Research Article

On the Generalized Hyers-Ulam-Rassias Stability of Quadratic Functional Equations

M. Eshaghi Gordji and H. Khodaei

Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

Correspondence should be addressed to M. Eshaghi Gordji, maj.ess@yahoo.com

Received 17 December 2008; Revised 19 February 2009; Accepted 10 March 2009

Recommended by John Rassias

We achieve the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities for quadratic functional equations $f(ax + by) + f(ax - by) = (b(a+b)/2)f(x + y) + (b(a+b)/2)f(x - y) + (2a^2 - ab - b^2)f(x) + (b^2 - ab)f(y)$ where $a, b$ are nonzero fixed integers with $b \neq \pm a, \pm 3a$, and $f(ax+by) + f(ax-by) = 2a^2f(x) + 2b^2f(y)$ for fixed integers $a, b$ with $a, b \neq 0$ and $a \pm b \neq 0$.

Copyright © 2009 M. E. Gordji and H. Khodaei. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In 1940, Ulam [1] proposed the stability problem for functional equations in the following question regarding to the stability of group homomorphism.

Let $(G_1, \cdot)$ be a group and let $(G_2, \ast)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) \ast h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$, for all $x \in G_1$? In other words, under what conditions does a homomorphism exist near an approximately homomorphism? Generally, the concept of stability for a functional equation comes up when we the functional equation is replaced by an inequality which acts as a perturbation of that equation. Hyers [2] answered to the question affirmatively in 1941 so if $f : E \rightarrow E'$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta, \tag{1.1}$$

for all $x, y \in E$, and for some $\delta > 0$ where $E, E'$ are Banach spaces; then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta, \tag{1.2}$$
for all $x \in E$. However, if $f(tx)$ is a continuous mapping at $t \in \mathbb{R}$ for each fixed $x \in E$ then $T$ is linear. In 1950, Hyers’s theorem was generalized by Aoki [3] for additive mappings and independently, in 1978, by Rassias [4] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. This stability phenomenon is called the Hyers-Ulam-Rassias stability.

On the other hand, Rassias [5–10] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruţa [11]. This stability phenomenon is called the Ulam-Găvruţa-Rassias stability (see also [12, 13]). In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [14]. This stability is called JMRassias mixed product-sum stability (see also [15–22]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

is related to symmetric biadditive function and is called a quadratic functional equation naturally, and every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function $f$ between two real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x) = B(x, x)$ for all $x$ where

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y))$$

(see [23, 24]). Skof proved Hyers-Ulam-Rassias stability problem for quadratic functional equation (1.3) for a class of functions $f : A \to B$, where $A$ is normed space and $B$ is a Banach space, (see [25]). Cholewa [26] noticed that Skof’s theorem is still true if relevant domain $A$ alters to an abelian group. In 1992, Czerwik proved the Hyers-Ulam-Rassias stability of (1.3) (see [27]) and four years later, Grabiec [28] generalized the result mentioned above.

Throughout this paper, assume that $a, b$ are fixed integers with $a, b \neq 0$, we introduce the following functional equations, which are different from (1.3):

$$f(ax + by) + f(ax - by) = \frac{b(a + b)}{2} f(x + y) + \frac{b(a + b)}{2} f(x - y)$$

$$+ \left(2a^2 - ab - b^2\right) f(x) + \left(b^2 - ab\right) f(y),$$

(1.5)

where $b \neq a, -3a,$ and

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y),$$

(1.6)

where $b \neq a$.

In this paper, we establish the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Găvruţa-Rassias stabilities problem for (1.5), (1.6) which are equivalent to (1.3).
2. Solution of (1.5), (1.6)

Let $X$ and $Y$ be real vector spaces. We here present the general solution of (1.5), (1.6).

**Theorem 2.1.** A function $f : X \to Y$ satisfies the functional equation (1.3) if and only if $f : X \to Y$ satisfies the functional equation (1.5). Therefore, every solution of functional equation (1.5) is also a quadratic function.

**Proof.** Let $f$ satisfy the functional equation (1.3). Putting $x = y = 0$ in (1.3), we get $f(0) = 0$. Set $x = 0$ in (1.3) to get $f(-y) = f(y)$. Letting $y = x$ and $y = 2x$ in (1.3), respectively, we obtain that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in X$. By induction, we lead to $f(kx) = k^2f(x)$ for all positive integers $k$. Replacing $x$ and $y$ by $2x + y$ and $2x - y$ in (1.3), respectively, gives

$$f(2x + y) + f(2x - y) = 8f(x) + 2f(y)$$

(2.1)

for all $x, y \in X$. Using (1.3) and (2.1), we lead to

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y)$$

(2.2)

for all $x, y \in X$. Suppose that $k \neq 0$ is a fixed integer by using (1.3), we get

$$kf(x + y) + kf(x - y) - 2kf(x) - 2kf(y) = 0$$

(2.3)

for all $x, y \in X$. Using (2.2) and (2.3), we obtain

$$f(2x + y) + f(2x - y) = (2 + k)f(x + y) + (2 + k)f(x - y) + 2(2 - k)f(x) - 2(1 + k)f(y)$$

(2.4)

for all $x, y \in X$. Replacing $x$ and $y$ by $3x + y$ and $3x - y$ in (1.3), respectively, then using (1.3) and (2.3), we have

$$f(3x + y) + f(3x - y) = (3 + k)f(x + y) + (3 + k)f(x - y) + 2(6 - k)f(x) - 2(2 + k)f(y)$$

(2.5)

for all $x, y \in X$. By using the above method, by induction, we infer that

$$f(ax + y) + f(ax - y) = (a + k)f(x + y) + (a + k)f(x - y) + 2(a^2 - a - k)f(x) - 2(a + k - 1)f(y)$$

(2.6)
for all \( x, y \in X \) and each positive integer \( a \geq 1 \). For a negative integer \( a \leq -1 \), replacing \( a \) by \(-a\) one can easily prove the validity of (2.6). Therefore (1.3) implies (2.6) for any integer \( a \neq 0 \). First, it is noted that (2.6) also implies the following equation

\[
f(\beta x + y) + f(\beta x - y) = (\beta + k)f(x + y) + (\beta + k)f(x - y)
+ 2(\beta^2 - \beta - k)f(x) - 2(\beta + k - 1)f(y)
\]  

(2.7)

for all integers \( b \neq 0 \). Setting \( y = 0 \) in (2.7) gives \( f(\beta x) = \beta^2 f(x) \). Substituting \( y \) by \( bx \) into (2.7), one gets

\[
(\beta + k)f(x + by) + (\beta + k)f(x - by) = \beta^2 f(x + y) + \beta^2 f(x - y)
- 2(\beta^2 - \beta - k)f(x) + 2\beta^2(\beta + k - 1)f(y)
\]  

(2.8)

for all \( x, y \in X \). Replacing \( y \) by \( by \) in (2.6), we observe that

\[
f(\alpha x + by) + f(\alpha x - by) = (\alpha + k)f(x + by) + (\alpha + k)f(x - by)
+ 2(\alpha^2 - \alpha - k)f(x) - 2(\alpha + k - 1)f(by)
\]  

(2.9)

for all \( x, y \in X \). Hence, according to (2.8) and (2.9), we get

\[
(\beta + k)f(\alpha x + by) + (\beta + k)f(\alpha x - by) = \beta^2(\alpha + k)f(x + y) + \beta^2(\alpha + k)f(x - y)
+ 2(\beta^2(\alpha + k) - \beta^2(\alpha + k))f(x) - 2\beta^2(\alpha - \beta)f(y)
\]  

(2.10)

for all \( x, y \in X \). In particular, if we substitute \( k := b \) in (2.10) and dividing it by \( 2b \), we conclude that \( f \) satisfies (1.5).

Let \( f \) satisfy the functional equation (1.5), for nonzero fixed integers \( a, b \) with \( b \neq \pm a, -3a \). Putting \( x = y = 0 \) in (1.5), we get

\[
\left(2a^2 - ba + b^2 - 2\right)f(0) = 0,
\]  

(2.11)

so

\[
\left(2a - \frac{b + \sqrt{16 - 7b^2}}{2}\right)\left(a - \frac{b - \sqrt{16 - 7b^2}}{4}\right)f(0) = 0,
\]  

(2.12)
but since \(a, b \neq 0\) and \(b \neq \pm a, -3a\), therefore \(f(0) = 0\). Setting \(y = 0\) in (1.5) gives \(f(ax) = a^2f(x)\) for all \(x \in X\). Letting \(y = -y\) in (1.5), we get

\[
f(ax - by) + f(ax + by) = \frac{b(a + b)}{2} f(x - y) + \frac{b(a + b)}{2} f(x + y)
+ \left(2a^2 - ab - b^2\right) f(x) + \left(b^2 - ab\right) f(-y)
\]

(2.13)

for all \(x, y \in X\). If we compare (1.5) with (2.13), then since \(a, b \neq 0\) and \(b \neq \pm a, -3a\), we conclude that \(f(-y) = f(y)\) for all \(y \in X\). Letting \(x = 0\) in (1.5) and using the evenness of \(f\) give \(f(by) = b^2f(y)\) for all \(y \in X\). Therefore for all \(x \in X\), we get \(f(abx) = a^2b^2f(x)\). Replacing \(x\) and \(y\) by \(bx\) and \(ay\) in (1.5), respectively, we have

\[
a^2b^2 f(x + y) + a^2b^2 f(x - y) = \frac{b(a + b)}{2} f(bx + ay) + \frac{b(a + b)}{2} f(bx - ay)
+ b^2\left(2a^2 - ab - b^3\right) f(x) + a^2\left(b^2 - ab\right) f(y)
\]

(2.14)

for all \(x, y \in X\). On the other hand, if we interchange \(x\) with \(y\) in (1.5), we obtain

\[
f(ay + bx) + f(ay - bx) = \frac{b(a + b)}{2} f(y + x) + \frac{b(a + b)}{2} f(y - x)
+ \left(2a^2 - ab - b^3\right) f(y) + \left(b^2 - ab\right) f(x)
\]

(2.15)

for all \(x, y \in X\). But since \(f\) is even, it follows from (2.15) that

\[
f(bx + ay) + f(bx - ay) = \frac{b(a + b)}{2} f(x + y) + \frac{b(a + b)}{2} f(x - y)
+ \left(b^2 - ab\right) f(x) + \left(2a^2 - ab - b^2\right) f(y)
\]

(2.16)

for all \(x, y \in X\). Hence, according to (2.14) and (2.16), we obtain that

\[
a^2b^2 f(x + y) + a^2b^2 f(x - y) = \frac{b(a + b)}{2} \left[\frac{b(a + b)}{2} (f(x + y) + f(x - y))
+ (b^2 - ab) f(x) + \left(2a^2 - ab - b^2\right) f(y)\right]
+ b^2\left(2a^2 - ab - b^2\right) f(x) + a^2\left(b^2 - ab\right) f(y)
\]

(2.17)
for all $x, y \in X$. So from (2.17), we have

$$
\frac{b^2}{4} \left( 4a^2 - (a + b)^2 \right) (f(x + y) + f(x - y)) = \frac{b^2}{2} \left( 3a^2 - 2ab - b^2 \right) f(x) + \frac{b^2}{2} \left( 3a^2 - 2ab - b^2 \right) f(y)
$$

(2.18)

for all $x, y \in X$. But since $a, b \neq 0$ and $b \neq a, -3a$, we conclude that

$$
f(x + y) + f(x - y) = 2f(x) + 2f(y)
$$

(2.19)

for all $x, y \in X$. Therefore, $f$ satisfies (1.3). \qed

**Theorem 2.2.** A function $f : X \to Y$ satisfies the functional equation (1.3) if and only if $f : X \to Y$ satisfies the functional equation (1.6). Therefore, every solution of functional equation (1.6) is also a quadratic function.

**Proof.** If $f$ satisfies the functional equation (1.3), then $f$ satisfies the functional equation (1.5). Now combining (1.3) with (1.5), we have

$$
f(ax + by) + f(ax - by) = \frac{b(a + b)}{2} (2f(x) + 2f(y)) + \left( 2a^2 - ab - b^2 \right) f(x) + \left( b^2 - ab \right) f(y)
$$

(2.20)

for all $x, y \in X$. So from (2.20), we conclude that $f$ satisfies (1.6).

Let $f$ satisfy the functional equation (1.6) for fixed integers $a, b$ with $a \neq 0, b \neq 0$ and $a \pm b \neq 0$. Putting $x = y = 0$ in (1.6), we get $(2(a^2 + b^2) - 2)f(0) = 0$, and since $a \neq 0, b \neq 0$, therefore $f(0) = 0$. Setting $y = 0$ in (1.6) gives $f(ax) = a^2 f(x)$ for all $x \in X$. Letting $y := -y$ in (1.6), we have

$$
f(ax - by) + f(ax + by) = 2a^2 f(x) + 2b^2 f(-y)
$$

(2.21)

for all $x, y \in X$. If we compare (1.6) with (2.21), then since $a, b \neq 0$ and $a \pm b \neq 0$, we obtain that $f(-y) = f(y)$ for all $y \in X$. Letting $x = 0$ in (1.6) and using the evenness of $f$ gives $f(by) = b^2 f(y)$ for all $y \in X$. Therefore for all $x \in X$, we get $f(abx) = a^2 b^2 f(x)$. Replacing $x$ and $y$ by $bx$ and $ay$ in (1.6), respectively, we have

$$
f(abx - aby) + f(abx + aby) = 2a^2 f(bx) + 2b^2 f(ay)
$$

(2.22)

for all $x, y \in X$. Now, by using $f(ax) = a^2 f(x), f(bx) = b^2 f(x)$ and $f(abx) = a^2 b^2 f(x)$, it follows from (2.22) that

$$
f(x + y) + f(x - y) = 2f(x) + 2f(y)
$$

(2.23)

for all $x, y \in X$. Which completes the proof of the theorem. \qed
Corollary 2.3 ([29, Proposition 2.1]). A function \( f : X \to Y \) satisfies the following functional equation:

\[
f(ax + y) + f(ax - y) = 2a^2 f(x) + 2f(y)
\]

(2.24)

for all \( x, y \in X \) if and only if \( f : X \to Y \) satisfies the functional equation (1.3) for all \( x, y \in X \).

Proof. Assume that \( b = 1 \) in functional equation (1.6) and apply Theorem 2.2.

\[\square\]

3. Stability

We now investigate the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities problem for functional equations (1.5), (1.6). From this point on, let \( X \) be a real vector space and let \( Y \) be a Banach space. Before taking up the main subject, we define the difference operator \( \Delta_f : X \times X \to Y \) by

\[
\Delta_f(x, y) = f(ax + by) + f(ax - by) - \frac{b(a + b)}{2} f(x + y) - \frac{b(a + b)}{2} f(x - y) - \left(2a^2 - ab - b^2\right) f(x) - \left(b^2 - ab\right) f(y)
\]

(3.1)

for all \( x, y \in X \) and \( a, b \) fixed integers such that \( a, b \neq 0 \) and \( a \pm b \neq 0 \) where \( f : X \to Y \) is a given function.

Theorem 3.1. Let \( j \in \{-1, 1\} \) be fixed, and let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\bar{\varphi}(x) := \sum_{i=(1-j)/2}^{\infty} \frac{1}{a^{2ij}} \varphi\left(a^{ij} x, 0\right) < \infty
\]

(3.2)

\[
\lim_{n \to \infty} \frac{1}{a^{2nj}} \varphi\left(a^{nj} x, a^{nj} y\right) = 0
\]

(3.3)

for all \( x, y \in X \). Suppose that \( f : X \to Y \) be a function satisfies

\[
\|\Delta_f(x, y)\| \leq \varphi(x, y)
\]

(3.4)

for all \( x, y \in X \). Furthermore, assume that \( f(0) = 0 \) in (3.4) for the case \( j = 1 \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{2a^{1+j}} \bar{\varphi}\left(\frac{x}{a^{(1-j)/2}}\right),
\]

(3.5)

for all \( x \in X \).
Proof. For $j = 1$, putting $y = 0$ in (3.4), we have

$$
\|2f(ax) - 2a^2f(x)\| \leq \varphi(x, 0)
$$

(3.6)

for all $x \in X$. So

$$
\|f(x) - \frac{1}{a^2}f(ax)\| \leq \frac{1}{2a^2} \varphi(x, 0)
$$

(3.7)

for all $x \in X$. Replacing $x$ by $ax$ in (3.7) and dividing by $a^2$ and summing the resulting inequality with (3.7), we get

$$
\left\| f(x) - \frac{1}{a^2} f(ax) \right\| \leq \frac{1}{2a^2} \left( \varphi(x, 0) + \frac{\varphi(ax, 0)}{a^2} \right)
$$

(3.8)

for all $x \in X$. Hence

$$
\left\| \frac{1}{a^2} f(ax) - \frac{1}{a^{2m}} f(a^m x) \right\| \leq \frac{1}{2a^2} \sum_{i=k}^{m-1} \frac{1}{a^i} \varphi(a^i x, 0)
$$

(3.9)

for all nonnegative integers $m$ and $k$ with $m > k$ and for all $x \in X$. It follows from (3.2) and (3.9) that the sequence $\{(1/a^{2n})f(a^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{(1/a^{2n})f(a^n x)\}$ converges. So one can define the function $Q : X \to Y$ by

$$
Q(x) := \lim_{n \to \infty} \frac{1}{a^{2n}} f(a^n x)
$$

(3.10)

for all $x \in X$. By (3.3) for $j = 1$ and (3.4),

$$
\| \Delta Q(x, y) \| = \lim_{n \to \infty} \frac{1}{a^{2n}} \| \Delta f(a^n x, a^n y) \| \leq \lim_{n \to \infty} \frac{1}{a^{2n}} \varphi(a^n x, a^n y) = 0
$$

(3.11)

for all $x, y \in X$. So $\Delta Q(x, y) = 0$. By Theorem 2.1, the function $Q : X \to Y$ is quadratic. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.9), we get the inequality (3.5) for $j = 1$.

Now, let $Q' : X \to Y$ be another quadratic function satisfying (1.5) and (3.5). Then we have

$$
\left\| Q(x) - Q'(x) \right\| = \frac{1}{a^{2n}} \left\| Q(a^n x) - Q'(a^n x) \right\|
$$

$$
\leq \frac{1}{a^{2n}} \left( \left\| Q(a^n x) - f(a^n x) \right\| + \left\| Q'(a^n x) - f(a^n x) \right\| \right)
$$

(3.12)

$$
\leq \frac{1}{a^2 a^{2n}} \tilde{\varphi}(a^n x, 0),
$$
Let $\phi(x, y)$ be real numbers such that $p, q \leq 2$ and $r + s \neq 2$. Suppose that a function $f : X \to Y$ satisfies

$$\|\Delta f(x, y)\| \leq \epsilon(\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s)$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\epsilon}{2(a^2 - a^p)} \|x\|^p$$

for all $x \in X$.

**Proof.** In Theorem 3.1, put $j := 1$ and $\varphi(x, y) := \epsilon(\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s)$.
Corollary 3.3. Let \( \varepsilon, p, q \geq 0 \) and \( r, s > 0 \) be real numbers such that \( p, q > 2 \) and \( r + s \neq 2 \). Suppose that a function \( f : X \to Y \) with \( f(0) = 0 \) satisfies (3.17) for all \( x, y \in X \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
\| f(x) - Q(x) \| \leq \frac{\varepsilon}{2(a^p - a^2)} \| x \|^p
\]  

(3.19)

for all \( x \in X \).

Proof. In Theorem 3.1, put \( j := -1 \) and \( \varphi(x, y) := \varepsilon(\| x \|^p + \| y \|^q + \| x \|^r \| y \|^s) \).

Theorem 3.4. Let \( j \in \{-1, 1\} \) be fixed, and let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\bar{\varphi}(x) := \sum_{i=(1-j)/2}^{\infty} \frac{1}{a^{2n_j}} \varphi(a^{n_j}x, 0) < \infty,
\]  

(3.20)

\[
\lim_{n \to \infty} \frac{1}{a^{2n_j}} \varphi(a^{n_j}x, a^{n_j}y) = 0
\]

for all \( x, y \in X \). Suppose that \( f : X \to Y \) be a function satisfies

\[
\| f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y) \| \leq \varphi(x, y)
\]  

(3.21)

for all \( x, y \in X \). Furthermore, assume that \( f(0) = 0 \) in (3.21) for the case \( j = 1 \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
\| f(x) - Q(x) \| \leq \frac{\varepsilon}{2a^{1+j}} \bar{\varphi}\left( \frac{x}{a^{1-j/2}} \right),
\]  

(3.22)

for all \( x \in X \).

Proof. The proof is similar to the proof of Theorem 3.1.

Acknowledgments

The authors would like to thank the referees for their valuable suggestions. Also, the second author would like to thank the office of gifted students at Semnan University for its financial support.

References


