Research Article

A Hybrid Iterative Scheme for a Maximal Monotone Operator and Two Countable Families of Relatively Quasi-Nonexpansive Mappings for Generalized Mixed Equilibrium and Variational Inequality Problems

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We introduce a new hybrid iterative scheme for finding a common element of the set of common fixed points of two countable families of relatively quasi-nonexpansive mappings, the set of the variational inequality for an $\alpha$-inverse-strongly monotone operator, the set of solutions of the generalized mixed equilibrium problem and zeros of a maximal monotone operator in the framework of a real Banach space. We obtain a strong convergence theorem for the sequences generated by this process in a 2 uniformly convex and uniformly smooth Banach space. The results presented in this paper improve and extend some recent results.

1. Introduction

Let $E$ be a Banach space with norm $\| \cdot \|$, $C$ a nonempty closed convex subset of $E$, and let $E^*$ denote the dual of $E$. Let $\theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function, and $B : C \to E^*$ a mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

$$
\theta(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \quad (1.1)
$$

The set of solutions to (1.1) is denoted by $\text{GMEP}(\theta, B, \varphi)$, that is,

$$
\text{GMEP}(\theta, B, \varphi) = \{ x \in C : \theta(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C \}. \quad (1.2)
$$
If $B \equiv 0$, the problem (1.1) reduces into the mixed equilibrium problem for $\theta$, denoted by MEP($\theta, \varphi$), which is to find $x \in C$ such that
\[
\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.
\tag{1.3}
\]

If $\theta \equiv 0$, the problem (1.1) reduces into the mixed variational inequality of Browder type, denoted by VI($C, B, \varphi$), which is to find $x \in C$ such that
\[
\langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.
\tag{1.4}
\]

If $B \equiv 0$ and $\varphi \equiv 0$ the problem (1.1) reduces into the equilibrium problem for $\theta$, denoted by EP($\theta$), which is to find $x \in C$ such that
\[
\theta(x, y) \geq 0, \quad \forall y \in C.
\tag{1.5}
\]

If $\theta \equiv 0$, the problem (1.3) reduces into the minimize problem, denoted by Argmin($\varphi$), is to find $x \in C$ such that
\[
\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.
\tag{1.6}
\]

The above formulation (1.4) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP($\theta$). In other words, the EP($\theta$) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, and so forth. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP($\theta$); see, for example, [1, 2] and references therein. Some solution methods have been proposed to solve the EP($\theta$); see, for example, [1, 3–11] and references therein.

A Banach space $E$ is said to be strictly convex if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. Then a Banach space $E$ is said to be smooth if the limit
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
exists for each $x, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly for $x, y \in S(E)$. Let $E$ be a Banach space. The modulus of convexity of $E$ is the function $\delta : [0, 2] \to [0, 1]$ defined by
\[
\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \left\| x - y \right\| \geq \varepsilon \right\}.
\tag{1.8}
\]

A Banach space $E$ is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let $p$ be a fixed real number with $p \geq 2$. A Banach space $E$ is said to be $p$-uniformly convex if there exists

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

for each $x, y \in S(E)$.
a constant \( c > 0 \) such that \( \delta(e) \geq ce^p \) for all \( e \in [0, 2] \); see [12, 13] for more details. Observe that every \( p \)-uniformly convex is uniformly convex. One should note that no Banach space is \( p \)-uniformly convex for \( 1 < p < 2 \). It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each \( p > 1 \), the generalized duality mapping \( J_p : E \rightarrow 2^E \) is defined by \( J_p(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \} \) for all \( x \in E \). In particular, \( J = J_2 \) is called the normalized duality mapping. If \( E \) is a Hilbert space, then \( J = I \), where \( I \) is the identity mapping.

A set valued mapping \( T : E \rightrightarrows E^* \) with graph \( G(T) = \{(x, x^*) : x^* \in Tx \} \), domain \( D(T) = \{ x \in E : Tx \neq \emptyset \} \), and range \( R(T) = \cup \{ Tx : x \in D(T) \} \). \( T \) is said to be monotone if \( \langle x^* - y^*, x - y \rangle \geq 0 \), whenever \( x^* \in Tx, y^* \in Ty \). A monotone operator \( T \) is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator on the same space. We know that if \( T \) is maximal monotone, then the solution set \( T^{-1}0 = \{ x \in D(T) : 0 \in Tx \} \) is closed and convex. It is known that \( T \) is a maximal monotone if and only if \( R(J + rT) = E^* \) for all \( r > 0 \) when \( E \) is a reflexive, strictly convex and smooth Banach space (see Rockafellar [14]).

Let \( E \) be a smooth, strictly convex and reflexive Banach space, let \( C \) be a nonempty closed convex subset of \( E \) and let \( T : E \rightrightarrows E^* \) be a monotone operator satisfying \( D(T) \subset C \subset J^{-1}(\bigcap_{r>0} R(J + rT)) \). Then we define the resolvent \( T \) by \( J_r x = \{ z \in D(T) : \langle x, z \rangle = Jz + rTx \} \), for all \( x \in E \). In other words, \( J_r = (I + rT)^{-1}J \) for all \( r > 0 \). \( J_r \) is a single-valued mapping from \( E \) to \( D(T) \). Also, we know that \( T^{-1}0 = F(J_r) \) for all \( r > 0 \), where \( F(J_r) \) is the set of all fixed points of \( J_r \). We can define, for \( r > 0 \), the Yosida approximation of \( T \) by \( T_r x = (Jx - JFJ_r)x / r \) for all \( x \in E \). We know that \( T_r x \in T(J_r x) \) for all \( r > 0 \) and \( x \in E \).

It is also known that if \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \). We consider the problem of finding:

\[
v \in E \quad \text{such that} \quad 0 \in Tv,
\]

where \( T \) is an operator from \( E \) into \( E^* \). Such \( v \in E \) is called a zero point of \( T \). Such a problem contains numerous problems in economics, optimization and physics. When \( T \) is a maximal monotone operator, a well-known method for solving (1.9) in a Hilbert space \( H \) is the proximal point algorithm: \( x_1 = x \in H \) and,

\[
x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots,
\]

where \( \{r_n\} \subset (0, \infty) \) and \( J_{r_n} = (I + r_nT)^{-1} \), then Rockafellar [15] proved that the sequence \( \{x_n\} \) converges weakly to an element of \( T^{-1}0 \). Let \( E \) be a real Banach space and let \( C \) be a nonempty closed convex subset of \( E \) and \( A : C \rightarrow E^* \) be an operator. The classical variational inequality problem for an operator \( A \) is to find \( \overline{x} \in C \) such that

\[
\langle A\overline{x}, y - \overline{x} \rangle \geq 0, \quad \forall y \in C.
\]

The set of solution of (1.11) is denote by \( \text{VI}(A, C) \). Recall that let \( A : C \rightarrow E^* \) be a mapping.
Then $A$ is called

(i) **monotone** if

$$
\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C,
$$

(1.12)

(ii) **$\alpha$-inverse-strongly monotone** if there exists a constant $\alpha > 0$ such that

$$
\langle Ax - Ay, x - y \rangle \geq \alpha \| Ax - Ay \|^2, \quad \forall x, y \in C.
$$

(1.13)

The class of inverse-strongly monotone mappings has been studied by many researchers to approximating a common fixed point; see [6, 7, 16, 17] for more details.

Let $C$ be a closed convex subset of $E$, a mapping $S : C \to C$ is said to be **nonexpansive** if $\| Sx - Sy \| \leq \| x - y \|$, for all $x, y \in C$. A point $x \in C$ is a **fixed point** of $S$ provided $Sx = x$. Denote by $F(S)$ the set of fixed points of $S$; that is, $F(S) = \{ x \in C : Sx = x \}$.

Consider the functional defined by

$$
\phi(x, y) = \| x \|^2 - 2 \langle x, Jy \rangle + \| y \|^2, \quad \text{for } x, y \in E.
$$

(1.14)

Recall that a point $p$ in $C$ is said to be an **asymptotic fixed point** of $S$ [18] if $C$ contains a sequence \( \{x_n\} \) which converges weakly to $p$ such that $\lim_{n \to \infty} \| x_n - Sx_n \| = 0$. The set of asymptotic fixed points of $S$ will be denoted by $\overline{F}(S)$. A mapping $S$ from $C$ into itself is said to be **relatively nonexpansive** [19–21] if $\overline{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [22–24]. $S$ is said to be **$\phi$-nonexpansive**, if $\phi(Sx, Sy) \leq \phi(x, y)$ for $x, y \in C$. $S$ is said to be relatively quasi-nonexpansive (or quasi-$\phi$-nonexpansive) if $F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(S)$. We note that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [22–26] which requires the strong restriction: $F(S) = \overline{F}(S)$.

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $P_C : H \to C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [27] recently introduced a **generalized projection** $\Pi_C$ from $E$ in to $C$ as follows:

$$
\Pi_C(x) = \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E.
$$

(1.15)

It is obvious from the definition of function $\phi$ that

$$
(\| y \| - \| x \|)^2 \leq \phi(y, x) \leq (\| y \| + \| x \|)^2, \quad \forall x, y \in E.
$$

(1.16)

If $E$ is a Hilbert space, then $\phi(y, x) = \| y - x \|^2$ and $\Pi_C$ becomes the metric projection of $E$ onto $C$. Let $\Pi_C$ be the generalized projection from a smooth, strictly convex and reflexive Banach space $E$ onto a nonempty closed convex subset $C$ of $E$. Then, $\Pi_C$ is a closed relatively quasi-nonexpansive mapping from $E$ onto $C$ with $F(\Pi_C) = C$. On the author hand, the generalized
projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x).$$

(1.17)

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping $J$ (see, e.g., [27–31]).

**Remark 1.1.** If $E$ is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.14), we have $\|x\| = \|y\|$. This implies that $\langle x, J y \rangle = \|x\|^2 = \|J y\|^2$. From the definition of $J$, one has $J x = J y$. Therefore, we have $x = y$; see [29, 31] for more details.

In 2004, Matsushita and Takahashi [32] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n),$$

(1.18)

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, $T$ is a relatively nonexpansive mapping and $\Pi_C$ denotes the generalized projection from $E$ onto a closed convex subset $C$ of $E$. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of $T$.

In 2005, Matsushita and Takahashi [25] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping $T$ in a Banach space $E$:

$$x_0 \in C, \text{ chosen arbitrarily,}$$
$$y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n),$$
$$C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\},$$
$$Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\},$$
$$x_{n+1} = \Pi_{C_n \cap Q_n} x_0.$$

(1.19)

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from $C$ onto $F(T)$. In 2008, Iiduka and Takahashi [33] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator $A$ in a 2-uniformly convex and uniformly smooth Banach space $E$: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n A x_n),$$

(1.20)

for every $n = 1, 2, 3, \ldots$, where $\Pi_C$ is the generalized metric projection from $E$ onto $C$, $J$ is the duality mapping from $E$ into $E^*$ and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.20) converges weakly to some element of $VI(A, C)$. 

Abstract and Applied Analysis
Recently, Takahashi and Zembayashi [34, 35], studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. In 2008, Cholamjiak [36], proved the following iteration

\[ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \]

\[ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n Jz_n), \]

\[ u_n \in C \quad \text{such that } \theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \quad (1.21) \]

\[ C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \]

\[ x_{n+1} = \Pi_{C_n \cap x_0}, \]

where \( J \) is the duality mapping on \( E \). Assume that \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequences in \([0,1]\) such that \( \alpha_n + \beta_n + \gamma_n = 1, \lim \inf_{n \to \infty} \alpha_n \beta_n > 0 \) and \( \lim \inf_{n \to \infty} \alpha_n \gamma_n > 0 \). Then \( \{ x_n \} \) converges strongly to \( q = \Pi_{F \cap x_0} \), where \( F := F(T) \cap F(S) \cap EP(\theta) \cap VI(A, C) \). In 2009, Wei et al. [37] proved the following iteration for two relatively nonexpansive mappings in a Banach space \( E \):

\[ x_0 \in C, \]

\[ Jz_n = \alpha_n Jx_n + (1 - \alpha_n) JT x_n, \]

\[ J u_n = \beta_n Jx_n + (1 - \beta_n) Jz_n, \]

\[ H_n = \{ v \in C : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1 - \beta_n) \phi(v, z_n) \leq \phi(v, x_n) \}, \]

\[ W_n = \{ z \in C : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0 \}, \]

\[ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \]

if \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences in \([0,1]\) such that \( \alpha_n \leq 1 - \delta_1 \) and \( \beta_n \leq 1 - \delta_2 \) for some \( \delta_1, \delta_2 \in (0,1) \), then \( \{ x_n \} \) generated by (1.22) converges strongly to a point \( \Pi_{Q := F(T) \cap F(S) \cap x_0} \). Where the mapping \( \Pi_{Q} \) of \( E \) onto \( Q \) is the generalized projection operator. Inoue et al. [38] proved strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the hybrid method. After that, Klin-eam et al. [2], extend Inoue et al. [38] to obtain the strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using a new hybrid method.

On the other hand, Nakajo et al. [39] introduced the following condition. Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \), let \( \{ S_n \} \) be a family of mappings of \( C \) into itself with \( F := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset \) and \( \omega_{w}(z_n) \) denotes the set of all weak subsequential limits of a bounded sequence \( \{ z_n \} \) in \( C \). \( \{ S_n \} \) is said to satisfy the NST-condition if for every bounded sequence \( \{ z_n \} \) in \( C \),

\[ \lim_{n \to \infty} \| z_n - S_n z_n \| = 0 \quad \text{implies that } \omega_{w}(z_n) \subset F. \quad (1.23) \]
Recall that a mapping $S : C \to C$ is closed if for each $\{x_n\}$ in $C$, if $x_n \to x$ and $Sx_n \to y$, then $Sx = y$. Let $\{S_n\}$ be a family of mappings of $C$ in to itself with $F := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, $\{S_n\}$ is said to satisfy the $(\ast)$-condition if for each bounded sequence $\{z_n\}$ in $C$,

$$\lim_{n \to \infty} \|z_n - S_nz_n\| = 0, \quad z_n \to z \quad \text{imply} \quad z \in F. \quad (1.24)$$

It follows directly from the definitions above that if $\{S_n\}$ satisfies NST-condition, then $\{S_n\}$ satisfies $(\ast)$-condition. If $S_n \equiv S$ and $S$ is closed, then $\{S_n\}$ satisfies $(\ast)$-condition (see [40] for more details).

In this paper, we introduce a new hybrid projection method for finding a common solution of the set of common fixed points of two countable families of relatively quasi nonexpansive mappings, the set of the variational inequality for an $\alpha$-inverse-strongly monotone operator, the set of solutions of the generalized mixed equilibrium problem and zeros of a maximal monotone operator in a real uniformly smooth and 2-uniformly convex Banach space.

2. Preliminaries

We also need the following lemmas for the proof of our main results.

**Lemma 2.1** (Xu [41]). If $E$ be a 2-uniformly convex Banach space and $0 < c \leq 1$, then for all $x, y \in E$, one has

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|, \quad (2.1)$$

where $J$ is the normalized duality mapping of $E$.

The best constant $1/c$ in lemma is called the 2-uniformly convex constant of $E$.

**Lemma 2.2** (Chidume [42, Corollary 4.17 pages 36-37]). If $E$ be a $p$-uniformly convex Banach space and let $p$ be a given real number with $p \geq 2$, then for all $x, y \in E$, $j_x \in J_p x$ and $j_y \in J_p y$

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2p - 2} \|x - y\|^p, \quad (2.2)$$

where $J_p$ is the generalized duality mapping of $E$ and $1/c$ is the $p$-uniformly convexity constant of $E$.

**Lemma 2.3** (Kamimura and Takahashi [30]). Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$. 

**Lemma 2.4** (Alber [27]). Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

**Lemma 2.5** (Alber [27]). Let $E$ be a reflexive, strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.4}$$

**Lemma 2.6** (Qin et al. [9]). Let $E$ be a real uniformly smooth and strictly convex Banach space, and $C$ be a nonempty closed convex subset of $E$. Let $S : C \to C$ be a relatively quasi-nonexpansive mapping. Then $F(S)$ is a closed convex subset of $C$.

Let $E$ be a reflexive, strictly convex, smooth Banach space and $J$ the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single valued, one-to-one, surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [27]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}x^*)$.

**Lemma 2.7** (Kohsaka and Takahashi [43, Lemma 3.2]). Let $E$ be a reflexive, strictly convex smooth Banach space and let $V$ be as in (2.5). Then

$$V(x, x^*) + 2 \langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \tag{2.6}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

**Lemma 2.8** (Kohsaka and Takahashi [44]). Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $T : E \to E^*$ be a monotone operator satisfying $D(T) \subset C \subset J^{-1}(\bigcap_{r>0} R(J + rT))$. Let $r > 0$, let $J_r$ and $T_r$ be the resolvent and the Yosida approximation of $A$, respectively. Then the following hold:

(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$, for all $x \in C, u \in T^{-1}0$;

(ii) $(J_r x, T_r x) \in T$, for all $x \in C$;

(iii) $F(J_r) = T^{-1}0$.

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be an inverse-strongly monotone mapping of $C$ into $E^*$ which is said to be hemicontinuous if for all $x, y \in C$, the mapping $F$ of $[0, 1]$ into $E^*$, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak$^*$ topology of $E^*$. We define by $N_C(v)$ the normal cone for $C$ at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \quad \forall y \in C\}. \tag{2.7}$$
Lemma 2.9 (Rockafellar [14]). Let $C$ be a nonempty, closed convex subset of a Banach space $E$ and $A$ a monotone, hemicontinuous operator of $C$ into $E^*$. Let $U : E \rightharpoonup E^*$ be an operator defined as follows:

$$UV = \begin{cases} Av + NC(v), & v \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$ \hfill (2.8)

Then $U$ is maximal monotone and $U^{-1}0 = VI(A, C)$.

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that $\theta$ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) $\theta$ is monotone, that is, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$ \hfill (2.9)

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

For example, let $B$ be a continuous and monotone operator of $C$ into $E^*$ and define

$$\theta(x, y) = \langle Bx, y - x \rangle, \quad \forall x, y \in C.$$ \hfill (2.10)

Then, $\theta$ satisfies (A1)–(A4).

The following result is in Takahashi and Zembayashi ([34, 35, Lemma 2.7]).

Lemma 2.10 (see [34, 35, Lemma 2.7]). Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $\theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that

$$\theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$ \hfill (2.11)

Motivated by Combettes and Hirstoaga [4] in a Hilbert space and Takahashi and Zembayashi [34] in a Banach space, Zhang [45] obtain the following lemma.
Lemma 2.11 (Zhang [45, Lemma 1.5]). Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$. Let $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous and $\theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in E$ such that

$$\theta(u, y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.12)$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{ u \in C : \theta(u, y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C \right\} \quad (2.13)$$

for all $x \in C$. Then the followings hold:

1. $K_r$ is single-valued;
2. $K_r$ is firmly nonexpansive, that is, for all $x, y \in E$, \( \langle K_r x - K_r y, J K_r x - J K_r y \rangle \leq \langle K_r x - K_r y, J x - J y \rangle \);
3. $F(K_r) = \tilde{F}(K_r) = \text{GMEP}(\theta, B, \varphi)$;
4. $\text{GMEP}(\cdot, B, \cdot)$ is closed and convex;
5. $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$, for all $p \in F(K_r)$ and $z \in E$.

3. Main Results

In this section, by using the $(\ast)$-condition, we prove the new convergence theorems for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of two countable families of relatively quasi-nonexpansive mappings, zeros of maximal monotone operators and the solution set of variational inequalities for an $\alpha$-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.1. Let $C$ be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space $E$. Let $T : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_r = (J + rT)^{-1}$ for all $r > 0$, where $J$ is the duality mapping on $E$. Let $\theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $E^*$ satisfy $\| Ay \| \leq \| Ay - Au \|$ for all $y \in C$ and $u \in \text{VI}(A, C) \neq \emptyset$ and let $B : C \to E^*$ be a continuous and monotone mapping. Let $S_n, T_n : C \to C$ be two families of relatively quasi-nonexpansive mappings with satisfy the $(\ast)$-condition such that

$$\Theta := \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \cap T^{-1} \text{VI}(A, C) \neq \emptyset. \quad (3.1)$$
For an initial point $x_0 \in E$ with $x_1 = \Pi_C x_0$ and $C_1 = C$, we define the sequence \{x_n\} as follows:

\[
\begin{align*}
\omega_n &= \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
z_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_n f_n \omega_n), \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Js_n z_n), \\
u_n &\in C \text{ such that } \theta(u_n, y) + \langle Bu_n, y - u_n \rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \\
&\geq 0, \quad \forall y \in C,
\end{align*}
\]

(3.2)

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b$ with $0 < a < b < c^2/2$, where $1/c$ is the 2-uniformly convexity constant of $E$. If $\lim inf_{n \to \infty} (1 - \alpha_n) > 0$ and $\lim inf_{n \to \infty} (1 - \beta_n) > 0$, then \{x_n\} converges strongly to $p = \Pi_\Theta x_0$.

**Proof.** We split the proof into seven steps.

**Step 1.** We first show that $C_{n+1}$ is closed and convex for each $n \geq 1$.

By Lemma 2.6, we know that $(\bigcap_{m=1}^{\infty} F(S_m)) \cap (\bigcap_{n=1}^{\infty} F(T_n))$ is closed and convex. We also know that if $T^{-1}0$ and VI(A,C) are closed and convex. From Lemma 2.11 (4), we have GMEP(γ, B, B') is closed and convex. Hence $\Theta := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap T^{-1}0 \cap$ GMEP(γ, B, B') ∩ VI(A,C) is a nonempty, closed and convex subset of C. Consequently, $\Pi_\Theta x_0$ is well defined.

Next, we prove that $C_n$ is closed and convex for each $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that $C_n$ is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

\[
2 \langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2.
\]

(3.3)

So, $C_{n+1}$ is closed and convex.

**Step 2.** We show that $\Theta \subset C_n$ for all $n \geq 1$.

Next, we show by induction that $\Theta \subset C_n$ for all $n \in \mathbb{N}$. Indeed, let $u_n = K_{r_n} y_n$ and $v_n = J_{r_n} w_n$ for all $n \geq 1$. On the other hand, from Lemma 2.11 one has $K_{r_n}$ is relatively
quasi-nonexpansive mapping and $\Theta \subset C_1 = C$. Suppose that $\Theta \subset C_n$ for some $n \geq 1$. Let $q \in \Theta \subset C_n$. Since $S_n$ is relatively quasi-nonexpansive mapping, we have

$$
\phi(q, y_n) = \phi \left( q, J^{-1} \left( \alpha_n Jx_n + (1 - \alpha_n) JS_n z_n \right) \right)
$$

$$
= \|q\|^2 - 2\langle q, \alpha_n Jx_n + (1 - \alpha_n) JS_n z_n \rangle + \|\alpha_n Jx_n + (1 - \alpha_n) JS_n z_n\|^2
$$

$$
\leq \|q\|^2 - 2\alpha_n \langle q, Jx_n \rangle - 2(1 - \alpha_n) \langle q, JS_n z_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|S_n z_n\|^2
$$

$$
= \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, S_n z_n)
$$

$$
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, z_n),
$$

by nonexpansiveness of $J_n$ (see [31, Theorem 4.6.3, page 130]) and $T_n$ is relatively quasi-nonexpansive mappings, we also have

$$
\phi(q, z_n) = \phi \left( q, J^{-1} \left( \beta_n Jx_n + (1 - \beta_n) JT_n v_n \right) \right)
$$

$$
= \|q\|^2 - 2\langle q, \beta_n Jx_n + (1 - \beta_n) JT_n v_n \rangle + \|\beta_n Jx_n + (1 - \beta_n) JT_n v_n\|^2
$$

$$
\leq \|q\|^2 - 2\beta_n \langle q, Jx_n \rangle - 2(1 - \beta_n) \langle q, JT_n v_n \rangle + \beta_n \|x_n\|^2 + (1 - \beta_n) \|T_n v_n\|^2
$$

$$
= \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, T_n v_n)
$$

$$
\leq \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, v_n)
$$

$$
= \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, J_n w_n)
$$

$$
\leq \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, w_n).
$$

So, it follows that

$$
\phi(q, u_n) = \phi(q, K_{r_n} y_n)
$$

$$
\leq \phi(q, y_n)
$$

$$
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, z_n),
$$

$$
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \left[ \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, w_n) \right].
$$
It follows from Lemmas 2.5 and 2.7, that
\[
\phi(q, w_n) = \phi(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
\leq \phi(q, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
= V(q, Jx_n - \lambda_n Ax_n) \\
\leq V(q, Jx_n - \lambda_n Ax_n) + 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n \rangle \\
= V(q, Jx_n) - 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n \rangle \\
= \phi(q, x_n) - 2\lambda_n \langle x_n - q, Ax_n \rangle + 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle.
\]

Thus, since \( q \in VI(A, C) \) and \( A \) is \( \alpha \)-inverse-strongly monotone, we have
\[
-2\lambda_n \langle x_n - q, Ax_n \rangle = -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle - 2\lambda_n \langle x_n - q, Aq \rangle \\
\leq -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle \\
= -2\lambda_n \| Ax_n - Aq \|^2.
\]

By Lemma 2.1 and the fact that \( \| Ay \| \leq \| Ay - Au \| \) for all \( y \in C \) and \( u \in \Theta \), we obtain
\[
2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle = 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n \rangle \\
\leq 2\| J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n) \| \| \lambda_n Ax_n \| \\
\leq \frac{4}{c^2} \| J J^{-1}(Jx_n - \lambda_n Ax_n) - J J^{-1}(Jx_n) \| \| \lambda_n Ax_n \| \\
= \frac{4}{c^2} \| Jx_n - \lambda_n Ax_n - Jx_n \| \| \lambda_n Ax_n \| \\
= \frac{4}{c^2} \| \lambda_n Ax_n \|^2 \\
= \frac{4}{c^2} \lambda_n^2 \| Ax_n \|^2 \\
\leq \frac{4}{c^2} \lambda_n^2 \| Ax_n - Aq \|^2.
\]
Substituting (3.8) and (3.9) into (3.7), we have

\[
\phi(q, w_n) \leq \phi(q, x_n) - 2\alpha_n \|Ax_n - Aq\|^2 + \frac{4}{c^2} \lambda_n^2 \|Ax_n - Aq\|^2
\]

\[
= \phi(q, x_n) + 2\lambda_n \left( \frac{2}{c^2} \lambda_n - \alpha \right) \|Ax_n - Aq\|^2
\]

\[
\leq \phi(q, x_n).
\]

(3.10)

Substituting (3.10) into (3.6), we get

\[
\phi(q, u_n) \leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \left[ \beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, x_n) \right] = \phi(q, x_n).
\]

(3.11)

This shows that \( q \in C_{n+1} \) which implies that \( \Theta \subset C_{n+1} \) and hence, \( \Theta \subset C_n \) for all \( n \geq 1 \). This implies that the sequence \( \{x_n\} \) is well defined.

**Step 3.** We prove that \( \{x_n\} \) is bounded.

Since \( x_n = \Pi_{C_n} x_0 \) and \( x_{n+1} = \Pi_{C_{n+1}} x_0 \subset C_{n+1} \subset C_n \), we have

\[
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1.
\]

(3.12)

By Lemma 2.5, we get

\[
\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0)
\]

\[
\leq \phi(p, x_0) - \phi(p, x_n)
\]

(3.13)

\[
\leq \phi(p, x_0), \quad \forall p \in \Theta.
\]

From (3.12) and (3.13), then \( \{\phi(x_n, x_0)\} \) are nondecreasing and bounded. So, we obtain that \( \lim_{n \rightarrow \infty} \phi(x_n, x_0) \) exists. In particular, by (1.16), the sequence \( \{(\|x_n\| - \|x_0\|)^2\} \) is bounded. This implies \( \{x_n\} \) is also bounded. So, we have \( \{u_n\}, \{z_n\}, \{y_n\} \) and \( \{w_n\} \) are bounded.

**Step 4.** We show that \( \{x_n\} \) is a Cauchy sequence in \( C \). Since \( x_m = \Pi_{C_m} x_0 \in C_m \subset C_n \) for \( m > n \), by Lemma 2.5, we have

\[
\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0)
\]

\[
\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0)
\]

(3.14)

\[
= \phi(x_m, x_0) - \phi(x_n, x_0).
\]

Taking \( m, n \rightarrow \infty \), we have \( \phi(x_m, x_n) \rightarrow 0 \). From Lemma 2.3, we get \( \|x_n - x_m\| \rightarrow 0 \). Hence \( \{x_n\} \) is a Cauchy sequence and by the completeness of \( E \) and the closedness of \( C \), we can assume that there exists \( p \in C \) such that \( x_n \rightarrow p \in C \) as \( n \rightarrow \infty \).
Abstract and Applied Analysis

Step 5. We show that \( \| J_u - J x_n \| \to 0 \), as \( n \to \infty \). We taking \( m = n + 1 \) in Step 4, we also have

\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.15}
\]

From Lemma 2.3, that

\[
\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0. \tag{3.16}
\]

Since \( J \) is uniformly norm-to-norm continuous on bounded subsets of \( E \), we have

\[
\lim_{n \to \infty} ||Jx_{n+1} - Jx_n|| = 0. \tag{3.17}
\]

Since \( x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n \) and the definition of \( C_{n+1} \), we have

\[
\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \quad \forall n \in \mathbb{N}. \tag{3.18}
\]

By (3.15), we obtain

\[
\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0. \tag{3.19}
\]

Again applying Lemma 2.3, we get

\[
\lim_{n \to \infty} ||x_{n+1} - u_n|| = 0. \tag{3.20}
\]

From

\[
||u_n - x_n|| = ||u_n - x_{n+1} + x_{n+1} - x_n||
\leq ||u_n - x_{n+1}|| + ||x_{n+1} - x_n|| \tag{3.21}
\]

It follows that

\[
\lim_{n \to \infty} ||u_n - x_n|| = 0. \tag{3.22}
\]

Since \( J \) is uniformly norm-to-norm continuous on bounded subsets of \( E \), we also have

\[
\lim_{n \to \infty} ||J u_n - J x_n|| = 0. \tag{3.23}
\]
Step 6. We will show that $x_n \to p \in \Theta$, where

$$\Theta := \left( \bigcap_{n=1}^{\infty} F(T_n) \right) \cap \left( \bigcap_{n=1}^{\infty} F(S_n) \right) \cap \text{GMEP}(\cdot, B, \cdot) \cap \text{VI}(A, C) \cap T^{-1} 0.$$  (3.24)

(a) We show that $x_n \to p \in (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{n=1}^{\infty} F(S_n))$. From definition of $C_{n+1}$, for any $z \in C_n$, we have

$$\phi(z, z_n) \leq \phi(z, x_n).$$  (3.25)

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we get $\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n)$. It follows from (3.15), that

$$\lim_{n \to \infty} \phi(x_{n+1}, z_n) = 0$$  (3.26)

again from Lemma 2.3, that

$$\lim_{n \to \infty} \|x_{n+1} - z_n\| = 0$$  (3.27)

it follows that since $J$ is uniformly norm-to-norm continuous, we also have

$$\lim_{n \to \infty} \|Jx_{n+1} - Jz_n\| = 0.$$  (3.28)

Since

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|$$  (3.29)

from (3.16) and (3.27), we also have

$$\lim_{n \to \infty} \|z_n - x_n\| = 0.$$  (3.30)

Since $J$ is uniformly norm-to-norm continuous, we obtain

$$\lim_{n \to \infty} \|Jz_n - Jx_n\| = 0.$$  (3.31)
From (3.4), (3.5) and (3.10), we get $\phi(p, y_n) \leq \phi(p, x_n)$. By Lemma 2.11 (5) and $u_n = K_{r_n}y_n$, we observe that

\[
\begin{align*}
\phi(u_n, y_n) &= \phi(K_{r_n}y_n, y_n) \\
&\leq \phi(p, y_n) - \phi(p, K_{r_n}y_n) \\
&\leq \phi(p, x_n) - \phi(p, K_{r_n}y_n) \\
&= \phi(p, x_n) - \phi(p, u_n) \\
&= \|p\|^2 - 2\langle p, Jx_n \rangle + \|x_n\|^2 - \left(\|p\|^2 - 2\langle p, Ju_n \rangle + \|u_n\|^2\right) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle \\
&\leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|p\| \|Jx_n - Ju_n\|. 
\end{align*}
\]

(3.32)

Since $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are bounded, it follows from (3.22), (3.23), and Lemma 2.3, we also have

\[
\lim_{n \to \infty} \|u_n - y_n\| = 0. 
\]

(3.33)

Since $J$ is uniformly norm-to-norm continuous, we have

\[
\lim_{n \to \infty} \|Ju_n - Ju_n\| = 0. 
\]

(3.34)

By using the triangle inequality, we obtain

\[
\|x_{n+1} - y_n\| = \|x_{n+1} - u_n + u_n - y_n\| \\
\leq \|x_{n+1} - u_n\| + \|u_n - y_n\|. 
\]

(3.35)

By (3.20) and (3.33), we get

\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. 
\]

(3.36)

Since $J$ is uniformly norm-to-norm continuous, we obtain

\[
\lim_{n \to \infty} \|Jx_{n+1} - Ju_n\| = 0. 
\]

(3.37)

Since

\[
\|y_n - z_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - z_n\|. 
\]

(3.38)
From (3.27) and (3.36), we have
\[
\lim_{n \to \infty} \|y_n - z_n\| = 0. \tag{3.39}
\]
Since \(J\) is uniformly norm-to-norm continuous, we also have
\[
\lim_{n \to \infty} \|Jy_n - Jz_n\| = 0. \tag{3.40}
\]
From (3.2), we get
\[
\|Jy_n - Jz_n\| = \|\alpha_n (Jx_n - Jz_n) + (1 - \alpha_n) (JS_n z_n - Jz_n)\|
= \|(1 - \alpha_n) (JS_n z_n - Jz_n) - \alpha_n (Jz_n - Jx_n)\| 
\geq (1 - \alpha_n) \|JS_n z_n - Jz_n\| - \alpha_n \|Jz_n - Jx_n\|, \tag{3.41}
\]
and hence
\[
(1 - \alpha_n) \|JS_n z_n - Jz_n\| \leq \|Jy_n - Jz_n\| + \alpha_n \|Jz_n - Jx_n\|, \tag{3.42}
\]
it follows that
\[
\|JS_n z_n - Jz_n\| \leq \frac{1}{1 - \alpha_n} (\|Jy_n - Jz_n\| + \alpha_n \|Jz_n - Jx_n\|). \tag{3.43}
\]
Since \(\lim \inf_{n \to \infty} (1 - \alpha_n) > 0\), (3.31) and (3.40), one has \(\lim_{n \to \infty} \|JS_n z_n - Jz_n\| = 0\). Since \(J^{-1}\) is uniformly norm-to-norm continuous, we get
\[
\lim_{n \to \infty} \|S_n z_n - z_n\| = 0. \tag{3.44}
\]
Since \(\|x_n - z_n\| \to 0\) and \(x_n \to p\), then we get \(z_n \to p\), hence it follows from (\(*\))-condition, that \(p \in \bigcap_{n=1}^{\infty} F(S_n)\).

Since \(v_n = J_{n} w_n\), we compute
\[
\|Jx_{n+1} - Jx_n\| = \|Jx_{n+1} - (\beta_n Jx_n + (1 - \beta_n) JT_n v_n)\|
= \|\beta_n Jx_{n+1} + (1 - \beta_n) Jx_n - (1 - \beta_n) JT_n v_n\|
= \|\beta_n (Jx_{n+1} - Jx_n) + (1 - \beta_n) (Jx_n - JT_n v_n)\| 
\geq (1 - \beta_n) \|Jx_{n+1} - JT_n v_n\| - \beta_n \|Jx_n - Jx_{n+1}\|. \tag{3.45}
\]
and hence
\[ \|J_{x_{n+1}} - J_{T_n}v_n\| \leq \frac{1}{1 - \beta_n} (\|J_{x_{n+1}} - J_{z_n}\| + \beta_n\|J_{x_n} - J_{x_{n+1}}\|). \] (3.46)

From (3.17), (3.28) and \( \lim \inf_{n \to \infty} (1 - \beta_n) > 0 \), we obtain that
\[ \lim_{n \to \infty} \|J_{x_{n+1}} - J_{T_n}v_n\| = 0. \] (3.47)

Since \( J^{-1} \) is uniformly norm-to-norm continuous on bounded sets, we have
\[ \lim_{n \to \infty} \|x_{n+1} - T_nv_n\| = 0. \] (3.48)

Using the triangle inequality, we have
\[ \|x_n - T_nv_n\| = \|x_n - x_{n+1} + x_{n+1} - T_nv_n\| \]
\[ \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_nv_n\|. \] (3.49)

From (3.16) and (3.48), we have
\[ \lim_{n \to \infty} \|x_n - T_nv_n\| = 0. \] (3.50)

On the other hand, for \( q \in \Theta \), we note that
\[ \phi(q, x_n) - \phi(q, u_n) = \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n \rangle \]
\[ \leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|q\| \|Jx_n - Ju_n\|. \] (3.51)
Since \( \{x_n\} \) and \( \{u_n\} \) are bounded, it follows from \( \|x_n - u_n\| \to 0 \) and \( \|Jx_n - Ju_n\| \to 0 \), that

\[
\phi(q, x_n) - \phi(q, u_n) \to 0. \tag{3.52}
\]

Furthermore, from (3.4), (3.5), (3.6) and (3.10), that

\[
\phi(q, u_n) \leq \phi(q, y_n)
\]

\[
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, z_n)
\]

\[
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) [\beta_n \phi(q, x_n) + (1 - \beta_n) \phi(q, w_n)]
\]

\[
= \alpha_n \phi(q, x_n) + (1 - \alpha_n) \beta_n \phi(q, x_n) + (1 - \alpha_n)(1 - \beta_n) \phi(q, w_n)
\]

\[
\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \beta_n \phi(q, x_n)
\]

\[
+ (1 - \alpha_n)(1 - \beta_n) \left[ \phi(q, x_n) - 2\lambda_n \left( \alpha - \frac{2}{c^2} \lambda_n \right) \|Ax_n - Aq\|^2 \right]
\]

\[
= \alpha_n \phi(q, x_n) + (1 - \alpha_n) \beta_n \phi(q, x_n) + (1 - \alpha_n)(1 - \beta_n) \phi(q, x_n)
\]

\[
- (1 - \alpha_n)(1 - \beta_n) 2\lambda_n \left( \alpha - \frac{2}{c^2} \lambda_n \right) \|Ax_n - Aq\|^2
\]

\[
= \phi(q, x_n) - (1 - \alpha_n)(1 - \beta_n) 2\lambda_n \left( \alpha - \frac{2}{c^2} \lambda_n \right) \|Ax_n - Aq\|^2,
\]

and hence

\[
2a \left( \alpha - \frac{2b}{c^2} \right) \|Ax_n - Aq\|^2 \leq 2\lambda_n \left( \alpha - \frac{2}{c^2} \lambda_n \right) \|Ax_n - Aq\|^2
\]

\[
\leq \frac{1}{(1 - \alpha_n)(1 - \beta_n)} (\phi(q, x_n) - \phi(q, u_n)). \tag{3.54}
\]

From (3.52), \( \lim_{n \to \infty} (1 - \alpha_n) > 0 \) and \( \lim_{n \to \infty} (1 - \beta_n) > 0 \), obtain that

\[
\lim_{n \to \infty} \|Ax_n - Aq\| = 0. \tag{3.55}
\]
Abstract and Applied Analysis

From Lemmas 2.5, 2.7, and (3.9), we compute

\[
\phi(x_n, w_n) = \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n))
\]
\[
\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n))
\]
\[
= V(x_n, Jx_n - \lambda_n Ax_n)
\]
\[
\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n \rangle
\]
\[
= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle
\]
\[
= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle
\]
\[
\leq \frac{4\lambda_n^2}{c^2} \|Ax_n - Aq\|^2
\]
\[
\leq \frac{4b^2}{c^2} \|Ax_n - Aq\|^2.
\]

Applying Lemma 2.3 and (3.55) it follows that

\[
\lim_{n \to \infty} \|x_n - w_n\| = 0.
\] (3.57)

Since \(J\) is uniformly norm-to-norm continuous, we also have

\[
\lim_{n \to \infty} \|Jx_n - Jw_n\| = 0.
\] (3.58)

Again by the triangle inequality, we get

\[
\|w_n - T_n v_n\| = \|w_n - x_n + x_n - T_n v_n\|
\leq \|w_n - x_n\| + \|x_n - T_n v_n\|.
\] (3.59)

From (3.50) and (3.57), we have

\[
\lim_{n \to \infty} \|w_n - T_n v_n\| = 0.
\] (3.60)
Abstract and Applied Analysis

From (3.5), we have \( \phi(q, v_n) \geq (1/(1 - \beta_n))(\phi(q, z_n) - \beta_n \phi(q, x_n)) \), it follows from Lemma 2.8 and (3.10), we note that

\[
\phi(v_n, w_n) = \phi(J_n w_n, w_n) \leq \phi(q, w_n) - \phi(q, J_n w_n) \\
= \phi(q, w_n) - \phi(q, v_n) \\
\leq \phi(q, w_n) - \frac{1}{1 - \beta_n}(\phi(q, z_n) - \beta_n \phi(q, x_n)) \\
\leq \phi(q, x_n) - \frac{1}{1 - \beta_n}(\phi(q, z_n) - \beta_n \phi(q, x_n)) \\
= \frac{1}{1 - \beta_n}(\phi(q, x_n) - \phi(q, z_n)) \\
= \frac{1}{1 - \beta_n} \left( \|x_n\|^2 - \|z_n\|^2 - 2\langle q, J_n x_n - J_n z_n \rangle \right) \\
\leq \frac{1}{1 - \beta_n} \left( \|x_n\|^2 - \|z_n\|^2 + 2|\langle q, J_n x_n - J_n z_n \rangle| \right) \\
\leq \frac{1}{1 - \beta_n} \left( \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|q\||\|J_n x_n - J_n z_n\| \right) \\
\leq \frac{1}{1 - \beta_n} \left( \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|q\||\|J_n x_n - J_n z_n\| \right).
\]

It follows from \( \lim \inf_{n \to \infty} (1 - \beta_n) > 0 \), (3.30) and (3.31), we get

\[
\lim_{n \to \infty} \phi(v_n, w_n) = 0. 
\]

(3.62)

From Lemma 2.3, it follows that

\[
\lim_{n \to \infty} \|v_n - w_n\| = 0. 
\]

(3.63)

By using the triangle inequality, we get

\[
\|v_n - T_n v_n\| = \|v_n - w_n + w_n - T_n v_n\| \\
\leq \|v_n - w_n\| + \|w_n - T_n v_n\|. 
\]

(3.64)

From (3.60) and (3.63), we have

\[
\lim_{n \to \infty} \|v_n - T_n v_n\| = 0. 
\]

(3.65)

Since \( \|x_n - v_n\| \leq \|x_n - w_n\| + \|w_n - v_n\| \), (3.57) and (3.63), then

\[
\lim_{n \to \infty} \|x_n - v_n\| = 0. 
\]

(3.66)
From (3.66) and since \( x_n \to p \), then \( v_n \to p \). By (3.60) it follows from (*)-condition, that 
\( p \in \bigcap_{n=1}^{\infty} F(T_n) \). Hence \( p \in \left( \bigcap_{n=1}^{\infty} F(S_n) \right) \cap \left( \bigcap_{n=1}^{\infty} F(T_n) \right) \).

(b) We show that \( x_n \to p \in \text{GMEP}(\theta, B, \varphi) \). Indeed, it follows from (A2), that

\[
\langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq -\theta(u_n, y) \geq \theta(y, u_n), \quad \forall y \in C,
\]

(3.67)

and hence

\[
\langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \left( y - u_n, \frac{Ju_n - Jy_n}{r_n} \right) \geq \theta(y, u_n), \quad \forall y \in C.
\]

(3.68)

For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1 - t)p \). Then, we get that \( y_t \in C \). From (3.68), it follows that

\[
\langle By_t, y_t - u_n \rangle \geq \langle By_t, y_t - u_n \rangle - \langle Bu_n, y_t - u_n \rangle - \varphi(y_t) + \varphi(u_n)
\]

\[
= \langle By_t - Bu_n, y_t - u_n \rangle - \varphi(y_t) + \varphi(u_n)
\]

\[
- \left( y_t - u_n, \frac{Ju_n - Jy_n}{r_n} \right) + \theta(y_t, u_n), \quad \forall y_t \in C.
\]

(3.69)

By the fact that \( y_n, u_n \to p \) as \( n \to \infty \), and \( \|Ju_n - Jy_n\|/r_n \to 0 \) as \( n \to \infty \). Since \( B \) is monotone, we know that \( \langle By_t - Bu_n, y_t - u_n \rangle \geq 0 \). Thus, it follows from (A4) that

\[
\theta(y_t, p) - \varphi(y_t) + \varphi(p) \leq \liminf_{n \to \infty} \theta(y_t, u_n) - \varphi(y_t) + \varphi(u_n) \leq \lim_{n \to \infty} \langle By_t, y_t - u_n \rangle
\]

\[
= \langle By_t, y_t - p \rangle.
\]

(3.70)

By the conditions (A1), (A4) and convexity of \( \varphi \), we have

\[
0 = \theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t)
\]

\[
\leq t\theta(y_t, y) + (1 - t)\theta(y_t, p) + t\varphi(y) + (1 - t)\varphi(p) - \varphi(y_t)
\]

\[
= t[\theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)[\theta(y_t, p) + \varphi(p) - \varphi(y_t)]
\]

\[
\leq t[\theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)[\langle By_t, y_t - p \rangle]
\]

\[
= t[\theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)t[\langle By_t, y - p \rangle]
\]

(3.71)
and hence

$$0 \leq \theta(y_t, y) + \phi(y) - \phi(y_t) + (1-t)(By_t, y - p). \quad (3.72)$$

From (A3) and the weakly lower semicontinuity of $\phi$, letting $t \to 0$, we also have

$$\theta(p, y) + (Bp, y - p) + \phi(y) - \phi(p) \geq 0, \quad \forall y \in C. \quad (3.73)$$

This implies that $p \in \text{GMEP}(\cdot, B, \cdot)$.

(c) We show that $x_n \to p \in \text{VI}(A, C)$. Indeed, define a set-valued $U : E \rightrightarrows E^*$ by Lemma 2.9, $U$ is maximal monotone and $U^{-1}0 = \text{VI}(A, C)$. Let $(v, w) \in G(U)$. Since $w \in UV = Av + NC(v)$, we get $w - Av \in NC(v)$.

From $w_n \in C$, we have

$$\langle v - w_n, w - Av \rangle \geq 0. \quad (3.74)$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then by Lemma 2.4, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0, \quad (3.75)$$

and thus

$$\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \rangle \leq 0. \quad (3.76)$$

It follows from (3.74) and (3.76), that

$$\langle v - w_n, w \rangle \geq \langle v - w_n, Av \rangle$$

$$\geq \langle v - w_n, Av \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \rangle$$

$$= \langle v - w_n, Av - Ax_n \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \rangle$$

$$= \langle v - w_n, Av - Aw_n \rangle + \langle v - w_n, Aw_n - Ax_n \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \rangle$$

$$\geq -\|v - w_n\| \|w_n - x_n\| - \|v - w_n\| \|Jx_n - Jw_n\|$$

$$\geq -M \left( \|w_n - x_n\| + \|Jx_n - Jw_n\| \right),$$

where $M = \sup_{n \geq 1} \|v - w_n\|$. Take the limit as $n \to \infty$, (3.57) and (3.58), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of $U$, we have $p \in U^{-1}0$ and hence $p \in \text{VI}(A, C)$. 

Letting $n \to p \in T^{-1}$. Since $J$ is uniformly norm-to-norm continuous on bounded sets, from (3.63), we get

$$\lim_{n \to \infty} \|Jw_n - Jv_n\| = 0. \quad (3.78)$$

From $r_n \geq d$, we have

$$\lim_{n \to \infty} \frac{1}{r_n} \|Jw_n - Jv_n\| = 0. \quad (3.79)$$

Since $J_n w_n = v_n$, therefore,

$$\lim_{n \to \infty} \|T_n w_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jw_n - JJ_n w_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jw_n - Jv_n\| = 0. \quad (3.80)$$

For $(w, w^*) \in G(T)$, from the monotonicity of $T$, we have $(w - v_n, w^* - T_n w_n) \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we get $(w - p, w^*) \geq 0$. From the maximality of $T$, we have $p \in T^{-1}$. Hence, from (a), (b), (c) and (d), we obtain $p \in \Theta$.

**Step 7.** we show that $p = \Pi_\Theta x_0$.

From $x_n = \Pi_{C_n} x_0$, we have $(Jx_0 - Jx_n, x_n - z) \geq 0$, for all $z \in C_n$. Since $\Theta \subset C_n$, we also have

$$(Jx_0 - Jx_n, x_n - y) \geq 0, \quad \forall y \in \Theta. \quad (3.81)$$

Taking limit $n \to \infty$, we obtain

$$(Jx_0 - Jp, p - y) \geq 0, \quad \forall y \in \Theta. \quad (3.82)$$

By Lemma 2.4, we can conclude that $p = \Pi_\Theta x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. \qed

**Theorem 3.2.** Let $C$ be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space $E$. Let $T : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_r := (J + rT)^{-1}J$ for all $r > 0$, where $J$ is the duality mapping on $E$. Let $\theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $\phi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $E^*$ satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in VI(A, C) \neq \emptyset$ and let $B : C \to E^*$ be a continuous and monotone mapping. Let $S_n, T_n : C \to C$ be two families of relatively quasi-nonexpansive mappings with satisfy the NST-condition such that $\Theta := (\bigcap_{n=1}^\infty F(S_n)) \cap (\bigcap_{n=1}^\infty F(T_n)) \cap T^{-1} \cap \text{GMEP}(\cdot, B, \cdot) \cap \text{VI}(A, C) \neq \emptyset$.

For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by (3.2) where $\{a_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ and $\{\lambda_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b$ with $0 < a < b < c^2/2$, where $1/c$ is the 2-uniformly convexity constant of $E$. If $\lim \inf_{n \to \infty} (1 - \alpha_n) > 0$ and $\lim \inf_{n \to \infty} (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$. 

Abstract and Applied Analysis 25
Proof. If \( \{T_n\}, \{S_n\} \) satisfy NST-condition, then \( \{T_n\}, \{S_n\} \) satisfy \((\ast)\)-condition. \(\square\)

Setting \( S_n \equiv S \) and \( T_n \equiv T \) in Theorem 3.1, then we obtain the following result.

**Corollary 3.3.** Let \( C \) be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space \( E \). Let \( \overline{T} : E \rightrightarrows E^* \) be a maximal monotone operator satisfying \( D(\overline{T}) \subset C \) and let \( J_r : = (J + r\overline{T})^{-1}J \) for all \( r > 0 \). Let \( \theta \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfying (A1)-(A4), and let \( \varphi : C \to \mathbb{R} \) be a proper lower semicontinuous and convex function. Let \( A \) be an \( \alpha \)-inverse-strongly monotone mapping of \( C \) into \( E^* \) satisfying \( \|Ay\| \leq \|Ay - Au\| \), for all \( y \in C \) and \( u \in VI(A, C) \neq \emptyset \) and let \( B : C \to E^* \) be a continuous and monotone mapping. Let \( T, S : C \to C \) be two closed relatively quasi-nonexpansive mappings such that \( \Theta : = F(S) \cap F(T) \cap \overline{T}^{-1}0 \cap \text{GMEP}(\cdot, B, \cdot) \cap VI(A, C) \neq \emptyset \). For an initial point \( x_0 \in E \) with \( x_1 = \Pi_{C_1} x_0 \) and \( C_1 = C \), we define the sequence \( \{x_n\} \) as follows:

\[
\begin{align*}
  w_n & = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
  z_n & = J^{-1}(\beta_n Jx_n + (1 - \beta_n) JT J_r w_n), \\
  y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JS z_n), \\
  u_n & \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bu_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \\
& \geq 0, \quad \forall y \in C, \\
  C_{n+1} & = \left\{ z \in C_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n) \right\}, \\
  x_{n+1} & = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{align*}
\]

where \( J \) is the duality mapping on \( E \), \( \{\alpha_n\}, \{\beta_n\} \) are sequences in \( [0, 1] \) and \( \{\lambda_n\} \subset [d, \infty) \) for some \( d > 0 \) and \( \{\lambda_n\} \subset [a, b] \) for some \( a, b \) with \( 0 < a < b < c^2/2 \), where \( 1/c \) is the 2-uniformly convexity constant of \( E \). If \( \lim \inf_{n \to \infty} (1 - \alpha_n) > 0 \) and \( \lim \inf_{n \to \infty} (1 - \beta_n) > 0 \), then \( \{x_n\} \) converges strongly to \( p \in \Theta \), where \( p = \Pi_{\Theta} x_0 \).

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of \( E \) into \( E^* \). Assume that \( A \) satisfies the conditions:

\begin{enumerate}
  \item[(C1)] \( A \) is \( \alpha \)-inverse-strongly monotone,
  \item[(C2)] \( A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset \).
\end{enumerate}

Hence, setting \( \overline{T}x \equiv 0 \), for all \( x \in C \) in Corollary 3.3, then \( J_r = I \), we also have the following result.

**Corollary 3.4.** Let \( E \) be a 2-uniformly convex and uniformly smooth Banach space. Let \( \theta \) be a bifunction from \( E \times E \) to \( \mathbb{R} \) satisfying (A1)-(A4), and let \( \varphi : E \to \mathbb{R} \) be a proper lower semicontinuous and convex function. Let \( A \) be an operator of \( E \) into \( E^* \) satisfying (C1) and (C2), and let \( B : E \to E^* \) be a continuous and monotone mapping. Let \( T, S : E \to E \) be two closed relatively quasi-nonexpansive mappings such that

\[
\Theta : = F(S) \cap F(T) \cap \text{GMEP}(\cdot, B, \cdot) \cap A^{-1}0 \neq \emptyset.
\]
For an initial point \( x_0 \in E \) with \( x_1 = \Pi_{E_1} x_0 \) and \( E_1 = E \), we define the sequence \( \{x_n\} \) as follows:

\[
\begin{align*}
  w_n &= \Pi_{C} J^{-1}(Jx_n - \lambda_n Ax_n), \\
  z_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) JT \omega_n), \\
  y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JS z_n),
\end{align*}
\]

\( u_n \in C \) such that \( \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bu_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in E, \)

\[
E_{n+1} = \{ z \in E_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n) \},
\]

\[
x_{n+1} = \Pi_{E_{n+1}} x_0, \quad \forall n \geq 1,
\]

(3.85)

where \( J \) is the duality mapping on \( E \), \( \{\alpha_n\}, \{\beta_n\} \) are sequences in \( [0,1] \) and \( \{r_n\} \subset [d, \infty) \) for some \( d > 0 \) and \( \{\lambda_n\} \subset [a, b] \) for some \( a, b > 0 \) with \( 0 < a < b < c^2 \alpha/2 \), where \( 1/c \) is the 2-uniformly convexity constant of \( E \). If \( \lim inf_{n \to \infty} (1 - \alpha_n) > 0 \) and \( \lim inf_{n \to \infty} (1 - \beta_n) > 0 \), then \( \{x_n\} \) converges strongly to \( p \in \Theta \), where \( p = \Pi_{\Theta} x_0 \).

Proof. Setting \( T x \equiv 0 \), for all \( x \in C \), then \( D(T) = E \) and hence \( C = E \) in Corollary 3.3, we also get \( \Pi_C = I \). We also have \( VI(A, C) = VI(A, E) = \{x \in E : Ax = 0\} \neq \emptyset \) and then the condition \( \|Ay\| \leq \|Ay - Au\| \) holds for all \( y \in E \) and \( u \in A^{-1} 0 \). So, we obtain the result. \( \square \)

Setting \( A \equiv 0 \) in Corollary 3.4, then we get \( \omega_n = x_n \). Hence we obtain the following Corollary.

**Corollary 3.5.** Let \( C \) be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space \( E \). Let \( \theta \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfying (A1)-(A4), and let \( \varphi : C \to \mathbb{R} \) be a proper lower semicontinuous and convex function. Let \( B : C \to E^* \) be a continuous and monotone mapping. Let \( T, S : C \to C \) be two closed relatively quasi-nonexpansive mappings such that

\[
\Theta := F(S) \cap F(T) \cap \text{GMEP}(\cdot, B, \cdot, \cdot) \neq \emptyset.
\]

(3.86)

For an initial point \( x_0 \in E \) with \( x_1 = \Pi_{C_1} x_0 \) and \( C_1 = C \), we define the sequence \( \{x_n\} \) as follows:

\[
\begin{align*}
  z_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) JT x_n), \\
  y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JS z_n),
\end{align*}
\]

\( u_n \in C \) such that \( \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bu_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \)

\[
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n) \},
\]

\[
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\]

(3.87)
where $J$ is the duality mapping on $E$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b$ with $0 < a < b < c^2/2$, where $1/c$ is the 2-uniformly convexity constant of $E$. If $\lim inf_{n \to \infty} (1 - \alpha_n) > 0$ and $\lim inf_{n \to \infty} (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $x \in \Theta$, where $x = \Pi_{\Theta} x_0$.

**Remark 3.6.** Theorem 3.1, Corollaries 3.4 and 3.5 improve and extend the corresponding results in Cholamjiak [36], Wei et al. [37] and Saewan et al. [26].

### 4. Application to Complementarity Problem

Let $K$ be a nonempty, closed convex cone in $E$. We define the polar $K^*$ of $K$ as follows:

$$K^* = \{y^* \in E^*: \langle x, y^* \rangle \geq 0, \forall x \in K\}. \quad (4.1)$$

If $A : K \to E^*$ is an operator, then an element $u \in K$ is called a solution of the complementarity problem ([31]) if

$$Au \in K^*, \quad \langle u, Au \rangle = 0. \quad (4.2)$$

The set of solutions of the complementarity problem is denoted by $C(A, K)$.

**Theorem 4.1.** Let $K$ be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space $E$. Let $\tilde{A} : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(\tilde{A}) \subset K$ and let $J_r = (J + r\tilde{A})^{-1}J$ for all $r > 0$. Let $\theta$ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $A : K \to E^*$ be an $\alpha$-inverse-strongly monotone mapping of $E$ into $E^*$ satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in K$ and $u \in C(A, K) \neq \emptyset$ and let $B : K \to E^*$ be a continuous and monotone mapping. Let $S_1, T : K \to K$ be two closed relatively quasi-nonexpansive mappings such that $\Theta := F(S) \cap F(T) \cap \tilde{A}^{-1}0 \cap GMEP(\tilde{G}, B^\ast) \cap C(A, K) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_K x_0$ and $K_1 = K$, we define the sequence $\{x_n\}$ as follows:

$$\omega_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n),$$

$$z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JTJ_r \omega_n),$$

$$y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSz_n),$$

$u_n \in K$ such that $\theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bu_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in K,$

$$K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{K_{n+1}} x_0, \quad \forall n \geq 1. \quad (4.3)$$

where $J$ is the duality mapping on $E$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b$ with $0 < a < b < c^2/2$, where $1/c$ is the 2-uniformly
convexity constant of $E$. If $\lim_{n \to \infty} (1 - \alpha_n) > 0$ and $\lim_{n \to \infty} (1 - \beta_n) > 0$, then \{$x_n$\} converges strongly to $p \in \Theta$, where $p = \Pi_{\Theta} x_0$.

Proof. As in the proof of Takahashi in [31, Lemma 7.11], we get that $VI(A, K) = C(A, K)$. So, we obtain the result.

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