Research Article

Some Identities of Symmetry for the Generalized Bernoulli Numbers and Polynomials

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By the properties of \( p \)-adic invariant integral on \( \mathbb{Z}_p \), we establish various identities concerning the generalized Bernoulli numbers and polynomials. From the symmetric properties of \( p \)-adic invariant integral on \( \mathbb{Z}_p \), we give some interesting relationship between the power sums and the generalized Bernoulli polynomials.

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1. Introduction

Let \( p \) be a fixed prime number. Throughout this paper, the symbols \( \mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \) and \( \mathbb{C}_p \) will denote the ring of rational integers, the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( v_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-v_p(p)} = 1/p \). Let \( \text{UD}(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( f \in \text{UD}(\mathbb{Z}_p) \), the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined as

\[
I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}
\]

(see [1]). From the definition (1.1), we have

\[
I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}, \quad f_1(x) = f(x + 1). \tag{1.2}
\]
Let $f_n(x) = f(x + n), \ (n \in \mathbb{N})$. Then we can derive the following equation from (1.2):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i),$$

(1.3)

(see [1]). It is well known that the ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

(1.4)

(see [1–25]), and the Bernoulli number $B_n$ are defined as $B_n = B_n(0)$. Let $d$ be a fixed positive integer. For $n \in \mathbb{N}$, we set

$$X = X_d = \lim_{N \to \infty} \left( \mathbb{Z}/dp^N \mathbb{Z} \right), \quad X_1 = \mathbb{Z}_p;$$

$$X^* = \bigcup_{(a,p)=1} (a + dp\mathbb{Z}_p);$$

$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\},$$

(1.5)

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. It is easy to see that

$$\int_X f(x) dx = \int_{\mathbb{Z}_p} f(x) dx, \quad \text{for } f \in \text{UD}(\mathbb{Z}_p).$$

(1.6)

In [14], the Witt’s formula for the Bernoulli numbers are given by

$$\int_{\mathbb{Z}_p} x^n dx = B_n, \quad n \in \mathbb{Z}_+.\quad (1.7)$$

Let $\chi$ be the Dirichlet’s character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to $\chi$ are defined as

$$\sum_{a=1}^{d} \chi(a) \frac{te^{at}}{e^{at} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!},$$

(1.8)

(see [22]), and the generalized Bernoulli numbers attached to $\chi$, $B_{n,\chi}$ are defined as $B_{n,\chi} = B_{n,\chi}(0)$.

In this paper, we investigate the interesting identities of symmetry for the generalized Bernoulli numbers and polynomials attached to $\chi$ by using the properties of $p$-adic invariant integral on $\mathbb{Z}_p$. Finally, we will give relationship between the power sum polynomials and the generalized Bernoulli numbers attached to $\chi$. 
2. Symmetry of Power Sum and the Generalized Bernoulli Polynomials

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{N} \). From (1.3), we note that

\[
\int_X \chi(x)e^{xt}dx = \frac{t}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_n \chi \frac{t^n}{n!}, \tag{2.1}
\]

where \( B_{n,\chi}(x) \) are the \( n \)th generalized Bernoulli numbers attached to \( \chi \). Now, we also see that the generalized Bernoulli polynomials attached to \( \chi \) are given by

\[
\int_X \chi(y)e^{(x+y)t}dy = \frac{t}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \tag{2.2}
\]

By (2.1) and (2.2), we easily see that

\[
\int_X \chi(x)x^n dx = B_{n,\chi}, \quad \int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x). \tag{2.3}
\]

From (2.2), we have

\[
B_{n,\chi}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} B_{\ell,\chi} x^{n-\ell}. \tag{2.4}
\]

From (2.2), we can also derive

\[
\int_X \chi(x)e^{xt}dx = \sum_{i=0}^{d-1} \chi(i) e^{(i/d)dt} = \sum_{n=0}^{\infty} \left( d^{n-1} \sum_{i=0}^{d-1} \chi(i) B_{n} \left( \frac{i}{d} \right) \right) \frac{t^n}{n!}. \tag{2.5}
\]

Therefore, we obtain the following lemma.

**Lemma 2.1.** For \( n \in \mathbb{Z}_+ \), one has

\[
\int_X \chi(x)x^n dx = B_{n,\chi} = d^{n-1} \sum_{i=0}^{d-1} \chi(i) B_{i} \left( \frac{i}{d} \right). \tag{2.6}
\]

We observe that

\[
\frac{1}{t} \left( \int_X \chi(x)e^{(nd+\epsilon)t}dx - \int_X e^{xt} \chi(x)dx \right) = \frac{nd}{t} \int_X \chi(x)e^{xt}dx = \frac{e^{ndt} - 1}{e^{dt} - 1} \left( \sum_{i=0}^{d-1} \chi(i) e^{it} \right). \tag{2.7}
\]

Thus, we have

\[
\frac{1}{t} \left( \int_X \chi(x)e^{(nd+\epsilon)t}dx - \int_X \chi(x)dx \right) = \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{d-1} \chi(\ell) e^{\epsilon\ell} \right) \frac{i^k}{k!}. \tag{2.8}
\]
Let us define the $p$-adic functional $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{\ell=0}^n \chi(\ell) e^k, \quad \text{for } k \in \mathbb{Z}_+. \quad (2.9)$$

By (2.8) and (2.9), we see that

$$\frac{1}{l} \left( \int_X \chi(x) e^{(nd+\chi)l} dx - \int_X \chi(x) e^{xl} dx \right) = \sum_{n=0}^{\infty} (T_k(\chi, nd - 1)) \frac{t^k}{k!}. \quad (2.10)$$

By using Taylor expansion in (2.10), we have

$$\int_X \chi(x) (dn + x)^k dx - \int_X \chi(x) x^k dx = kT_{k-1}(\chi, nd - 1), \quad \text{for } k, n, d \in \mathbb{N}. \quad (2.11)$$

That is,

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd - 1). \quad (2.12)$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we consider the following integral equation:

$$\frac{d \int_X \chi(x_1) \chi(x_2) e^{(w_1x_1+w_2x_2)t} dx_1 dx_2}{\int_X e^{t(dw_1+w_2)t} dx} = \frac{t(e^{dw_1} - 1)}{(e^{dw_1} - 1)(e^{dw_2} - 1)} \left( \sum_{a=0}^{d-1} \chi(a) e^{aw_1t} \right) \left( \sum_{b=0}^{d-1} \chi(b) e^{aw_2t} \right). \quad (2.13)$$

From (2.7) and (2.10), we note that

$$\frac{dw_1 \int_X \chi(x) e^{xl} dx}{\int_X e^{t(dw_1)t} dx} = \sum_{k=0}^{\infty} (T_k(\chi, dw_1 - 1)) \frac{t^k}{k!}. \quad (2.14)$$

Let us consider the $p$-adic functional $T_\chi(w_1, w_2)$ as follows:

$$T_\chi(w_1, w_2) = \frac{d \int_X \chi(x_1) \chi(x_2) e^{(w_1x_1+w_2x_2)t} dx_1 dx_2}{\int_X e^{t(w_1+w_2)t} dx_3}. \quad (2.15)$$

Then we see that $T_\chi(w_1, w_2)$ is symmetric in $w_1$ and $w_2$, and

$$T_\chi(w_1, w_2) = \frac{t(e^{dw_1} - 1)}{(e^{dw_1} - 1)(e^{dw_2} - 1)} \left( \sum_{a=0}^{d-1} \chi(a) e^{aw_1t} \right) \left( \sum_{b=0}^{d-1} \chi(b) e^{aw_2t} \right). \quad (2.16)$$
By (2.15) and (2.16), we have

\[
T_x(w_1, w_2) = \left( \frac{1}{w_1} \int_X x(x_1)e^{w_1(x_1+x_2)}dx_1 \right) \left( \frac{d\omega_1}{\int_X x\omega_1(x_1)e^{w_2x_2}dx_2} \right)
\]
\[
= \left( \frac{1}{w_1} \sum_{i=0}^{\infty} B_{i,x}(w_2x) \frac{w_1^i}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_1 - 1) \frac{w_2^k}{k!} \right)
\]
\[
= \frac{1}{w_1} \left( \sum_{i=0}^{\infty} \left( \sum_{\ell=0}^{\infty} \frac{\ell}{\ell} \frac{B_{i,x}(w_2x)w_1^iT_{\ell-i}(\chi, dw_1 - 1)w_2^{\ell-i}x^{\ell-i}}{\ell! \ell!} \right) \right) \frac{t^\ell}{\ell!}
\]
\[
= \sum_{\ell=0}^{\infty} \left( \sum_{i=0}^{\infty} \frac{\ell}{i} \frac{B_{i,x}(w_2x)w_1^iT_{\ell-i}(\chi, dw_1 - 1)w_2^{\ell-i}x^{\ell-i}}{\ell! \ell!} \right) \frac{t^\ell}{\ell!}
\]

(2.17)

From the symmetric property of \(T_x(w_1, w_2)\) in \(w_1\) and \(w_2\), we note that

\[
T_x(w_1, w_2) = \left( \frac{1}{w_2} \int_X x(x_2)e^{w_2(x_2+x_1)}dx_2 \right) \left( \frac{d\omega_2}{\int_X x\omega_1(x_1)e^{w_2x_2}dx_2} \right)
\]
\[
= \left( \frac{1}{w_2} \sum_{i=0}^{\infty} B_{i,x}(w_1x) \frac{w_2^i}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_2 - 1) \frac{w_1^k}{k!} \right)
\]
\[
= \frac{1}{w_2} \left( \sum_{i=0}^{\infty} \left( \sum_{\ell=0}^{\infty} \frac{\ell}{\ell} \frac{B_{i,x}(w_1x)w_2^iT_{\ell-i}(\chi, dw_2 - 1)w_1^{\ell-i}x^{\ell-i}}{\ell! \ell!} \right) \right) \frac{t^\ell}{\ell!}
\]
\[
= \sum_{\ell=0}^{\infty} \left( \sum_{i=0}^{\infty} \frac{\ell}{i} \frac{B_{i,x}(w_1x)w_2^iT_{\ell-i}(\chi, dw_2 - 1)w_1^{\ell-i}x^{\ell-i}}{\ell! \ell!} \right) \frac{t^\ell}{\ell!}
\]
\]
\[
(2.18)
\]

By comparing the coefficients on the both sides of (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.2.** For \(w_1, w_2, d \in \mathbb{N}\), one has

\[
\sum_{i=0}^{\ell} \frac{\ell}{i} B_{i,x}(w_2x)T_{\ell-i}(\chi, dw_1 - 1)w_1^{i-1}w_2^{\ell-i} = \sum_{i=0}^{\ell} \frac{\ell}{i} B_{i,x}(w_1x)T_{\ell-i}(\chi, dw_2 - 1)w_2^{i-1}w_1^{\ell-i}.
\]

(2.19)

Let \(x = 0\) in Theorem 2.2. Then we have

\[
\sum_{i=0}^{\ell} \frac{\ell}{i} B_{i,x}T_{\ell-i}(\chi, dw_1 - 1)w_1^{i-1}w_2^{\ell-i} = \sum_{i=0}^{\ell} \frac{\ell}{i} B_{i,x}T_{\ell-i}(\chi, dw_2 - 1)w_2^{i-1}w_1^{\ell-i}.
\]

(2.20)
By (2.14) and (2.16), we also see that

\[
T(x, w_1, w_2) = \left( \frac{e^{w_1 w_2 x}}{w_1} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \frac{d w_1 \chi(x_2) e^{w_2 x_2} dx_2}{\int x e^{d w_1 w_2 x} dx} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_1} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \frac{e^{d w_1 w_2 x} - 1}{e^{w_2 x} - 1} \right) \left( \sum_{i=0}^{d-1} \chi(i) e^{w_2 x} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_1} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \sum_{i=0}^{d-1} \chi(i) \sum_{\ell=0}^{i} e^{w_2 (i+\ell)d} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_1} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \sum_{i=0}^{d-1} e^{w_2 x} \chi(i) \right)
\]

(2.21)

\[
= \frac{1}{w_1} \sum_{i=0}^{d w_1 - 1} \chi(i) \int_x \chi(x_1) e^{d w_1 x_1} dx_1
\]

\[
= \frac{1}{w_1} \sum_{i=0}^{d w_1 - 1} \chi(i) \sum_{k=0}^{w_2 x} B_{k, x} \left( w_2 x, \frac{w_2}{w_1} \right) \frac{w_2^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{d w_1 - 1} \chi(i) B_{k, x} \left( w_2 x, \frac{w_2}{w_1} \right) w_2^{k-1} \right) \frac{w_2^k}{k!}
\]

From the symmetric property of \( T(x, w_1, w_2) \) in \( w_1 \) and \( w_2 \), we can also derive the following equation:

\[
T(x, w_1, w_2) = \left( \frac{e^{w_1 w_2 x}}{w_2} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \frac{d w_2 \chi(x_2) e^{w_2 x_2} dx_2}{\int x e^{d w_1 w_2 x} dx} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_2} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \frac{e^{d w_1 w_2 x} - 1}{e^{w_1 x} - 1} \right) \left( \sum_{i=0}^{d-1} \chi(i) e^{w_1 x} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_2} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \sum_{i=0}^{d-1} \chi(i) \sum_{\ell=0}^{i} e^{w_1 (i+\ell)d} \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x}}{w_2} \int_x \chi(x_1) e^{w_1 x_1} dx_1 \right) \left( \sum_{i=0}^{d-1} e^{w_1 x} \chi(i) \right)
\]

(2.22)

\[
= \frac{1}{w_2} \sum_{i=0}^{d w_2 - 1} \chi(i) \int_x \chi(x_2) e^{d w_2 x_2} dx_2
\]

\[
= \frac{1}{w_2} \sum_{i=0}^{d w_2 - 1} \chi(i) \sum_{k=0}^{w_1 x} B_{k, x} \left( w_1 x, \frac{w_1}{w_2} \right) \frac{w_1^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{d w_2 - 1} \chi(i) B_{k, x} \left( w_1 x, \frac{w_1}{w_2} \right) w_1^{k-1} \right) \frac{w_1^k}{k!}
\]

By comparing the coefficients on the both sides of (2.21) and (2.22), we obtain the following theorem.
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Theorem 2.3. For \( w_1, w_2, d \in \mathbb{N} \), one has

\[
\sum_{i=0}^{dw_1-1} \chi(i) B_{k,x} \left( \frac{w_2 x + \frac{w_2}{w_1} i}{w_1} \right) w_1^{k-1} = \sum_{i=0}^{dw_1-1} \chi(i) B_{k,x} \left( \frac{w_1 x + \frac{w_1}{w_2} i}{w_2} \right) w_2^{k-1}.
\] (2.23)

Remark 2.4. Let \( x = 0 \) in Theorem 2.3. Then we see that

\[
\sum_{i=0}^{dw_1-1} \chi(i) B_{k,x} \left( \frac{w_2}{w_1} i \right) w_1^{k-1} = \sum_{i=0}^{dw_1-1} \chi(i) B_{k,x} \left( \frac{w_1}{w_2} i \right) w_2^{k-1}.
\] (2.24)

If we take \( w_2 = 1 \), then we have

\[
\sum_{i=0}^{dw_1-1} \chi(i) B_{k,x} \left( \frac{i}{w_1} \right) w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,x} \left( w_1 i \right).
\] (2.25)

Remark 2.5. Let \( \chi \) be trivial character. Then we can easily derive the “multiplication theorem for Bernoulli polynomials” from Theorems 2.2 and 2.3 (see [14]).

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References


