| \right\}$$

$$= \frac{R_2 C_2 \min_{\theta_1 \leq \theta_2} \phi_1(t)}{2C_1} \geq B + 1 > B.$$ (3.24)

Now,

$$\| Tx(t) \| \geq \max_{\theta_1 \leq \theta_2} \left\{ C_2 \phi_1(t) H(s,s)p(s)q(s) \left[ f(s, [x(s) - v(s)]^*) + g(s) \right] \nabla s \right\}$$

$$\geq \max_{\theta_1 \leq \theta_2} \lambda C_2 \phi_1(t) \int_{p(0)}^{\alpha_1} H(s,s)p(s)q(s)f(s, [x(s) - v(s)]^*) \nabla s$$

$$\geq \lambda C_2 \min_{\theta_1 \leq \theta_2} \phi_1(t) \int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s)f(s, x(s) - v(s)) \nabla s$$

$$\geq \lambda C_2 \min_{\theta_1 \leq \theta_2} \phi_1(t) \int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s)N(x(s) - v(s)) \nabla s$$

$$\geq \lambda C_2 \min_{\theta_1 \leq \theta_2} \phi_1(t) \int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s) \frac{N}{2} x(s) \nabla s$$
Let \( u \) satisfy conditions (H1)–(H3). Then boundary value problem (1.1) has at least two positive solutions for \( \lambda > 0 \) sufficiently small.

**Theorem 3.3.** Suppose that (H1)–(H3) hold. Then boundary value problem (1.1) has at least two positive solutions for \( \lambda > 0 \) sufficiently small.
Proof. On the hand, fix \( \delta \in (0, 1) \). From (H₂), let \( 0 < \varepsilon < \min\{1, r/2\} \) be such that
\[
f(t, z) \geq \delta f(t, 0), \quad \text{for } \rho(0) \leq t \leq \sigma(1), \ 0 \leq z \leq \varepsilon.
\] (3.28)

Choose
\[
\frac{\varepsilon}{2c \tilde{f}(\varepsilon)} := \bar{\lambda},
\] (3.29)

where \( \tilde{f}(\varepsilon) = \max_{p(0) \leq t \leq \sigma(1), 0 \leq z \leq \varepsilon} \{ f(t, z) + g(t) \} \), \( c = C_1 \int_{p(0)}^{\sigma(1)} H(s, s) p(s) q(s) \nabla s \), and \( r = (C_1 C_3 / C_2) \int_{p(0)}^{\sigma(1)} q(s) g(s) \nabla s \).

On the other hand, set \( \Omega_1 = \{ x \in P : ||x|| < R_1 \} \), where \( R_1 = 1 + r \). Choose
\[
\lambda^* = \min \left\{ 1, R_1 \left[ C_1 \int_{p(0)}^{\sigma(1)} H(s, s) p(s) q(s) \left[ \max_{0 \leq z \leq R_1} f(s, z) + g(s) \right] \nabla s \right]^{-1} \right\}.
\] (3.30)

So, let
\[
0 < \lambda < \min \{ \bar{\lambda}, \lambda^* \}.
\] (3.31)

From \( 0 < \lambda < \min \{ \bar{\lambda}, \lambda^* \} \), we have \( 0 < \lambda < \bar{\lambda} \), from proof of Theorem 3.1, we know that (1.1) has a positive solution \( u_1 \) and \( ||u_1|| \leq ||u_0|| \leq R_0 < r/2 \). Further, also from \( 0 < \lambda < \min \{ \bar{\lambda}, \lambda^* \} \), we have \( 0 < \lambda < \lambda^* \), from proof of Theorem 3.2, we know that (1.1) has a positive solution \( u_2 \) and \( ||u_2|| \geq R_1/2 > r/2 \). Then (1.1) has at least two positive solutions \( u_1 \) and \( u_2 \). This completes the proof of Theorem 3.3. \( \square \)

4. Examples

To illustrate the usefulness of the results, we give some examples.

Example 4.1. Consider the boundary value problem
\[
u_0^{\Delta^V}(t) + a(t) u_0^{\Delta}(t) + b(t) u(t) + \lambda \left( u^{\Delta}(t) + \frac{1}{(t-t^2)^{1/2}} \cos(2\pi u(t)) \right) = 0, \quad \lambda > 0,
\] (4.1)
\[
u(0) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} a_i u(\eta_i),
\]

where \( a > 1 \). Then, if \( \lambda > 0 \) is sufficiently small, (4.1) has a positive solution \( u \) with \( u(t) > 0 \) for \( t \in (0, 1) \).
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To see this, we will apply Theorem 3.2 with

\[ q(t) = 1, \quad f(t, u) = u^a(t) + \frac{1}{(t - t^2)^{1/2}} \cos(2\pi u(t)), \quad g(t) = \frac{1}{(t - t^2)^{1/2}}. \]  

(4.2)

Clearly

\[ f(t, 0) = \frac{1}{(t - t^2)^{1/2}} > 0, \quad f(t, u) + g(t) \geq u^a(t) > 0, \quad \lim_{u \to \infty} \frac{f(t, u)}{u} = +\infty, \]  

(4.3)

for \( t \in (\rho(0), \sigma(1)), u > 0 \). Namely, \((H_1^*)\) and \((H_2)-(H_4)\) hold. From \( \int_{\rho(0)}^{\sigma(1)} (1/((\sigma(1) - \rho(0))s - (s - \rho(0))^2)^{1/2}) \nabla s = \pi \), set \( R_1 = C_1, \pi \) and \( m = \max_{\rho(0) \leq s \leq \sigma(1)} \{ p(t) \} + 1 \), we have

\[
\int_{\rho(0)}^{\sigma(1)} C_1 H(s, s)p(s) \left[ \max_{0 \leq s \leq R_1} \nabla f(s, z) + g(s) \right] \nabla s
\]

\[
\leq \int_{\rho(0)}^{\sigma(1)} mC_1 \| \phi_1 \| \| \phi_2 \| \left[ (C_1, \pi)^a + \frac{1}{((\sigma(1) - \rho(0))s - (s - \rho(0))^2)^{1/2}} \right] \nabla s
\]

\[
\leq \frac{mC_1 \| \phi_1 \| \| \phi_2 \| }{\phi_1^a (\rho(0))} ((C_1, \pi)^a + \pi)
\]

and \( \lambda^* = \min \{ 1, \phi_1^a (\rho(0)) / m \| \phi_1 \| \| \phi_2 \| ((C_1, \pi)^a - 1) \} \). Now, if \( \lambda < \lambda^* \), Theorem 3.2 guarantees that (4.1) has a positive solution \( u \) with \( \| u \| \geq 2 \).

Example 4.2. Consider the boundary value problem

\[
u^N(t) + a(t)u^a(t) + b(t)u(t) + \lambda \left( u^2(t) - \frac{9}{2} u(t) + 2 \right) = 0, \quad 0 < t < 1, \quad \lambda > 0,
\]

\[ u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{n} \alpha_i u(\eta_i). \]

Then, if \( \lambda > 0 \) is sufficiently small, (4.5) has two solutions \( u_i \) with \( u_i(t) > 0 \) for \( t \in (0, 1), i = 1, 2 \).

To see this, we will apply Theorem 3.3 with

\[ f(t, u) = u^2(t) - \frac{9}{2} u(t) + 2, \quad g(t) = 4. \]  

(4.6)

Clearly

\[ q(t) = 0, \quad f(t, 0) = 2 > 0, \quad f(t, u) + g(t) \geq \frac{15}{16} > 0, \quad \lim_{u \to \infty} \frac{f(t, u)}{u} = +\infty. \]  

(4.7)
Namely, (H_1)–(H_3) hold. Let δ = 1/4, ε = 1/4 and c = \int^{(1)}_{\rho(0)} C_1 H(s, s)p(s) \, ds, then we may have

\[
\bar{\lambda} = \frac{1}{8c(\max_{0 \leq z \leq \delta} f(t, z) + 4)} = \frac{1}{48c}. \tag{4.8}
\]

Now, if \( \lambda < \bar{\lambda} \), Theorem 3.2 guarantees that (4.5) has a positive solution \( u_1 \) with \( \|u_1\| \leq 1/4 \).

Next, let \( R_1 = p \), where \( p = 4cC_3/C_2 + 1 \) is a constant, then we have

\[
\int^{(1)}_{\rho(0)} C_1 H(s, s)p(s)\left[ \max_{0 \leq z \leq R_1} f(s, z) + g(s) \right] \, ds = \int^{(1)}_{\rho(0)} C_1 H(s, s)p(s)\left[ \frac{9}{2} + 4 \right] \, ds = \frac{17c}{2} \tag{4.9}
\]

and \( \lambda^* = \min\{1, 2p/17c\} \). Now, if \( 0 < \lambda < \lambda^* \), Theorem 3.1 guarantees that (4.5) has a positive solution \( u_2 \) with \( \|u_2\| \geq p \).

So, since all the conditions of Theorem 3.3 are satisfied, if \( \lambda < \min\{\bar{\lambda}, \lambda^*\} \), Theorem 3.3 guarantees that (4.5) has two solutions \( u_i \) with \( u_i(t) > 0 \) \( (i = 1, 2) \).

**Example 4.3.** Consider the boundary value problem

\[
u^\Delta \psi(t) + a(t)u^\Delta(t) + b(t)u(t) + \lambda(u^\Delta(t) + \cos(2\pi u(t))) = 0, \quad 0 < t < 1, \quad \lambda > 0,
\]

\[u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i \eta_i, \tag{4.10}\]

where \( a > 1 \). Then, if \( \lambda > 0 \) is sufficiently small, (4.10) has two solutions \( u_i \) with \( u_i(t) > 0 \) for \( t \in (0, 1), \ i = 1, 2 \).

To see this we will apply Theorem 3.3 with

\[
f(t, u) = u^\Delta(t) + \cos(2\pi u(t)), \quad g(t) = 2. \tag{4.11}\]

Clearly

\[
f(t, 0) = 1 > 0, \quad f(t, u) + g(t) \geq u^\Delta(t) + 1 > 0, \quad \lim_{u \to +\infty} \frac{f(t, u)}{u} = +\infty, \quad \text{for } t \in (\rho(0), \sigma(1)). \tag{4.12}\]

Namely, (H_1)–(H_3) hold. Let δ = 1/2, ε = 1/8 and c = \int^{(1)}_{\rho(0)} C_1 H(s, s)p(s) \, ds, then we may have

\[
\frac{\varepsilon}{2c(\max_{0 \leq x \leq \delta} f(t, x) + 2)} \geq \frac{1}{16c((1/8)^a + 3)} := \bar{\lambda}. \tag{4.13}\]

Now, if \( 0 < \lambda < \bar{\lambda} \) then \( 0 < \lambda < \varepsilon/2c(\max_{0 \leq x \leq \delta} f(t, x) + 2) \), Theorem 3.2 guarantees that (4.10) has a positive solution \( u_1 \) with \( \|u_1\| \leq 1/8 \).
Next, let \( R_1 = p \), where \( p = 4cC_3/C_2 + 1 \) is a constant, then we have

\[
\int_{p(0)}^{t(1)} C_1 H(s, s)p(s) \left[ \max_{0 \leq z \leq R_1} \delta(s, z) + g(s) \right] \nabla s \leq \int_{p(0)}^{t(1)} C_1 H(s, s)p(s) [p^a + 2] \nabla s = (p^a + 2)c
\]

(4.14)

and \( \lambda^* = \min\{1, p/(p^a + 2)c\} \). Now, if \( 0 < \lambda < \lambda^* \), then \( 0 < \lambda < p \times (\int_{p(0)}^{t(1)} C_1 H(s, s)p(s)[\max_{0 \leq z \leq R_1} \delta(s, z) + g(s)] \nabla s)^{-1} \), Theorem 3.1 guarantees that (4.10) has a positive solution \( u_2 \) with \( \|u_2\| \geq 1 \).

So, if \( \lambda < \min\{1, \lambda^*\} \), Theorem 3.3 guarantees that (4.10) has two solutions \( u_i \) with \( u_i(t) > 0 \) \((i = 1, 2)\).

Example 4.4. Let \( T = \{0, 1/4, 2/4, 3/4, 1, 5/4, \ldots\} \). We consider the following four point boundary value problem:

\[
\begin{align*}
\Delta_4 u(t) + \frac{12}{5} u^4(t) - \frac{16}{5} u(t) + \lambda (u^4(t) + \cos(2\pi u(t))) &= 0, \quad \lambda > 0, \\
\text{u(0) = 0,} & \quad \text{u} \left( \frac{5}{4} \right) = \frac{1}{2} u \left( \frac{1}{4} \right) + \frac{1}{4} u \left( \frac{1}{2} \right).
\end{align*}
\]

(4.15)

where \( a(t) = 12/5 \), \( b(t) = -16/5 \), and \( q(t) = 1 \). Then, if \( \lambda > 0 \) is sufficiently small, (4.15) has two solutions \( u_i \) with \( u_i(t) > 0 \) \((i = 1, 2)\).

Let \( \phi_1 \) and \( \phi_2 \) be the solutions of the following linear boundary value problems, respectively,

\[
\begin{align*}
\Delta_4 u(t) + \frac{12}{5} u^4(t) - \frac{16}{5} u(t) &= 0, \quad u(0) = 0, \quad u \left( \frac{5}{4} \right) = 1, \\
\Delta_4 u(t) + \frac{12}{5} u^4(t) - \frac{16}{5} u(t) &= 0, \quad u(0) = 1, \quad u \left( \frac{5}{4} \right) = 0.
\end{align*}
\]

(4.16)

It is evident (form the Corollaries 4.24 and 4.25 and Theorem 4.28 of [17]) that

\[
\phi_1(t) = \frac{(5/4)^4 - (1/2)^4t}{(5/4)^3 - (1/2)^3}, \quad \phi_2(t) = \frac{(5/4)^4(1/2)^4 - (1/2)^4(5/4)^4t}{(5/4)^3 - (1/2)^3}.
\]

(4.17)

Also \( \phi_1 \) satisfies (C3). Green’s function is

\[
H(t, s) = \frac{1024}{9279} \begin{cases} 
\left( \frac{5}{4} \right)^4 \left( \frac{1}{2} \right)^4 \left( \frac{5}{4} \right)^5 \left( \frac{1}{2} \right)^5 \left( \frac{1}{2} \right)^4 - \left( \frac{1}{2} \right)^5 \left( \frac{5}{4} \right)^4 t, & s \leq t, \\
\left( \frac{5}{4} \right)^4 \left( \frac{1}{2} \right)^4 \left( \frac{5}{4} \right)^5 \left( \frac{1}{2} \right)^5 \left( \frac{1}{2} \right)^4 - \left( \frac{1}{2} \right)^5 \left( \frac{5}{4} \right)^4 s, & t \leq s,
\end{cases}
\]

(4.18)

and \( p(t) = (2/5)^{t-1} \) follows from \( c_{sa}(t, t_0) = (1 + ah)^{(t-t_0)/h} \) on \( T = h\mathbb{N} \).
To see this, we will apply Theorem 3.3 with
\[ f(t,u) = u^2(t) + \cos(2\pi u(t)), \quad g(t) = 2. \]  
(4.19)

Clearly
\[ f(t,0) = 1 > 0, \quad f(t,u) + g(t) \geq u^2(t) + 1 > 0, \quad \lim_{u \to +\infty} \frac{f(t,u)}{u} = +\infty, \quad \text{for } t \in (\rho(0), \sigma(1)). \]  
(4.20)

Namely, (H₁)–(H₃) hold. Let \( \delta = 1/2, \ \varepsilon = 1/8 \) and \( c = \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s) \nabla s, \) then we may have
\[ \varepsilon \geq \frac{1}{16c((1/8)^a + 3)} := \bar{\lambda}. \]  
(4.21)

Now, if \( 0 < \lambda < \bar{\lambda} \) then \( 0 < \lambda < \varepsilon/2c(\max_{0 \leq s \leq 2} f(t,x) + 2), \) Theorem 3.2 guarantees that (4.15) has a positive solution \( u_1 \) with \( \|u_1\| \leq 1/8. \)

Next, let \( R_1 = 4cC_3/C_2 + 1 \) is a constant, then we have
\[ \int_{\rho(0)}^{\sigma(1)} \max_{0 \leq z \leq R_1} f(s,z) + g(s) \nabla s \leq \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s) [R_1^a + 2] \nabla s = (p^a + 2)c \]  
(4.22)

and \( \lambda^* = \min\{1, R_1/(R_1^a + 2)c\}. \) Now, if \( 0 < \lambda < \lambda^* \), then \( 0 < \lambda < R_1 \times (\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)[\max_{0 \leq z \leq R_1} f(s,z) + g(s)] \nabla s)^{-1} \), Theorem 3.1 guarantees that (4.15) has a positive solution \( u_2 \) with \( \|u_2\| \geq 1. \)

So, if \( \lambda < \min\{\bar{\lambda}, \lambda^*\} \), Theorem 3.3 guarantees that (4.15) has two solutions \( u_i \) with \( u_i(t) > 0 \ (i = 1,2). \)

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**References**


