Research Article

Global Behavior of the Difference Equation

\[ x_{n+1} = \frac{(p + x_{n-1})}{(q x_n + x_{n-1})} \]

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We study the following difference equation \( x_{n+1} = \frac{(p + x_{n-1})}{(q x_n + x_{n-1})} \), \( n = 0,1,\ldots, \) where \( p, q \in (0, +\infty) \) and the initial conditions \( x_{-1}, x_0 \in (0, +\infty) \). We show that every positive solution of the above equation either converges to a finite limit or to a two cycle, which confirms that the Conjecture 6.10.4 proposed by Kulenović and Ladas (2002) is true.

1. Introduction

Kulenović and Ladas in [1] studied the following difference equation:

\[ x_{n+1} = \frac{p + x_{n-1}}{q x_n + x_{n-1}}, \quad n = 0,1,\ldots, \quad (1.1) \]

where \( p, q \in (0, +\infty) \) and the initial conditions \( x_{-1}, x_0 \in (0, +\infty) \), and they obtained the following theorems.

Theorem A (see [1, Theorem 6.6.2]). Equation (1.1) has a prime period-two solution

\[ \ldots, \phi, \varphi, \phi, \varphi, \ldots \quad (1.2) \]

if and only if \( q > 1 + 4p \). Furthermore, when \( q > 1 + 4p \), the prime period-two solution is unique and the values of \( \phi \) and \( \varphi \) are the positive roots of the quadratic equation

\[ t^2 - t + \frac{p}{q - 1} = 0. \quad (1.3) \]
Theorem B (see [1, Theorem 6.6.4]). Let \( \{x_n\}_{n=1}^{\infty} \) be a solution of (1.1). Let \( I \) be the closed interval with end points 1 and \( p/q \) and let \( J \) and \( K \) be the intervals which are disjoint from \( I \) and such that

\[
I \cup J \cup K = (0, +\infty).
\]  

Then either all the even terms of the solution lie in \( J \) and all odd terms lie in \( K \), or vice-versa, or for some \( N \geq 0 \),

\[
x_n \in I \quad \text{for } n \geq N,
\]

when (E1) holds, except for the length of the first semicycle of the solution, if \( p < q \), the length is one; if \( p > q \), the length is at most two.

Theorem C (see [1, Theorem 6.6.5]).

(a) Assume \( q \leq 1 + 4p \). Then the equilibrium \( \bar{x} = (1 + \sqrt{1 + 4p(1 + q)}) / (2(1 + q)) \) of (1.1) is global attractor.

(b) Assume \( q > 1 + 4p \). Then every solution of (1.1) eventually enters and remains in the interval \( [p/q, 1] \).

In [1], they proposed the following conjecture.

Conjecture 1 (see [1, Conjecture 6.10.4]). Assume that \( p, q \in (0, +\infty) \). Show that every positive solution of (1.1) either converges to a finite limit or to a two cycle.

Gibbons et al. in [2] trigged off the investigation of the second-order difference equations \( x_{n+1} = f(x_n, x_{n-1}) \) such that the function \( f(x, y) \) is increasing in \( y \) and decreasing in \( x \). Motivated by [2], Berg [3] and Stević [4] obtained some important results on the existence of monotone solutions of such equations which was later considerably developed in a series of papers [5–14] (for related papers see also [15–19]). The monotonous character of solutions of the equations was explained by Stević in [20]. For some other papers in the area, see also [1, 17–19, 21–26] and the references cited therein. In this paper, we shall confirm that the Conjecture 1 is true. The main idea used in this paper can be found in papers [24, 26].

2. Global behavior of (1.1)

Theorem 2.1. Let \( \{x_n\}_{n=1}^{\infty} \) be a nonoscillatory solution of (1.1); then \( \{x_n\}_{n=1}^{\infty} \) converges to the unique positive equilibrium \( \bar{x} \) of (1.1).

\[
\frac{p}{q\bar{x} + x} + \frac{1}{q + 1} = \frac{p + \bar{x}}{q\bar{x} + x} = \bar{x} \geq x_{n+2} = \frac{p + x_n}{q_{n+1} + x_n} = \frac{p + x_n}{q_{n} + x_n} = \frac{p}{q_{n} + x_n} + \frac{1}{q + 1},
\]

which implies \( x_n \geq \bar{x} \); this is a contradiction. Let \( \lim_{n \to \infty} x_n = a \); then \( a = (p + a)/(qa + a) \) and \( a = \bar{x} \). The proof is complete. \( \square \)
In the sequel, let \( q > 1 + 4p \) and \( \ldots, \phi, \varphi, \phi, \varphi, \ldots \) the unique prime period-two solution of (1.1) with \( \phi < \varphi \). Define \( f \in C(\varphi, \varphi, \phi, \phi) \) by

\[
f(x, y) = \frac{p + y}{q + y} \tag{2.2}
\]

for any \( x, y \in \varphi, \varphi \) and \( g \in C(\varphi, \varphi, \phi, \phi) \) by

\[
y^* = g(y) = \frac{p + y - y^2}{qy} \tag{2.3}
\]

for any \( y \in \varphi, \varphi \). Then

\[
f(y^*, y) = y. \tag{2.4}
\]

**Lemma 2.2.** Let \( q > 1 + 4p \), then the following statements are true.

(i) \( f(x, y) > y \) if and only if \( x < y^* \).

(ii) \( x > y \) if and only if \( x^* < y^* \).

(iii) If \( \varphi < y < \varphi \), then \( f(y, y^*) < y^* \) and \( y > y^{**} \). If \( \phi < y < \varphi \), then \( f(y, y^*) > y^* \) and \( y^{**} > y \).

**Proof.** (i) Since \( f \) is decreasing in \( x \) and \( f(y^*, y) = y, x < y^* \) if and only if \( f(x, y) > f(y^*, y) = y \).

(ii) Since \( y^* = g(y) \) is a decreasing function for \( y, x > y \) if and only if \( x^* < y^* \).

(iii) Since

\[
f(y, y^*) - y^* = \frac{p + ((p + y - y^2) / qy)}{qy + ((p + y - y^2) / qy)} - \frac{p + y - y^2}{qy} \tag{2.5}
\]

\[
= \frac{(q^2 - 1)[y - (1 - \sqrt{1 + 4p + 4pq})/2(q + 1)](y - \phi)(y - \varphi)(y - \psi)}{qy[(q^2 - 1)y^2 + p + y]},
\]

it follows that

\[
\varphi < y < \varphi \Rightarrow f(y, y^*) < y^*,
\]

\[
\phi < y < \varphi \Rightarrow f(y, y^*) > y^*. \tag{2.6}
\]

By (i), we obtain \( y > y^{**} \) if \( \varphi < y < \varphi \) and \( y^{**} > y \) if \( \phi < y < \varphi \). The proof is complete. \( \square \)
Lemma 2.3. Let $q > 1 + 4p$ and $\{x_n\}_{n=1}^{+\infty}$ is a positive solution of (1.1); then $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ do exactly one of the following.

(i) Eventually, they are both monotonically increasing.
(ii) Eventually, they are both monotonically decreasing.
(iii) Eventually, one of them is monotonically increasing and the other is monotonically decreasing.

Proof. See [20] (also see [27]).

Remark 2.4. Stević in [20] noticed the relationship between the monotonicity of the subsequences $x_{2n}$ and $x_{2n-1}$ of solution $\{x_n\}_{n=1}^{+\infty}$ of a second-order difference equation $x_{n+1} = f(x_n, x_{n-1})$ and the monotonicity of the function $f(x, y)$ in variables $x$ and $y$. A simple observation shows that Stević’s proof works in the general case if the function $y/x$ is replaced by $f(x, y)$. The result was later used for many times by Stević and his collaborators (see, e.g., [21, 23–26]).

Lemma 2.5. Let $q > 1 + 4p$. Assume that there exists some $i$ such that $\psi \geq x_i \geq x_{i+2} > \overline{x} > x_{i+1} \geq \phi$; then $x_{i+1} \geq x_{i+3}$.

Proof. Since $x_{i+2} = f(x_{i+3}, x_i) \leq x_i = f(x_i^*, x_i)$, it follows that $x_{i+1} \geq x_i^*$. By Lemma 2.2(ii), we get $x_i^{**} \geq x_{i+1}^{**}$ which with Lemma 2.2(iii) implies $x_{i} \geq x_{i}^{**} \geq x_{i+1}^{*}$. Since $f(x, y)$ is increasing in $y$ ($x, y \in [\phi, \psi]$) and $x_i \geq x_{i+1}^{**}$, it follows that

$$x_{i+2} = f(x_{i+1}, x_i) \geq f(x_{i+1}, x_{i+1}^*) .$$

By Lemma 2.2(iii), we have $x_{i+2} \geq f(x_{i+1}, x_{i+1}^*) \geq x_{i}^{*} \geq x_{i+1} \geq \phi$. Thus $x_{i+1} = f(x_{i+1}^*, x_{i+1}) \geq f(x_{i+2}, x_{i+2}) = x_{i+3}$. The proof is complete.

Theorem 2.6. Let $q > 1 + 4p$ and $\{x_n\}_{n=1}^{+\infty}$ be an oscillatory solution of (1.1); then $\{x_n\}_{n=1}^{+\infty}$ converges to the unique prime period-two solution of (1.1).

Proof. It follows from Theorem C(b) that there exists $N > 0$ such that for any $n \geq N$,

$$x_n \in \left[\frac{p}{q}, 1\right] ,$$

and $x_N \geq \overline{x}$ and $x_{N+1} < \overline{x}$. We assume without loss of generality that

$$x_n \in \left[\frac{p}{q}, 1\right] \quad \text{for any } n \geq -1 ,$$

and $x_{-1} \geq \overline{x}$ and $x_0 < \overline{x}$. Since

$$h(x, y) = \frac{p + y}{q + y} \quad \left( x, y \in \left[\frac{p}{q}, 1\right] \right)$$

is decreasing in $x$ and increasing in $y$, it follows that $x_{2n-1} > \overline{x}$ and $x_{2n} < \overline{x}$ for any $n \geq 1$. 


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References


