Research Article

On an Integral-Type Operator from Zygmund-Type Spaces to Mixed-Norm Spaces on the Unit Ball

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The boundedness and compactness of an integral-type operator recently introduced by the author from Zygmund-type spaces to the mixed-norm space on the unit ball are characterized here.

1. Introduction

Let \( \mathbb{B} = \{ z \in \mathbb{C}^n : |z| < 1 \} \) be the open unit ball in \( \mathbb{C}^n \), \( \partial \mathbb{B} \) its boundary, \( dV_N \) the normalized volume measure on \( \mathbb{B} \), and \( H(\mathbb{B}) \) the class of all holomorphic functions on \( \mathbb{B} \). Strictly positive, bounded, continuous functions on \( \mathbb{B} \) are called weights.

For an \( f \in H(\mathbb{B}) \) with the Taylor expansion \( f(z) = \sum_{|\beta| \geq 0} a_\beta z^\beta \), let

\[
\Re f(z) = \sum_{|\beta| \geq 0} |\beta| a_\beta z^\beta
\]

be the radial derivative of \( f \), where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a multi-index, \( |\beta| = \beta_1 + \cdots + \beta_n \), and \( z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n} \).

A positive, continuous function \( v \) on the interval \([0,1]\) is called normal \([1]\) if there are \( \delta \in [0,1) \) and \( a \) and \( b \), \( 0 < a < b \) such that

\[
\frac{v(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \quad \lim_{r \to 1^-} \frac{v(r)}{(1-r)^a} = 0, \\
\frac{v(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \quad \lim_{r \to 1^-} \frac{v(r)}{(1-r)^b} = \infty.
\]

(1.2)
If we say that a function \( \nu : \mathbb{B} \to [0, \infty) \) is normal, we also assume that it is radial, that is, 
\[ \nu(z) = \nu(|z|), \quad z \in \mathbb{B}. \]

Let \( \mu \) be a weight. By \( \mathcal{Z}_\mu(\mathbb{B}) = \mathcal{Z}_\mu \), we denote the class of all \( f \in H(\mathbb{B}) \) such that

\[ z(f) := \sup_{z\in\mathbb{B}} \mu(z) |R^2 f(z)| < \infty, \tag{1.3} \]

and call it the Zygmund-type class. The quantity \( z(f) \) is a seminorm. A norm on \( \mathcal{Z}_\mu \) can be introduced by \( \|f\|_\mathcal{Z} = |f(0)| + z(f) \). Zygmund-type class with this norm will be called the Zygmund-type space.

The little Zygmund-type space on \( \mathbb{B} \), denoted by \( \mathcal{Z}_{\mu,0}(\mathbb{B}) = \mathcal{Z}_{\mu,0} \), is the closed subspace of \( \mathcal{Z}_\mu \) consisting of functions \( f \) satisfying the following condition

\[ \lim_{|z|\to 1} \mu(z) |R^2 f(z)| = 0. \tag{1.4} \]

For \( 0 < p, q < \infty \), and \( \phi \) normal, the mixed-norm space \( H(p,q,\phi)(\mathbb{B}) = H(p,q,\phi) \) consists of all functions \( f \in H(\mathbb{B}) \) such that

\[ \|f\|_{H(p,q,\phi)} = \left( \int_0^1 M_q^p(f,r) \phi^n(r) \frac{dr}{1-r} \right)^{1/p} < \infty, \tag{1.5} \]

where

\[ M_q(f,r) = \left( \int_{\partial \mathbb{B}} |f(r \zeta)|^q \, d\sigma(\zeta) \right)^{1/q}, \tag{1.6} \]

and \( d\sigma \) is the normalized surface measure on \( \partial \mathbb{B} \). For \( p = q \), \( \phi(r) = (1 - r^2)^{(\alpha+1)/p} \), and \( \alpha > -1 \), the space is equivalent with the weighted Bergman space \( A^p_\alpha(\mathbb{B}) \).

In [2], the present author has introduced products of integral and composition operators on \( H(\mathbb{B}) \) as follows (see also [3–5]). Assume \( g \in H(\mathbb{B}) \), \( g(0) = 0 \), and \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \), then we define an operator on \( H(\mathbb{B}) \) by

\[ P^g_\varphi(f)(z) = \int_0^1 f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}. \tag{1.7} \]

The operator is an extension of the operator introduced in [6]. Here, we continue to study operator \( P^g_\varphi \) by characterizing the boundedness and compactness of the operator between Zygmund-type spaces and the mixed-norm space. For some results on related integral-type operators mostly in \( \mathbb{C}^n \), see, for example, [3, 6–27] and the references therein.

In this paper, constants are denoted by \( C \); they are positive and may differ from one occurrence to the other. The notation \( a \lesssim b \) means that there is a positive constant \( C \) such that \( a \leq Cb \). If both \( a \leq b \) and \( b \leq a \) hold, then one says that \( a \asymp b \).
2. Auxiliary Results

In this section, we quote several lemmas which are used in the proofs of the main results.

The first lemma was proved in [2].

Lemma 2.1. Assume that $\varphi$ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, and $g(0) = 0$. Then, for every $f \in H(\mathbb{B})$, it holds

$$\Re \left[ P^g_\varphi(f) \right](z) = f(\varphi(z))g(z).$$ (2.1)

The next Schwartz-type characterization of compactness [28] is proved in a standard way (see, e.g., the proof of the corresponding lemma in [11]), hence we omit its proof.

Lemma 2.2. Assume $p, q > 0$, $\varphi$ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi$ is normal, and $\mu$ is a weight. Then, the operator $P^g_\varphi : \mathcal{L}_\mu (\text{or } \mathcal{L}_{\mu,0}) \rightarrow H(p,q,\varphi)$ is compact if and only if for every bounded sequence $(f_k)_{k \in \mathbb{N}} \subset \mathcal{L}_\mu (\text{or } \mathcal{L}_{\mu,0})$ converging to 0 uniformly on compacts of $\mathbb{B}$ we have

$$\lim_{k \rightarrow \infty} \| P^g_\varphi f_k \|_{H(p,q,\varphi)} = 0.$$ (2.2)

The next lemma is folklore and can be found, for example, in [6] (one-dimensional case for standard power weights is due to Flett [29, Theorems 6 and 7]).

Lemma 2.3. Assume that $0 < p, q < \infty$, $\varphi$ is normal, and $m \in \mathbb{N}$. Then, the following asymptotic relationship holds for every $f \in H(\mathbb{B})$,

$$\int_0^1 M^p_\varphi(f,r) \frac{\phi^p(r)}{1-r} dr \asymp |f(0)|^p + \int_0^1 M^p_\varphi(\Re^m f,r) (1 - r)^{mp} \frac{\phi^p(r)}{1-r} dr.$$ (2.3)

Lemma 2.4. Assume that $\mu$ is normal and $f \in \mathcal{L}_\mu$. Then,

$$|f(z)| \leq C \| f \|_{\mathcal{L}_\mu} \left( 1 + \int_0^{|z|} \frac{ds}{\mu(s)} dt \right), \quad z \in \mathbb{B}.$$ (2.4)

Moreover, if

$$\int_0^1 \int_0^t \frac{ds}{\mu(s)} dt < \infty,$$ (2.5)

then

$$|f(z)| \leq C \| f \|_{\mathcal{L}_\mu},$$ (2.6)

for any $z \in \mathbb{B}$.
Proof. By Lemma 2.3.1 in [21] applied to $\Re f$ we have that
\[ |\Re f(z)| \leq C \|f\|_{2_n} \left( 1 + \int_0^{\|z\|} \frac{ds}{\mu(s)} \right). \] (2.6)

Hence, for $|z| \geq 1/2$, we have that
\[ |f(z) - f\left(\frac{z}{2}\right)| \leq \int_{1/2}^{1} |\Re f(tz)| \frac{dt}{t} \leq C \|f\|_{2_n} \left( 1 + \int_0^{\|z\|} \frac{ds}{\mu(s)} \right) d(|z|), \] (2.7)
so that
\[ |f(z)| \leq M_{\infty}\left(f, \frac{1}{2}\right) + C \|f\|_{2_n} \left( 1 + \int_0^{\|z\|} \frac{ds}{\mu(s)} \right), \] (2.8)
where $M_{\infty}(f, 1/2) = \max_{|z| \leq 1/2} |f(z)|$.

If $|z| \leq 1/2$, then by the mean value property of the function $f(z) - f(0)$ (see [30]), Jensen’s inequality, and Parseval’s formula, we obtain
\[
\max_{|z| \leq 1/2} |f(z) - f(0)|^2 \leq 4^n \int_{|z| \leq 3/4} |f(w) - f(0)|^2 dV_N(w)
\leq 4^n \int_{|z| \leq 3/4} |\Re f(w)|^2 dV_N(w)
\leq 3^n \max_{|z| \leq 3/4} |\Re f(z)|^2.
\] (2.9)

From (2.9) and (2.6), we obtain
\[
M_{\infty}(f, 1/2) \leq |f(0)| + \left(\sqrt{3}\right)^n \max_{|z| \leq 3/4} |\Re f(z)|
\leq |f(0)| + \left(\sqrt{3}\right)^n C \|f\|_{2_n} \left( 1 + \int_0^{3/4} \frac{ds}{\mu(s)} \right)
\leq C \|f\|_{2_n}.
\] (2.10)

From (2.8) and (2.10), (2.3) follows, from which by (2.4) the second statement follows. \qed

Lemma 2.5. Assume $\mu$ is normal and (2.4) holds. Then, for every bounded sequence $(f_k)_{k \in \mathbb{N}} \subset \mathcal{L}_{\mu}$ converging to 0 uniformly on compacts of $\mathbb{B}$, we have that
\[
\lim_{k \to \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0.
\] (2.11)
Proof. From (2.4), we have that for every \( \varepsilon > 0 \), there is a \( \delta \in (0, \min\{\varepsilon, 1/2\}) \) such that

\[
\int_{|z|}^{1|z|} \int_{0}^{1} \frac{ds}{\mu(s)} dt < \varepsilon, \tag{2.12}
\]

for \( |z| > 1 - \delta \).

Hence, from (2.12) it follows that for each \( k \in \mathbb{N} \) and \( |z| \geq 1 - \delta \)

\[
|f_k(z) - f_k((1-\delta)z)| \leq \int_{1-\delta}^{1} \left| \mathfrak{R} f_k(tz) \right| \frac{dt}{t} \\
\leq C \|f_k\|_{L_{p,q,\phi}} \int_{1-\delta}^{1} \left( 1 + \int_{0}^{1|z|} \frac{ds}{\mu(s)} \right) dt \\
\leq C \|f_k\|_{L_{p,q,\phi}} \left( \varepsilon + \int_{(1-\delta)|z|}^{1|z|} \frac{ds}{\mu(s)} dt \right). \tag{2.13}
\]

From (2.12) and (2.13), we obtain

\[
|f_k(z)| \leq \sup_{|w| \leq 1-\delta} |f_k(w)| + 2C\varepsilon \sup_{k \in \mathbb{N}} \|f_k\|_{L_{p,q,\phi}}. \tag{2.14}
\]

Letting \( k \to \infty \) in this inequality, using the assumption that \( f_k \) converges to 0 on the compact \( |w| \leq 1 - \delta \), and using the fact that \( \varepsilon \) is an arbitrary positive number, the lemma follows. \( \square \)

3. The Boundedness and Compactness of \( P_{\psi}^{g} : \mathfrak{L}_{\mu} \) (or \( \mathfrak{L}_{\mu,0} \)) \( \to H(p,q,\phi) \)

The boundedness and compactness of the operator \( P_{\psi}^{g} : \mathfrak{L}_{\mu} \) (or \( \mathfrak{L}_{\mu,0} \)) \( \to H(p,q,\phi) \) are characterized in this section.

**Theorem 3.1.** Assume that \( p,q > 0, \phi \) is a holomorphic self-map of \( \mathbb{B} \), \( g \in H(\mathbb{B}) \), \( g(0) = 0 \), \( \phi \) and \( \mu \) are normal, and \( \mu \) satisfies condition (2.4). Let

\[
G(z) = \int_{0}^{1} g(tz) \frac{dt}{t}. \tag{3.1}
\]

Then, the following statements are equivalent:

(a) \( P_{\psi}^{g} : \mathfrak{L}_{\mu,0} \to H(p,q,\phi) \) is bounded;
(b) \( P_{\psi}^{g} : \mathfrak{L}_{\mu} \to H(p,q,\phi) \) is bounded;
(c) \( P_{\psi}^{g} : \mathfrak{L}_{\mu,0} \to H(p,q,\phi) \) is compact;
(d) \( P_{\psi}^{g} : \mathfrak{L}_{\mu} \to H(p,q,\phi) \) is compact;
(e) \( G \in H(p,q,\phi) \).

Moreover, if \( P_{\psi}^{g} : \mathfrak{L}_{\mu} \to H(p,q,\phi) \) is bounded, then the following asymptotic relations hold:

\[
\|P_{\psi}^{g}\|_{\mathfrak{L}_{\mu} \to H(p,q,\phi)} \asymp \|P_{\psi}^{g}\|_{\mathfrak{L}_{\mu,0} \to H(p,q,\phi)} \asymp \|G\|_{H(p,q,\phi)}, \tag{3.2}
\]

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Proof. The implications (d) ⇒ (b), (b) ⇒ (a), (d) ⇒ (c), and (c) ⇒ (a) are obvious.

(a) ⇒ (e) Since $P^\varphi_q : \mathcal{L}_{\mu,0} \to H(p,q,\phi)$ is bounded and $f(z) \equiv 1 \in \mathcal{L}_{\mu,0}$, by Lemma 2.1 we have that $G(z) = P^\varphi_q (1)(z) \in H(p,q,\phi)$. Moreover,

$$\|G\|_{H(p,q,\phi)} = \|P^\varphi_q (1)\|_{H(p,q,\phi)} \leq \|P^\varphi_q\|_{\mathcal{L}_{\mu,0} \to H(p,q,\phi)},$$

(3.3)

(e) ⇒ (d) Assume that $(f_k)_{k \in \mathbb{N}} \subset \mathcal{L}_{\mu}$ is a bounded sequence converging to 0 uniformly on compacts of $\mathbb{B}$. Then, by Lemmas 2.1, 2.3, and 2.5, we have

$$\left\| P^\varphi_q f_k \right\|_{H(p,q,\phi)} = \left | P^\varphi_q f_k (0) \right | + \left ( \int_0^1 M_q^p (g f_k \circ \varphi, r) \frac{\phi^p (r)}{(1-r)^{1-p}} dr \right )^{1/p} \leq C \|G\|_{H(p,q,\phi)} \sup_{z \in \mathbb{B}} | f_k (z) | \to 0, \; \text{as} \; k \to \infty,$$

(3.4)

which along with Lemma 2.2 implies the compactness of $P^\varphi_q : \mathcal{L}_{\mu} \to H(p,q,\phi)$.

From (2.4) and by Lemmas 2.3 and 2.4, we have

$$\left\| P^\varphi_q f \right\|_{H(p,q,\phi)} \leq C \left ( \int_0^1 M_q^p (g \circ \varphi, r) \frac{\phi^p (r)}{(1-r)^{1-p}} dr \right )^{1/p} \leq C \|f\|_{Z_q} \left ( \int_0^1 M_q^p (g, r) \frac{\phi^p (r)}{(1-r)^{1-p}} dr \right )^{1/p} \leq C \|f\|_{Z_q} \|G\|_{H(p,q,\phi)},$$

(3.5)

This, together with (3.3) and the inequality

$$\left\| P^\varphi_q \right\|_{\mathcal{L}_{\mu} \to H(p,q,\phi)} \leq \left\| P^\varphi_q \right\|_{\mathcal{L}_{\mu} \to H(p,q,\phi)'},$$

(3.6)

implies the asymptotic relations in (3.2), as desired.

\[\square\]

References


