Research Article

On Perfectly Homogeneous Bases in Quasi-Banach Spaces

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For $0 < p < \infty$ the unit vector basis of $\ell_p$ has the property of perfect homogeneity: it is equivalent to all its normalized block basic sequences, that is, perfectly homogeneous bases are a special case of symmetric bases. For Banach spaces, a classical result of Zippin (1966) proved that perfectly homogeneous bases are equivalent to either the canonical $c_0$-basis or the canonical $\ell_p$-basis for some $1 \leq p < \infty$. In this note, we show that (a relaxed form of) perfect homogeneity characterizes the unit vector bases of $\ell_p$ for $0 < p < 1$ as well amongst bases in nonlocally convex quasi-Banach spaces.

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1. Introduction and Background

Let us first review the relevant elementary concepts and definitions. Further details can be found in the books [1, 2] and the paper [3]. A (real) quasi-normed space $X$ is a locally bounded topological vector space. This is equivalent to saying that the topology on $X$ is induced by a quasi-norm, that is, a map $\| \cdot \| : X \to [0, \infty)$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}$, $x \in X$;
(iii) there is a constant $\kappa \geq 1$ so that for any $x_1$ and $x_2 \in X$ we have

$$\|x_1 + x_2\| \leq \kappa(\|x_1\| + \|x_2\|).$$  \hspace{1cm} (1.1)

The best constant $\kappa$ in inequality (1.1) is called the modulus of concavity of the quasi-norm. If $\kappa = 1$, the quasi-norm is a norm. A quasi-norm on $X$ is $p$-subadditive if

$$\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p, \quad x_1, x_2 \in X.$$  \hspace{1cm} (1.2)
A theorem by Aoki [4] and Rolewicz [5] asserts that every quasi-norm has an equivalent \( p \)-subadditive quasi-norm, where \( 0 < p \leq 1 \) is given by \( \kappa = 2^{1/p-1} \). A \( p \)-subadditive quasi-norm \( \| \cdot \| \) induces an invariant metric on \( X \) by the formula \( d(x, y) = \|x - y\|^p \). The space \( X \) is called quasi-Banach space if \( X \) is complete for this metric. A quasi-Banach space is isomorphic to a Banach space if and only if it is locally convex.

A basis \( (x_n)_{n=1}^{\infty} \) of a quasi-Banach space \( X \) is symmetric if \( (x_n)_{n=1}^{\infty} \) is equivalent to \( (x_{\pi(n)})_{n=1}^{\infty} \) for any permutation \( \pi \) of \( \mathbb{N} \). Symmetric bases are unconditional and so there exists a nonnegative constant \( K \) such that for all \( x = \sum_{n=1}^{\infty} a_n x_n \) the inequality

\[
\left\| \sum_{n=1}^{\infty} \theta_n a_n x_n \right\| \leq K \left\| \sum_{n=1}^{\infty} a_n x_n \right\|
\]  

holds for any bounded sequence \( (\theta_n)_{n=1}^{\infty} \in B_{\ell_1} \). The least such constant \( K \) is called the unconditional constant of \( (x_n)_{n=1}^{\infty} \).

For instance, the canonical basis of the spaces \( \ell_p \) for \( 0 < p < \infty \) is symmetric and 1-unconditional. What is more, it is the only symmetric basis of \( \ell_p \) up to equivalence, that is, whenever \( (x_n)_{n=1}^{\infty} \) is another normalized symmetric basis of \( \ell_p \), there is a constant \( C \) such that

\[
\frac{1}{C} \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq C \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p},
\]  

for any finitely nonzero sequence of scalars \( (a_n)_{n=1}^{\infty} \) [6, 7].

The spaces \( \ell_p \) for \( 0 < p < 1 \) share the property of uniqueness of symmetric basis with all natural quasi-Banach spaces whose Banach envelope (i.e., the smallest containing Banach space) is isomorphic to \( \ell_1 \), as was recently proved in [8]. For other results on uniqueness of unconditional or symmetric basis in nonlocally convex quasi-Banach spaces the reader can consult the papers [9, 10].

This article illustrates how Zippin’s techniques can also be used to characterize the unit vector bases of \( \ell_p \) for \( 0 < p < 1 \) as the only, up to equivalence, perfectly homogeneous bases in nonlocally convex quasi-Banach spaces. We use standard Banach space theory terminology and notation throughout, as may be found in [11, 12].

## 2. Perfectly Homogeneous Bases in Quasi-Banach Spaces

Let \( (x_i)_{i=1}^{\infty} \) be a basis for a quasi-Banach space \( X \). A block basic sequence \( (u_n)_{n=1}^{\infty} \) of \( (x_i)_{i=1}^{\infty} \),

\[
u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i,
\]  

is said to be a constant coefficient block basic sequence if for each \( n \) there is a constant \( c_n \) so that \( a_i = c_n \) or \( a_i = 0 \) for \( p_{n-1} + 1 \leq i \leq p_n \).

**Definition 2.1.** A basis \( (x_i)_{i=1}^{\infty} \) of a quasi-Banach space \( X \) is almost perfectly homogeneous if every normalized constant coefficient block basic sequence of \( (x_i)_{i=1}^{\infty} \) is equivalent to \( (x_i)_{i=1}^{\infty} \).
Let us notice that using a uniform boundedness argument we obtain that, in fact, if \((x_i)_{i=1}^\infty\) is almost perfectly homogeneous then it is uniformly equivalent to all its normalized constant coefficient block basic sequences. That is, there is a constant \(M \geq 1\) such that for any normalized constant coefficient block basic sequence \((u_n)_{n=1}^\infty\) of \((x_i)_{i=1}^\infty\) we have

\[
M^{-1} \left\| \sum_{k=1}^n a_k x_k \right\| \leq \left\| \sum_{k=1}^n a_k u_k \right\| \leq M \left\| \sum_{k=1}^n a_k x_k \right\|,
\]

for all choices of scalars \((a_k)_{k=1}^n\) and \(n \in \mathbb{N}\). Equation (2.2) also yields that for any increasing sequence of integers \((k_j)_{j=1}^\infty\),

\[
M^{-1} \left\| \sum_{j=1}^n x_j \right\| \leq \left\| \sum_{j=1}^n x_{k_j} \right\| \leq M \left\| \sum_{j=1}^n x_j \right\|.
\]

This is our main result (cf. [13]).

**Theorem 2.2.** Let \(X\) be a nonlocally convex quasi-Banach space with normalized basis \((x_i)_{i=1}^\infty\). Suppose that \((x_i)_{i=1}^\infty\) is almost perfectly homogeneous. Then \((x_i)_{i=1}^\infty\) is equivalent to the canonical basis of \(\ell_q\) for some \(0 < q < 1\).

**Proof.** Let \(\kappa\) be the modulus of concavity of the quasi-norm. Since \(X\) is nonlocally convex, \(\kappa > 1\). By the Aoki-Roléwicz theorem we can assume that the quasi-norm is \(p\)-subadditive for \(0 < p < 1\) such that \(\kappa = 2^{1/p-1}\). We will show that \((x_i)_{i=1}^\infty\) is equivalent to the canonical \(\ell_q\)-basis for some \(p \leq q < 1\).

By renorming, without loss of generality we can assume \((x_i)_{i=1}^\infty\) to be 1-unconditional. For each \(n\) put,

\[
\lambda(n) = \left\| \sum_{i=1}^n x_i \right\|.
\]

Note that

\[
1 \leq \lambda(n) \leq n^{1/p}, \quad n \in \mathbb{N},
\]

and that, by the 1-unconditionality of the basis, the sequence \((\lambda(n))_{n=1}^\infty\) is nondecreasing.

We are going to construct disjoint blocks of length \(n\) of the basis \((x_i)_{i=1}^\infty\) as follows:

\[
v_1 = \sum_{i=1}^n x_i, \quad v_2 = \sum_{i=n+1}^{2n} x_i, \ldots, \quad v_j = \sum_{i=(j-1)n+1}^{jn} x_i, \ldots
\]

Equation (2.3) says that

\[
M^{-1} \lambda(n) \leq \|v_j\| \leq M \lambda(n), \quad j \in \mathbb{N},
\]
and so by the 1-unconditionality of $(x_i)_{i=1}^{\infty}$,

\[
\frac{1}{M\lambda(n)} \left\| \sum_{j=1}^{m} v_j \right\| \leq \left\| \sum_{j=1}^{m} \|v_j\|^{-1} v_j \right\| \leq \frac{M}{\lambda(n)} \left\| \sum_{j=1}^{m} v_j \right\|, \quad m \in \mathbb{N}.
\] (2.8)

On the other hand, by (2.2) we know that

\[
\frac{\lambda(m)}{M} \leq \left\| \sum_{j=1}^{m} \|v_j\|^{-1} v_j \right\| \leq M \lambda(m), \quad m \in \mathbb{N}.
\] (2.9)

If we put these last two inequalities together we obtain

\[
\frac{1}{M^2} \lambda(m) \lambda(n) \leq \lambda(mn) \leq M^2 \lambda(m) \lambda(n), \quad m, n \in \mathbb{N}.
\] (2.10)

Substituting in (2.10) integers of the form $m = 2^k$ and $n = 2^j$ give

\[
\frac{1}{M^2} \lambda\left(2^k\right) \lambda\left(2^j\right) \leq \lambda\left(2^{k+j}\right) \leq M^2 \lambda\left(2^k\right) \lambda\left(2^j\right), \quad k, j \in \mathbb{N}.
\] (2.11)

For $k = 0, 1, 2, \ldots$, let $h(k) = \log_2 \lambda(2^k)$. From (2.11) it follows that

\[
|h(j) - h(k) - h(j + k)| \leq 2 \log_2 M.
\] (2.12)

We need the following well-known lemma from real analysis.

**Lemma 2.3.** Suppose that $(s_n)_{n=1}^{\infty}$ is a sequence of real numbers such that

\[
|s_{m+n} - s_m - s_n| \leq 1
\] (2.13)

for all $m, n \in \mathbb{N}$. Then there is a constant $c$ so that

\[
|s_n - cn| \leq 1, \quad n = 1, 2, \ldots
\] (2.14)

Lemma 2.3 yields a constant $c$ so that

\[
|h(k) - ck| \leq 2 \log_2 M, \quad k = 1, 2, \ldots
\] (2.15)

In turn, using (2.5) we have

\[
1 \leq \lambda\left(2^k\right) \leq 2^{k/p}, \quad k = 1, 2, \ldots
\] (2.16)
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which implies

\[ 0 \leq h(k) \leq \frac{k}{p}, \quad (2.17) \]

and so, combining with (2.15) we obtain that the range of possible values for \( c \) is

\[ 0 \leq c \leq \frac{1}{p}, \quad (2.18) \]

If \( c = 0 \) then \( (\lambda(n))_{n=1}^{\infty} \) would be (uniformly) bounded and so \( (x_i)_{i=1}^{\infty} \) would be equivalent to the canonical basis of \( c_0 \), a contradiction with the local nonconvexity of \( X \). Otherwise, if \( 0 < c \leq 1/p \) there is \( q \in [p, \infty) \) such that \( c = 1/q \). This way we can rewrite (2.15) in the form

\[ \left| h(k) - \frac{k}{q} \right| \leq 2 \log_2 M, \quad k \in \mathbb{N}, \quad (2.19) \]

or equivalently,

\[ M^{-2k/q} \leq \lambda \left( \frac{2^k}{q} \right) \leq 2^{k/q} M^2, \quad k \in \mathbb{N}. \quad (2.20) \]

Now, given \( n \in \mathbb{N} \) we pick the only integer \( k \) so that \( 2^{k-1} \leq n \leq 2^k \). Then,

\[ \lambda \left( \frac{2^{k-1}}{q} \right) \leq \lambda(n) \leq \lambda \left( \frac{2^k}{q} \right), \quad (2.21) \]

and so

\[ M^{-2^{-1/q} n^{1/q}} \leq \lambda(n) \leq M^{2^{1/q} n^{1/q}}. \quad (2.22) \]

If \( A \) is any finite subset of \( \mathbb{N} \), by (2.3) we have

\[ M^{-1} \lambda(|A|) \leq \left\| \sum_{j \in A} x_j \right\| \leq M \lambda(|A|), \quad (2.23) \]

hence

\[ C^{-1} |A|^{1/q} \leq \left\| \sum_{j \in A} x_j \right\| \leq C |A|^{1/q}, \quad (2.24) \]

where \( C = M^{3/2^{1/q}} \).
To prove the equivalence of $\{x_i\}_{i=1}^\infty$ with the canonical basis of $\ell_q$, given any $n \in \mathbb{N}$ we let $(a_i)_{i=1}^n$ be nonnegative scalars such that $a_i^q \in \mathbb{Q}$ and $\sum_{i=1}^n a_i^q = 1$. Each $a_i^q$ can be written in the form $a_i^q = m_i/m$ where $m_i \in \mathbb{N}$, $m$ is the common denominator of the $a_i^q$’s, and $\sum_{i=1}^n m_i = m$.

Let $A_1$ be interval of natural numbers $[1, m_1]$ and for $j = 2, \ldots, n$ let $A_i$ be the interval of natural numbers $[m_1 + \cdots + m_{i-1} + 1, m_1 + \cdots + m_i]$. The sets $A_1, \ldots, A_n$ are disjoint and have cardinality $|A_i| = m_i$ for each $i = 1, \ldots, n$. Consider the normalized constant coefficient block basic sequence defined as

$$u_i = c_i^{-1} \sum_{j \in A_i} x_j, \quad 1 \leq i \leq n,$$

where $c_i = \|\sum_{j \in A_i} x_k\|$. Equation (2.24) yields

$$C^{-1} m_1^{1/q} \leq c_i \leq C m_1^{1/q}, \quad 1 \leq i \leq n.$$

Therefore,

$$\left\| \sum_{i=1}^n \sum_{j \in A_i} x_j \right\| \leq \left\| \sum_{i=1}^n a_i u_i \right\| \leq \frac{C}{m_1^{1/q}} \left\| \sum_{i=1}^n \sum_{j \in A_i} x_k \right\|,$$

that is,

$$C^{-1} \frac{\lambda(m)}{m_1^{1/q}} \leq \left\| \sum_{i=1}^n a_i u_i \right\| \leq C \frac{\lambda(m)}{m_1^{1/q}}.$$

Thus,

$$\frac{1}{C^2 M} \leq \left\| \sum_{i=1}^n a_i u_i \right\| \leq C^2 M.$$

Using (2.2) again, we have

$$\frac{1}{C^2 M^2} \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq C^2 M^2.$$

We note that a simple density argument shows that (2.30) holds whenever $\sum_{i=1}^n |a_i|^q = 1$ (i.e., without the assumption that $|a_i|^q$ is rational), and this completes the proof that $\{x_i\}_{i=1}^\infty$ is equivalent to the canonical $\ell_q$-basis for some $p \leq q < \infty$. Since $X$ is not locally convex, we conclude that $p \leq q < 1$. \qed
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