Research Article

Multiplicity Results for $p$-Laplacian with Critical Nonlinearity of Concave-Convex Type and Sign-Changing Weight Functions

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The multiple results of positive solutions for the following quasilinear elliptic equation:

$$-\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\lambda > 0$, $\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian, $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $1 < q < p < N$, $p^* = Np/(N-p)$ is the so-called critical Sobolev exponent and the weight functions $f, g$ are satisfying the following conditions:

(f1) $f \in C(\overline{\Omega})$ and $f^* = \max \{f, 0\} \neq 0$;
(f2) there exist $\beta_0, \rho_0 > 0$ and $x_0 \in \Omega$ such that $B(x_0, 2\rho_0) \subset \Omega$ and $f(x) \geq \beta_0$ for all $x \in B(x_0, 2\rho_0)$.

Without loss of generality, we assume that $x_0 = 0$.

(g1) $g \in C(\overline{\Omega})$ and $g^* = \max \{g, 0\} \neq 0$;
(g2) $|g^*|_{\infty} = g(0) = \max_{x \in \Omega} g(x)$;
(g3) \( g(x) > 0 \) for all \( x \in B(0,2\rho_0) \);

(g4) there exists \( \beta > N/(p-1) \) such that

\[
g(x) = g(0) + o(|x|^\beta) \quad \text{as} \quad x \to 0.
\]

For the weight functions \( f=g=1 \), \((E_{f,g})\) has been studied extensively. Historically, the role played by such concave-convex nonlinearities in producing multiple solutions was investigated first in the work [1]. They studied the following semilinear elliptic equation:

\[
-\Delta u = \lambda u^{q-1} + u^{p-1} \quad \text{in} \quad \Omega,
\]

\[
\begin{align*}
&u > 0 \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

for \( 1 < q < 2 \) and showed the existence of \( \lambda_0 > 0 \) such that (1.2) admits at least two solutions for all \( \lambda \in (0,\lambda_0) \) and no solution for \( \lambda > \lambda_0 \). Subsequently, in the work [2, 3], the corresponding quasilinear version has been studied

\[
-\Delta_p u = \lambda u^{q-1} + u^{r-1} \quad \text{in} \quad \Omega,
\]

\[
\begin{align*}
&u > 0 \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( 1 < p < N \) and \( 1 < q < p \). They obtained results similar to the results of [1] above, but only for some ranges of the exponents \( p \) and \( q \). We summarize their results in what follows.

**Theorem 1.1** (see [2, 3]). Assume that either \( 2N/(N+2) < p < 3 \) or \( p > 3 \), \( p > q > p^* - 2/(p-1) \). Then there exists \( \lambda_0 > 0 \) such that (1.3) admits at least two solutions for all \( \lambda \in (0,\lambda_0) \) and no solution for \( \lambda > \lambda_0 \).

It is possible to get complete multiplicity result for problem (1.3) if \( \Omega \) is taken to be a ball in \( \mathbb{R}^N \). Prashanth and Sreenadh [4] have studied (1.3) in the unit ball \( B^N(0;1) \) in \( \mathbb{R}^N \) and obtained the following results.

**Theorem 1.2** (see [4]). Let \( \Omega = B^N(0;1), 1 < p < N, 1 < q < p \). Then there exists \( \lambda_0 > 0 \) such that (1.3) admits at least two solutions for all \( \lambda \in (0,\lambda_0) \) and no solution for \( \lambda > \lambda_0 \). Additionally, if \( 1 < p < 2 \), then (1.3) admits exactly two solutions for all small \( \lambda > 0 \).

For \( p = 2 \), Tang [5] has studied the exact multiplicity about the following semilinear elliptic equation:

\[
-\Delta u = \lambda u^{q-1} + u^{r-1} \quad \text{in} \quad B^N(0;1),
\]

\[
\begin{align*}
&u > 0 \quad \text{in} \quad B^N(0;1), \\
&u = 0 \quad \text{on} \quad \partial B^N(0;1),
\end{align*}
\]

where \( 1 < q < 2 < r \leq 2N/(N-2) \) and \( N \geq 3 \). We also mention his result below.
Theorem 1.3 (see [5]). There exists $\lambda_0 > 0$ such that (1.4) admits exactly two solutions for $\lambda \in (0, \lambda_0)$, exactly one solution for $\lambda = \lambda_0$, and no solution for $\lambda > \lambda_0$.

To proceed, we make some motivations of the present paper. Recently, in [6] the author has considered (1.2) with subcritical nonlinearity of concave-convex type, $g \equiv 1$, and $f$ is a continuous function which changes sign in $\Omega$, and showed the existence of $\lambda_0 > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ via the extraction of Palais-Smale sequences in the Nehari manifold. In a recent work [7], the author extended the results of [6] to the quasilinear case with the more general weight functions $f, g$ but also having subcritical nonlinearity of concave-convex type. In the present paper, we continue the study of [7] by considering critical nonlinearity of concave-convex type and sign-changing weight functions $f, g$.

In this paper, we use a variational method involving the Nehari manifold to prove the multiplicity of positive solutions. The Nehari method has been used also in [8] to prove the existence of multiple for a singular elliptic problem. The existence of at least one solution can be obtained by using the same arguments as in the subcritical case [7]. The existence of a second solution needs different arguments due to the lack of compactness of the Palais-Smale sequences. For what, we need additional assumptions $(f_2)$ and $(g_2)$ to prove the compactness of the extraction of Palais-Smale sequences in the Nehari manifold (see Theorem 4.4). The multiplicity result is proved only for the parameter $\lambda \in (0, (q/p)\Lambda_1)$ (see Theorem 1.5) but for all $1 < p < N$ and $1 \leq q < p$. This is not the case in the papers referred [2, 3] where the multiplicity is global but not with the full range of $p, q$ and with the weight functions $f \equiv g \equiv 1$. Finally, we mention a recent contribution on $p$-Laplacian equation with changing sign nonlinearity by Figuereido et al. [9] which gives the global multiplicity but not with the full range of $p$ and $q$. The method used in the paper by Figuereido et al. is similar to the method introduced in [1].

In order to represent our main results, we need to define the following constant $\Lambda_1$. Set

$$
\Lambda_1 = \left( \frac{p - q}{(p^* - q)|g^*|_{\infty}} \right)^{(p-q)/(p^*-p)} \left( \frac{p^* - p}{|\Omega|^{(q-p^*)/p^*} S(N/p) - (N/p^*)q/(q/p)} > 0, \right. (1.5)
$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$ and $S$ is the best Sobolev constant (see (2.2)).

Theorem 1.4. Assume $(f_1)$ and $(g_1)$ hold. If $\lambda \in (0, \Lambda_1)$, then $(E_{\lambda f, g})$ admits at least one positive solution $u_1 \in C^{1, \alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

Theorem 1.5. Assume that $(f_1)$-$(f_2)$ and $(g_1)$-$(g_4)$ hold. If $\lambda \in (0, (q/p)\Lambda_1)$, then $(E_{\lambda f, g})$ admits at least two positive solutions $u_1, U_1 \in C^{1, \alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

This paper is organized as follows. In Section 2, we give some preliminaries and some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.4 and 1.5.

2. Preliminaries and Nehari Manifold

Throughout this paper, $(f_1)$ and $(g_1)$ will be assumed. The dual space of a Banach space $E$ will be denoted by $E^{-1}$. $W^{1,p}_0(\Omega)$ denotes the standard Sobolev space with the following
norm:
\[ \|u\|^p = \int_{\Omega} |\nabla u|^p \, dx. \]  \hspace{1cm} (2.1)

\( W_0^{1,p}(\Omega) \) with the norm \( \| \cdot \| \) is simply denoted by \( W \). We denote the norm in \( L^p(\Omega) \) by \( |\cdot|_p \) and the norm in \( L^p(\mathbb{R}^N) \) by \( |\cdot|_{L^p(\mathbb{R}^N)} \). \(|\Omega|\) is the Lebesgue measure of \( \Omega \). \( B(x, r) \) is a ball centered at \( x \) with radius \( r \). \( O(\epsilon^t) \) denotes \( |O(\epsilon^t)|/\epsilon^t \leq C \), \( o(\epsilon^t) \) denotes \( |o(\epsilon^t)|/\epsilon^t \to 0 \) as \( \epsilon \to 0 \), and \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \). \( C, C_i \) will denote various positive constants; the exact values of which are not important. \( S \) is the best Sobolev embedding constant defined by

\[ S = \inf_{u \in W \setminus \{0\}} \frac{|\nabla u|_p^p}{|u|_{p'}^p}. \]  \hspace{1cm} (2.2)

**Definition 2.1.** Let \( c \in \mathbb{R} \), \( E \) be a Banach space and \( I \in C^1(E, \mathbb{R}) \).

(i) \( \{u_n\} \) is a \((PS)_c\)-sequence in \( E \) for \( I \) if \( I(u_n) = c + o_n(1) \) and \( I'(u_n) = o_n(1) \) strongly in \( E^{-1} \) as \( n \to \infty \).

(ii) We say that \( I \) satisfies the \((PS)_c\) condition if any \((PS)_c\)-sequence \( \{u_n\} \) in \( E \) for \( I \) has a convergent subsequence.

Associated with \((E_{lf,g})\), we consider the energy functional \( J_\lambda \) in \( W \), for each \( u \in W \),

\[ J_\lambda (u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{q} \int_{\Omega} f|u|^q \, dx - \frac{1}{p'} \int_{\Omega} g|u|^{p'} \, dx. \]  \hspace{1cm} (2.3)

It is well known that \( J_\lambda \) is of \( C^1 \) in \( W \) and the solutions of \((E_{lf,g})\) are the critical points of the energy functional \( J_\lambda \) (see Rabinowitz [10]).

As the energy functional \( J_\lambda \) is not bounded below on \( W \), it is useful to consider the functional on the Nehari manifold

\[ \mathcal{N}_\lambda = \{ u \in W \setminus \{0\} : \langle J_\lambda'(u), u \rangle = 0 \}. \]  \hspace{1cm} (2.4)

Thus, \( u \in \mathcal{N}_\lambda \) if and only if

\[ \langle J_\lambda'(u), u \rangle = \|u\|^p - \lambda \int_{\Omega} f|u|^q \, dx - \int_{\Omega} g|u|^{p'} \, dx = 0. \]  \hspace{1cm} (2.5)

Note that \( \mathcal{N}_\lambda \) contains every nonzero solution of \((E_{lf,g})\). Moreover, we have the following results.

**Lemma 2.2.** The energy functional \( J_\lambda \) is coercive and bounded below on \( \mathcal{N}_\lambda \).
Proof. If $u \in \mathcal{N}_\lambda$, then by (f1), (2.5), and the Hölder inequality and the Sobolev embedding theorem we have

\[
J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^{p^*} - \lambda \left( \frac{p^* - q}{p^* q} \right) \int_\Omega f |u|^q \, dx
g \geq \frac{1}{N} \|u\|^{p^*} - \lambda \left( \frac{p^* - q}{p^* q} \right) S^{-q/p} |\Omega|^{(p^* - q)/p^*} \|u\|^p \left\| f^+ \right\|_\infty.
\]

Thus, $J_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$.

Define

$$q_\lambda(u) = \langle J'_\lambda(u), u \rangle.$$  \hfill (2.8)

Then for $u \in \mathcal{N}_\lambda$,

$$\langle q'_\lambda(u), u \rangle = p \|u\|^{p - q} - \lambda q \int_\Omega f |u|^q \, dx - p^* \int_\Omega g |u|^{p^*} \, dx$$

$$= (p - q) \|u\|^{p - q} - (p^* - q) \int_\Omega g |u|^{p^*} \, dx$$

$$= \lambda (p^* - q) \int_\Omega f |u|^q \, dx - (p^* - p) \|u\|^p.$$  \hfill (2.10)

Similar to the method used in Tarantello [11], we split $\mathcal{N}_\lambda$ into three parts:

$$\mathcal{N}^+_\lambda = \{ u \in \mathcal{N}_\lambda : \langle q'_\lambda(u), u \rangle > 0 \},$$

$$\mathcal{N}^0_\lambda = \{ u \in \mathcal{N}_\lambda : \langle q'_\lambda(u), u \rangle = 0 \},$$

$$\mathcal{N}^-_\lambda = \{ u \in \mathcal{N}_\lambda : \langle q'_\lambda(u), u \rangle < 0 \}.$$  \hfill (2.12)

Then, we have the following results.

Lemma 2.3. Assume that $u_\lambda$ is a local minimizer for $J_\lambda$ on $\mathcal{N}_\lambda$ and $u_\lambda \notin \mathcal{N}^0_\lambda$. Then $J'_\lambda(u_\lambda) = 0$ in $W^{-1}$.

Proof. Our proof is almost the same as that in Brown and Zhang [12, Theorem 2.3] (or see Binding et al. [13]).

Lemma 2.4. One has the following.

(i) If $u \in \mathcal{N}^+_\lambda$, then $\int_\Omega f |u|^q \, dx > 0$.

(ii) If $u \in \mathcal{N}^0_\lambda$, then $\int_\Omega f |u|^q \, dx > 0$ and $\int_\Omega g |u|^{p^*} \, dx > 0$.

(iii) If $u \in \mathcal{N}^-_\lambda$, then $\int_\Omega g |u|^{p^*} \, dx > 0$.

Proof. The proof is immediate from (2.10) and (2.11).
Moreover, we have the following result.

**Lemma 2.5.** If \( \lambda \in (0, \Lambda_1) \), then \( \mathcal{N}_1^\lambda = \emptyset \) where \( \Lambda_1 \) is the same as in (1.5).

**Proof.** Suppose otherwise that there exists \( \lambda \in (0, \Lambda_1) \) such that \( \mathcal{N}_1^\lambda \neq \emptyset \). Then by (2.10) and (2.11), for \( u \in \mathcal{N}_1^\lambda \), we have

\[
\|u\| = \frac{p^* - q}{p - q} \int_\Omega g|u|^{p'} dx, \quad (2.13)
\]

\[
\|u\| = \frac{p^* - q}{p^* - p} \int_\Omega f|u|^q dx.
\]

Moreover, by (f1), (g1), and the Hölder inequality and the Sobolev embedding theorem, we have

\[
\|u\| \geq \left( \frac{p - q}{(p^* - q)|g^*|_\infty} \right)^{1/(p' - p)},
\]

\[
\|u\| \leq \left( \frac{p^* - q}{p^* - p} S^{-q/p} |\Omega|^{(p' - q)/p'} \right)^{1/(p - q)}.
\]

This implies

\[
\lambda \geq \left( \frac{p - q}{(p^* - q)|g^*|_\infty} \right)^{(p - q)/(p' - p)} \int_\Omega |u|^{(q - p')/p'} S |\Omega|^{(q - p')/(p' - p)} = \Lambda_1, \quad (2.15)
\]

which is a contradiction. Thus, we can conclude that if \( \lambda \in (0, \Lambda_1) \), we have \( \mathcal{N}_1^\lambda = \emptyset \). \( \square \)

By Lemma 2.5, we write \( \mathcal{N}_1 = \mathcal{N}_1^+ \cup \mathcal{N}_1^- \) and define

\[
\alpha_1 = \inf_{u \in \mathcal{N}_1} J_1(u), \quad \alpha_1^+ = \inf_{u \in \mathcal{N}_1^+} J_1(u), \quad \alpha_1^- = \inf_{u \in \mathcal{N}_1^-} J_1(u). \quad (2.16)
\]

Then we get the following result.

**Theorem 2.6.** (i) If \( \lambda \in (0, \Lambda_1) \) and \( u \in \mathcal{N}_1^- \), then one has \( J_1(u) < 0 \) and \( \alpha_1 \leq \alpha_1^- < 0 \).

(ii) If \( \lambda \in (0, (q/p)\Lambda_1) \), then \( \alpha_1^- > d_0 \) for some positive constant \( d_0 \) depending on \( \lambda, p, q, N, S, |f^*|_\infty, |g^*|_\infty \), and \( |\Omega| \).

**Proof.** (i) Let \( u \in \mathcal{N}_1^- \). By (2.10), we have

\[
\frac{p - q}{p^* - q} \|u\|^p > \int_\Omega g|u|^{p'} dx, \quad (2.17)
\]
and so

\[ J_\lambda (u) = \left( \frac{1}{p} - \frac{1}{q} \right) \| u \|^p + \left( \frac{1}{q} - \frac{1}{p^*} \right) \int_\Omega g |u|^{p^*} dx \]

\[ < \left[ \left( \frac{1}{p} - \frac{1}{q} \right) + \left( \frac{1}{q} - \frac{1}{p^*} \right) \frac{p - q}{p^* - q} \| u \| \right] \| u \|^p \]

\[ = -\frac{p - q}{qN} \| u \|^p < 0. \quad (2.18) \]

Therefore, from the definition of \( a_\lambda, a_\lambda^+ \), we can deduce that \( a_\lambda \leq a_\lambda^+ < 0 \).

(ii) Let \( u \in \mathcal{N}_\lambda^- \). By (2.10), we have

\[ \frac{p - q}{p^* - q} \| u \|^p < \int_\Omega g |u|^{p^*} dx. \quad (2.19) \]

Moreover, by (g1) and the Sobolev embedding theorem, we have

\[ \int_\Omega g |u|^{p^*} dx \leq S^{-p'/p} \| u \|^p |g^+|_\infty. \quad (2.20) \]

This implies

\[ \| u \| > \left( \frac{p - q}{(p^* - q) |g^+|_\infty} \right)^{1/(p' - p)} S^{N/p^*}, \quad \forall u \in \mathcal{N}_\lambda^-. \quad (2.21) \]

By (2.7) in the proof of Lemma 2.2, we have

\[ J_\lambda (u) \geq \| u \|^q \left[ \frac{p^* - p}{p^* p} \| u \|_{p^* - q} - \lambda S^{-q/p} \frac{p^* - q}{p^* q} |\Omega|^{(p^* - q)/p^*} |f^+|_\infty \right] \]

\[ > \left( \frac{p - q}{(p^* - q) |g^+|_\infty} \right)^{q/(p' - p)} S^{N/p^*} \]

\[ \times \left[ \frac{p^* - p}{p^* p} S^{(p^* - q)N/p^*} \left( \frac{p - q}{(p^* - q) |g^+|_\infty} \right)^{(p - q)/(p' - p)} - \lambda S^{-q/p} \frac{p^* - q}{p^* q} |\Omega|^{(p^* - q)/p^*} |f^+|_\infty \right]. \quad (2.22) \]

Thus, if \( \lambda \in (0, (q/p)\Lambda_1) \), then

\[ J_\lambda (u) > d_0, \quad \forall u \in \mathcal{N}_\lambda^-, \quad (2.23) \]

for some positive constant \( d_0 = d_0(\lambda, p, q, N, S, |f^+|_\infty, |g^+|_\infty, |\Omega|) \). This completes the proof. □
For each \( u \in W \) with \( \int_{\Omega} g|u|^{p'} \, dx > 0 \), we write

\[
t_{\text{max}} = \left( \frac{(p-q)\|u\|^{p}}{(p^* - q)\int_{\Omega} g|u|^{p'} \, dx} \right)^{1/(p'-p)} > 0.
\] (2.24)

Then the following lemma holds.

**Lemma 2.7.** Let \( \lambda \in (0, \Lambda_1) \). For each \( u \in W \) with \( \int_{\Omega} g|u|^{p'} \, dx > 0 \), one has the following:

(i) if \( \int_{\Omega} f|u|^q \, dx \leq 0 \), then there exists a unique \( t^* > t_{\text{max}} \) such that \( t^* u \in \mathcal{N}_\lambda^- \) and

\[
f_\lambda(t^* u) = \sup_{t \geq 0} f_\lambda(tu),
\] (2.25)

(ii) if \( \int_{\Omega} f|u|^q \, dx > 0 \), then there exists unique \( 0 < t^* < t_{\text{max}} < t^- \) such that \( t^* u \in \mathcal{N}_\lambda^+ \), \( t^- u \in \mathcal{N}_\lambda^- \), and

\[
f_\lambda(t^* u) = \inf_{0 \leq t < t_{\text{max}}} f_\lambda(tu); \quad f_\lambda(t^- u) = \sup_{t \geq 0} f_\lambda(tu).
\] (2.26)

**Proof.** Fix \( u \in W \) with \( \int_{\Omega} g|u|^{p'} \, dx > 0 \). Let

\[
k(t) = t^{p-q}\|u\|^{p} - t^{p'-q}\int_{\Omega} g|u|^{p'} \, dx \quad \text{for } t \geq 0.
\] (2.27)

It is clear that \( k(0) = 0, k(t) \to -\infty \) as \( t \to \infty \). From

\[
k'(t) = (p-q)t^{p-q-1}\|u\|^{p} - (p^*-q)t^{p'-q-1}\int_{\Omega} g|u|^{p'} \, dx,
\] (2.28)

we can deduce that \( k'(t) = 0 \) at \( t = t_{\text{max}} \), \( k'(t) > 0 \) for \( t \in (0, t_{\text{max}}) \) and \( k'(t) < 0 \) for \( t \in (t_{\text{max}}, \infty) \). Then \( k(t) \) that achieves its maximum at \( t_{\text{max}} \) is increasing for \( t \in [0, t_{\text{max}}) \) and decreasing for \( t \in (t_{\text{max}}, \infty) \). Moreover,

\[
k(t_{\text{max}}) = \left( \frac{(p-q)\|u\|^{p}}{(p^* - q)\int_{\Omega} g|u|^{p'} \, dx} \right)^{(p-q)/(p'-p)}\|u\|^{p'}
\]

\[
- \left( \frac{(p-q)\|u\|^{p}}{(p^* - q)\int_{\Omega} g|u|^{p'} \, dx} \right)^{(p'-q)/(p'-p)}\int_{\Omega} g|u|^{p'} \, dx
\]

\[
= \|u\|^{q} \left[ \left( \frac{p-q}{p^*-q} \right)^{(p-q)/(p'-p)} - \left( \frac{p-q}{p^*-q} \right)^{(p'-q)/(p'-p)} \right] \left( \frac{\|u\|^{p'}}{\int_{\Omega} g|u|^{p'} \, dx} \right)^{(p-q)/(p'-p)}
\]\n
\[
\geq \|u\|^{q} \left( \frac{p^*-p}{p^*-q} \right) \left( \frac{p-q}{p^*-q} \right)^{sp'/q} \left( \frac{p^*-q}{p^*-q} \right)^{(p-q)/(p'-p)}.
\] (2.29)
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(i) We have \( \int_{\Omega} f |u|^q dx \leq 0 \). There exists a unique \( t^- > t_{\max} \) such that

\[
\lambda = \int_{\Omega} f |u|^q dx
\]

and \( k'(t^-) < 0 \). Now,

\[
(p - q)(t^-)^p ||u||^p - (p^+ - q)(t^-)^p \int_{\Omega} g |u|^{p^+} dx = (t^-)^{1+q} \left[ (p - q)(t^-)^{p-q-1} ||u||^p - (p^+ - q)(t^-)^{p^+-q-1} \int_{\Omega} g |u|^{p^+} dx \right]
\]

\[
= (t^-)^{1+q} k'(t^-) < 0,
\]

\[
\langle f(t^- u), t^- u \rangle = (t^-)^q ||u||^p - (t^-)^p \int_{\Omega} g |u|^{p^+} dx - (t^-)^q \int_{\Omega} f |u|^q dx = (t^-)^q \left[ k(t^-) - \lambda \int_{\Omega} f |u|^q dx \right] = 0.
\]

Then we have that \( t^- u \in \mathcal{A}_\lambda \). For \( t > t_{\max} \), we have

\[
(p - q) ||tu||^p - (p^+ - q) \int_{\Omega} g |tu|^{p^+} dx < 0, \quad \frac{d^2}{dt^2} f_\lambda(tu) < 0,
\]

\[
\frac{d}{dt} f_\lambda(tu) = t^{p-1} ||u||^p - t^{p^+-1} \int_{\Omega} g |u|^{p^+} dx - t^{q-1} \int_{\Omega} f |u|^q dx = 0 \quad \text{for } t = t^-.
\]

Thus, \( f_\lambda(t^- u) = \sup_{t \geq 0} f_\lambda(tu) \).

(ii) We have \( \int_{\Omega} f |u|^q dx > 0 \). By (2.29) and

\[
k(0) = 0 < \lambda \int_{\Omega} f |u|^q dx
\]

\[
\leq \lambda S^{-q/p} |\Omega|^{(p^+ - q)/p} ||u||^q |f^+|_\infty
\]

\[
< ||u||^q \left( \frac{p^+ - p}{p^+ - q} \right) \left( \frac{p - q}{p^+ - q} \right) g^{p^+/p} |f^+|_\infty
\]

\[
\leq k(t_{\max}) \quad \text{for } \lambda \in (0, \Lambda_1),
\]

there are unique \( t^+ \) and \( t^- \) such that \( 0 < t^+ < t_{\max} < t^- \),

\[
k(t^+) = \lambda \int_{\Omega} f |u|^q dx = k(t^-),
\]

\[
k'(t^+) = 0 > k'(t^-).
\]
Lemma 3.1.

We have $t^* u \in \mathcal{N}_x, t^- u \in \mathcal{N}_x$, and $f_1(t^- u) \geq f_1(tu) \geq f_1(t^* u)$ for each $t \in [t^*, t^-]$ and $f_1(t^* u) \leq f_1(tu)$ for each $t \in [0, t^-]$. Thus,

$$f_1(t^* u) = \inf_{0 \leq s \leq f_{\max}} f_1(tu), \quad f_1(t^- u) = \sup_{t \geq 0} f_1(tu).$$  \hspace{1cm} (2.34)

This completes the proof. \hfill \square

3. Proof of Theorem 1.4

First, we will use the idea of Tarantello [11] to get the following results.

**Lemma 3.1.** If $\lambda \in (0, \Lambda_1)$, then for each $u \in \mathcal{N}_x$, there exist $\varepsilon > 0$ and a differentiable function $\xi : B(0; \varepsilon) \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u - v) \in \mathcal{N}_x$, and

$$\langle \xi'(0), v \rangle = \frac{p \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda q |u|^{p^2} uv dx - p^* \int_{\Omega} g|u|^{p-2}uv dx}{(p-q)|u|^p - (p^* - q) \int_{\Omega} g|u|^{p^*} dx}$$  \hspace{1cm} (3.1)

for all $v \in W$.

**Proof.** For $u \in \mathcal{N}_x$, define a function $F : \mathbb{R} \times W \rightarrow \mathbb{R}$ by

$$F_u(\xi, w) = \langle f'_1(\xi(u - w)), \xi(u - w) \rangle$$

$$= \xi^p \int_{\Omega} |\nabla(u - w)|^p dx - \xi^q \int_{\Omega} f|u - w|q dx$$

$$- \xi^{p^*} \int_{\Omega} g|u - w|^{p^*} dx. \hspace{1cm} (3.2)$$

Then $F_u(1, 0) = \langle f'_1(u), u \rangle = 0$ and

$$\frac{d}{d\xi} F_u(1, 0) = p||u||^p - \lambda q \int_{\Omega} f|u|^q dx - p^* \int_{\Omega} g|u|^{p^*} dx$$

$$= (p-q)||u||^p - (p^* - q) \int_{\Omega} g|u|^{p^*} dx \neq 0. \hspace{1cm} (3.3)$$

According to the implicit function theorem, there exist $\varepsilon > 0$ and a differentiable function $\xi : B(0; \varepsilon) \subset W \rightarrow \mathbb{R}$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v \rangle = \frac{p \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda q |u|^{p^2} uv dx - p^* \int_{\Omega} g|u|^{p-2}uv dx}{(p-q)|u|^p - (p^* - q) \int_{\Omega} g|u|^{p^*} dx},$$

$$F_u(\xi(v), v) = 0, \hspace{0.5cm} \forall v \in B(0; \varepsilon),$$
which is equivalent to

$$\langle f'_1(\xi(v)(u-v)), \xi(v)(u-v) \rangle = 0, \quad \forall v \in B(0; e),$$

(3.5)

that is, $\xi(v)(u-v) \in N_\lambda$.

**Lemma 3.2.** Let $1 \in (0, \Lambda_1)$, then for each $u \in N_\lambda^-$, there exist $e > 0$ and a differentiable function $\xi^- : B(0; e) \subset W \rightarrow \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in N^-_\lambda$, and

$$\left\langle \left(\xi^-(v)\right)'(0), v \right\rangle = \frac{p\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla v \, dx - \lambda q \int_{\Omega} f|u|^{p-2}uv \, dx - p^* \int_{\Omega} g|u|^{p^*-2}uv \, dx}{(p-q)\|u\|^p - (p^*-q)\int_{\Omega} g|u|^{p^*} \, dx}$$

(3.6)

for all $v \in W$.

**Proof.** Similar to the argument in Lemma 3.1, there exist $e > 0$ and a differentiable function $\xi^- : B(0; e) \subset W \rightarrow \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in N^-_\lambda$ for all $v \in B(0; e)$. Since

$$\langle \varphi_1'(u), u \rangle = (p-q)\|u\|^p - (p^*-q)\int_{\Omega} g|u|^{p^*} \, dx < 0.$$ 

(3.7)

Thus, by the continuity of the function $\xi^-$, we have

$$\left\langle \varphi_1'(\xi^-(v)(u-v)), \xi^-(v)(u-v) \right\rangle = (p-q)\|\xi^-(v)(u-v)\|^p$$

$$- (p^*-q)\int_{\Omega} g|\xi^-(v)(u-v)|^{p^*} \, dx < 0,$$

(3.8)

if $\epsilon$ sufficiently small, this implies that $\xi^-(v)(u-v) \in N^-_\lambda$.

**Proposition 3.3.** (i) If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{H_\lambda}$-sequence $\{u_n\} \subset N_\lambda$ in $W$ for $f_1$.

(ii) If $\lambda \in (0, (q/p)\Lambda_1)$, then there exists a $(PS)_{H^-_\lambda}$-sequence $\{u_n\} \subset N^-_\lambda$ in $W$ for $f_1$.

**Proof.** (i) By Lemma 2.2 and the Ekeland variational principle [14], there exists a minimizing sequence $\{u_n\} \subset N_\lambda$ such that

$$f_1(u_n) < \alpha_3 + \frac{1}{n},$$

$$f_1(u_n) < f_1(w) + \frac{1}{n}\|w - u_n\| \quad \text{for each } w \in N_\lambda.$$

(3.9)

By $\alpha_3 < 0$ and taking $n$ large, we have

$$f_1(u_n) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u_n\|^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)\lambda \int_{\Omega} f|u_n|^q \, dx$$

$$< \alpha_3 + \frac{1}{n} < \frac{\alpha_3}{p},$$

(3.10)
From (2.7), (3.10), $\alpha_1 < 0$, and the Hölder inequality, we deduce that

$$|f^+|_\infty \lambda S^{-q/p} |\Omega|^{(p^*-q)/p'} \|u_n\|^q \geq \lambda \int_\Omega f|u_n|^q \, dx > \frac{-p'q}{p(p^* - q)} \alpha_1 > 0. \quad (3.11)$$

Consequently, $u_n \neq 0$ and putting together (3.10), (3.11), and the Hölder inequality, we obtain

$$\|u_n\| > \left[ \frac{-p'q}{p(p^* - q)} \int_\infty^{\lambda S^{q/p} |\Omega|^{(q-p')/p'}} |f^+| \, dx \right]^{1/q}, \quad (3.12)$$
$$\|u_n\| < \left[ \frac{p(p^* - q)}{q(p^* - p)} \lambda S^{-q/p} |\Omega|^{(p^*-q)/p'} |f^+|_\infty \right]^{1/(p-q)}.$$

Now, we show that

$$\|f_\lambda'(u_n)\|_{W^{-1}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

Apply Lemma 3.1 with $u_n$ to obtain the functions $\xi_n : B(0; \epsilon_n) \rightarrow \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathcal{M}_1$. Choose $0 < \rho < \epsilon_n$. Let $u \in W$ with $u \neq 0$ and let $w_\rho = \rho u/\|u\|$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathcal{M}_\lambda$, we deduce from (3.9) that

$$f_\lambda(\eta_\rho) - f_\lambda(u_n) \geq -\frac{1}{n} \|\eta_\rho - u_n\|, \quad (3.14)$$

and by the mean value theorem, we have

$$\langle f_\lambda'(u_n), \eta_\rho - u_n \rangle + \mathcal{O}(\|\eta_\rho - u_n\|) \geq -\frac{1}{n} \|\eta_\rho - u_n\|. \quad (3.15)$$

Thus,

$$\langle f_\lambda'(u_n), -w_\rho \rangle + (\xi_n(w_\rho) - 1) \langle f_\lambda'(u_n), (u_n - w_\rho) \rangle \geq -\frac{1}{n} \|\eta_\rho - u_n\| + \mathcal{O}(\|\eta_\rho - u_n\|). \quad (3.16)$$

Since $\xi_n(w_\rho)(u_n - w_\rho) \in \mathcal{M}_1$ and (3.16) it follows that

$$-\rho \langle f_\lambda'(u_n), \frac{u}{\|u\|} \rangle + (\xi_n(w_\rho) - 1) \langle f_\lambda'(u_n) - f_\lambda'(\eta_\rho), (u_n - w_\rho) \rangle \geq -\frac{1}{n} \|\eta_\rho - u_n\| + \mathcal{O}(\|\eta_\rho - u_n\|). \quad (3.17)$$
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Thus,

\[
\left\langle J'_\lambda (u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{\|\eta_\rho - u_n\|}{n\rho} + \frac{\sigma(\|\eta_\rho - u_n\|)}{\rho} + \frac{(\xi_n(w_\rho) - 1)}{\rho} \left( J'_\lambda (u_n) - J'_\lambda (\eta_\rho), (u_n - w_\rho) \right).
\]

(3.18)

Since \(\|\eta_\rho - u_n\| \leq \rho \xi_n(w_\rho) + |\xi_n(w_\rho) - 1| \|u_n\|\) and

\[
\lim_{\rho \to 0} \frac{|\xi_n(w_\rho) - 1|}{\rho} \leq \|\xi'_n(0)\|,
\]

(3.19)

if we let \(\rho \to 0\) in (3.18) for a fixed \(n\), then by (3.12) we can find a constant \(C > 0\), independent of \(\rho\), such that

\[
\left\langle J'_\lambda (u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n} (1 + \|\xi'_n(0)\|).
\]

(3.20)

The proof will be complete once we show that \(\|\xi'_n(0)\|\) is uniformly bounded in \(n\). By (3.1), (3.12), \((f1)\), \((g1)\), and the Hölder inequality and the Sobolev embedding theorem, we have

\[
\left\langle \xi'_n(0), v \right\rangle \leq \frac{b \|v\|}{\left| (p - q) \|u_n\|^p - (p^* - q) \int_\Omega g |u_n|^{p'} dx \right|}
\]

for some \(b > 0\).

(3.21)

We only need to show that

\[
\left| (p - q) \|u_n\|^p - (p^* - q) \int_\Omega g |u_n|^{p'} dx \right| > C
\]

(3.22)

for some \(C > 0\) and \(n\) large enough. We argue by contradiction. Assume that there exists a subsequence \(\{u_n\}\) such that

\[
(p - q) \|u_n\|^p - (p^* - q) \int_\Omega g |u_n|^{p'} dx = o_n(1).
\]

(3.23)

By (3.23) and the fact that \(u_n \in \mathcal{M}_\lambda\), we get

\[
\|u_n\|^p = \frac{p^* - q}{p - q} \int_\Omega g |u_n|^{p'} dx + o_n(1),
\]

\[
\|u_n\|^p = \frac{\lambda}{p^* - p} \int_\Omega f |u_n|^q dx + o_n(1).
\]

(3.24)
Moreover, by (f1), (g1), and the Hőlder inequality and the Sobolev embedding theorem, we have

\[ \|u_n\| \geq \left[ \frac{p - q}{(p^* - q) S^{p^* - p}} \right]^{1/(p^* - p)} + o_n(1), \]

\[ \|u_n\| \leq \left[ \frac{\lambda (p^* - q) |f^*| \infty S^{-q/p} |\Omega|^{(p^* - q)/p^*}}{p^* - p} \right]^{1/(p^* - q)} + o_n(1). \]  \tag{3.25}

This implies \( \lambda \geq \Lambda_1 \) which is a contradiction. We obtain

\[ \left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n}. \]  \tag{3.26}

This completes the proof of (i).

(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit detailed proof here. \( \square \)

Now, we establish the existence of a local minimum for \( J_\lambda \) on \( \mathcal{N}_\lambda^{+} \).

**Theorem 3.4.** If \( \lambda \in (0, \Lambda_1) \), then \( J_\lambda \) has a minimizer \( u_\lambda \) in \( \mathcal{N}_\lambda \) and it satisfies that

(i) \( J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^{+} \),

(ii) \( u_\lambda \) is a positive solution of \( (E_{f,g}) \) in \( C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \).

**Proof.** By Proposition 3.3(i), there exists a minimizing sequence \( \{u_n\} \) for \( J_\lambda \) on \( \mathcal{N}_\lambda \) such that

\[ J_\lambda(u_n) = \alpha_\lambda + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } W^{-1}. \]  \tag{3.27}

Since \( J_\lambda \) is coercive on \( \mathcal{N}_\lambda \) (see Lemma 2.2), we get that \( \{u_n\} \) is bounded in \( W \). Going if necessary to a subsequence, we can assume that there exists \( u_\lambda \in W \) such that

\[ u_n \rightharpoonup u_\lambda \quad \text{weakly in } W, \]

\[ u_n \to u_\lambda \quad \text{almost every where in } \Omega, \]

\[ u_n \to u_\lambda \quad \text{strongly in } L^s(\Omega) \forall 1 \leq s < p^*. \]  \tag{3.28}

First, we claim that \( u_\lambda \) is a nontrivial solution of \( (E_{f,g}) \). By (3.27) and (3.28), it is easy to see that \( u_\lambda \) is a solution of \( (E_{f,g}) \). From \( u_n \in \mathcal{N}_\lambda \) and (2.6), we deduce that

\[ \lambda \int_\Omega f|u_n|^q dx = \frac{q(p^* - p)}{p(p^* - q)} \|u_n\|^p - \frac{p^* q}{p^* - q} J_\lambda(u_n). \]  \tag{3.29}

Let \( n \to \infty \) in (3.29), by (3.27), (3.28), and \( \alpha_\lambda < 0 \), we get

\[ \int_\Omega f|u_\lambda|^q dx \geq -\frac{p^* q}{p^* - q} \alpha_\lambda > 0. \]  \tag{3.30}
Thus, \( u_\lambda \in \mathcal{A}_\lambda \) is a nontrivial solution of \((E_{f,g})\). Now we prove that \( u_n \to u_\lambda \) strongly in \( W \) and \( J_\lambda(u_\lambda) = \alpha_\lambda \). By (3.29), if \( u \in \mathcal{A}_\lambda \), then

\[
J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^p - \frac{p^* - q}{p^* q} \lambda \int_\Omega |f(u)|^q dx. 
\]

(3.31)

In order to prove that \( J_\lambda(u_\lambda) = \alpha_\lambda \), it suffices to recall that \( u_\lambda \in \mathcal{A}_\lambda \), by (3.31), and applying Fatou’s lemma to get

\[
\alpha_\lambda \leq J_\lambda(u_\lambda) = \frac{p^* - p}{p^* p} \|u_\lambda\|^p - \frac{p^* - q}{p^* q} \lambda \int_\Omega |f(u_\lambda)|^q dx 
\leq \liminf_{n \to \infty} \left( \frac{p^* - p}{p^* p} \|u_n\|^p - \frac{p^* - q}{p^* q} \lambda \int_\Omega |f(u_n)|^q dx \right) 
\leq \liminf_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda. 
\]

(3.32)

This implies that \( J_\lambda(u_\lambda) = \alpha_\lambda \) and \( \lim_{n \to \infty} \|u_n\|^p = \|u_\lambda\|^p \). Let \( v_n = u_n - u_\lambda \), then Brézis and Lieb lemma [15] implies that

\[
\|v_n\|^p = \|u_n\|^p - \|u_\lambda\|^p + o_n(1). 
\]

(3.33)

Therefore, \( u_n \to u_\lambda \) strongly in \( W \). Moreover, we have \( u_\lambda \in \mathcal{M}^+_\lambda \). On the contrary, if \( u_\lambda \in \mathcal{M}^-\lambda \), then by Lemma 2.7, there are unique \( t^*_0 \) and \( t^-_0 \) such that \( t^*_0 u_\lambda \in \mathcal{A}^+_\lambda \) and \( t^-_0 u_\lambda \in \mathcal{A}^-\lambda \). In particular, we have \( t^*_0 < t^-_0 = 1 \). Since

\[
\frac{d}{dt} J_\lambda(t^*_0 u_\lambda) = 0, \quad \frac{d^2}{dt^2} J_\lambda(t^*_0 u_\lambda) > 0, 
\]

(3.34)

there exists \( t^*_0 < t^-_0 \) such that \( J_\lambda(t^*_0 u_\lambda) < J_\lambda(t^-_0 u_\lambda) \). By Lemma 2.7,

\[
J_\lambda(t^*_0 u_\lambda) < J_\lambda(t^-_0 u_\lambda) \leq J_\lambda(t^-_0 u_\lambda) = J_\lambda(u_\lambda), 
\]

(3.35)

which is a contradiction. Since \( J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|) \) and \(|u_\lambda| \in \mathcal{M}^+ \), by Lemma 2.3 we may assume that \( u_\lambda \) is a nontrivial nonnegative solution of \((E_{f,g})\). Moreover, from \( f, g \in L^\infty(\Omega) \), then using the standard bootstrap argument (see, e.g., [16]) we obtain \( u_\lambda \in L^\infty(\Omega) \); hence by applying regularity results [17, 18] we derive that \( u_\lambda \in C^{1,a}(\Omega) \) for some \( a \in (0,1) \) and finally, by the Harnack inequality [19] we deduce that \( u_\lambda > 0 \). This completes the proof.

Now, we begin the proof of Theorem 1.4. By Theorem 3.4, we obtain \((E_{f,g})\) that has a positive solution \( u_\lambda \in C^{1,a}(\Omega) \) for some \( a \in (0,1) \).

4. Proof of Theorem 1.5

Next, we will establish the existence of the second positive solution of \((E_{f,g})\) by proving that \( J_\lambda \) satisfies the \((PS)_{\alpha_\lambda} \) condition.
**Lemma 4.1.** Assume that (f1) and (g1) hold. If \( \{ u_n \} \subset W \) is a (PS)_c-sequence for \( J_\lambda \), then \( \{ u_n \} \) is bounded in \( W \).

**Proof.** We argue by contradiction. Assume that \( \| u_n \| \to \infty \). Let \( \tilde{u}_n = u_n / \| u_n \|/c \). We may assume that \( \tilde{u}_n \rightharpoonup \tilde{u} \) in \( W \). This implies that \( \tilde{u}_n \rightharpoonup \tilde{u} \) strongly in \( L^s(\Omega) \) for all \( 1 \leq s < p^* \) and

\[
\frac{\lambda}{q} \int_\Omega f|\tilde{u}_n|^q dx = \frac{\lambda}{q} \int_\Omega f|\tilde{u}|^q dx + o_n(1). \tag{4.1}
\]

Since \( \{ u_n \} \) is a (PS)_c-sequence for \( J_\lambda \) and \( \| u_n \| \to \infty \), there hold

\[
\frac{1}{p} \int_\Omega |\nabla \tilde{u}_n|^p dx - \frac{\lambda \| u_n \|^{q-p}}{q} \int_\Omega f|\tilde{u}_n|^q dx - \| u_n \|^{p\lambda - p} \int_\Omega g|\tilde{u}_n|^p dx = o_n(1), \tag{4.2}
\]

\[
\int_\Omega |\nabla \tilde{u}_n|^p dx - \lambda \| u_n \|^{q-p} \int_\Omega f|\tilde{u}_n|^q dx - \| u_n \|^{p\lambda - p} \int_\Omega g|\tilde{u}_n|^p dx = o_n(1).
\]

From (4.1)-(4.2), we can deduce that

\[
\int_\Omega |\nabla \tilde{u}_n|^p dx = \frac{p(p^* - q)}{q(p^* - p)} \| u_n \|^{p\lambda - p} \int_\Omega f|\tilde{u}|^q dx + o_n(1). \tag{4.3}
\]

Since \( 1 \leq q < 2 \) and \( \| u_n \| \to \infty \), (4.3) implies

\[
\int_\Omega |\nabla \tilde{u}_n|^p dx \to 0, \quad \text{as } n \to \infty, \tag{4.4}
\]

which is contrary to the fact \( \| \tilde{u}_n \| = 1 \) for all \( n \). \( \square \)

**Lemma 4.2.** Assume that (f1) and (g1) hold. If \( \{ u_n \} \subset W \) is a (PS)_c-sequence for \( J_\lambda \) with \( c \in (0, (1/N)]g^{(N-p)/pN} \), then there exists a subsequence of \( \{ u_n \} \) converging weakly to a nontrivial solution of \( (E_{f,g}) \).

**Proof.** Let \( \{ u_n \} \subset W \) be a (PS)_c-sequence for \( J_\lambda \) with \( c \in (0, (1/N)]g^{(N-p)/pN} \). We know from Lemma 4.1 that \( \{ u_n \} \) is bounded in \( W \), and then there exists a subsequence of \( \{ u_n \} \) (still denoted by \( \{ u_n \} \) and \( u_0 \in W \) such that

\[
u_n \rightharpoonup u_0 \quad \text{weakly in } W,
\]

\[
u_n \rightharpoonup u_0 \quad \text{almost everywhere in } \Omega,
\]

\[
u_n \rightharpoonup u_0 \quad \text{strongly in } L^s(\Omega) \forall 1 \leq s < p^*.
\]

It is easy to see that \( J'_\lambda(u_0) = 0 \) and

\[
\lambda \int_\Omega f(x)|u_0|^q dx = \lambda \int_\Omega f(x)|u_0|^q dx + o_n(1). \tag{4.6}
\]
Next we verify that $u_0 \neq 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Setting
\[
l = \lim_{n \to \infty} \int_{\Omega} g|u_n|^{p'} \, dx. \tag{4.7}
\]
Since $f'_1(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.6), we can deduce that
\[
0 = \left( \lim_{n \to \infty} f'_1(u_n), u_n \right) = \lim_{n \to \infty} \left( \|u_n\|^p - \int_{\Omega} g|u_n|^{p'} \right) = \lim_{n \to \infty} \|u_n\|^p - l, \tag{4.8}
\]
that is,
\[
\lim_{n \to \infty} \|u_n\|^p = l. \tag{4.9}
\]
If $l = 0$, then we get $c = \lim_{n \to \infty} f'_1(u_n) = 0$, which contradicts with $c > 0$. Thus we conclude that $l > 0$. Furthermore, the Sobolev inequality implies that
\[
\|u_n\|^p \geq S \left( \int_{\Omega} |u_n|^{p'} \right)^{p/p'} \geq S \left( \int_{\Omega} \frac{g}{\|g\|_{p}} |u_n|^{p'} \right)^{p/p'} = S |\|g\|_{p}^{-(N-p)/N} \left( \int_{\Omega} g|u_n|^{p'} \right)^{p/p'} . \tag{4.10}
\]
Then as $n \to \infty$ we have
\[
l = \lim_{n \to \infty} \|u_n\|^p \geq S |\|g\|_{p}^{-(N-p)/p} \|g\|_{p}^{N/N} . \tag{4.11}
\]
which implies that
\[
l \geq |\|g\|_{p}^{-(N-p)/p} \|g\|_{p}^{N/N} . \tag{4.12}
\]
Hence, from (4.6) to (4.12) we get
\[
c = \lim_{n \to \infty} f'_1(u_n)
= \frac{1}{p} \lim_{n \to \infty} \|u_n\|^p - \frac{1}{q} \lim_{n \to \infty} \int_{\Omega} f|u_n|^q \, dx - \frac{1}{p^*} \lim_{n \to \infty} \int_{\Omega} g|u_n|^{p'} \, dx 
= \left( \frac{1}{p} - \frac{1}{p^*} \right) l 
\geq \frac{1}{N} |\|g\|_{p}^{-(N-p)/p} \|g\|_{p}^{N/N} . \tag{4.13}
\]
This is a contradiction to $c < (1/N)|\|g\|_{p}^{-(N-p)/p} \|g\|_{p}^{N/N}$. Therefore $u_0$ is a nontrivial solution of $(E_{f,g})$. □
Lemma 4.3. Assume that (f1)-(f2) and (g1)-(g4) hold. Then for any $\lambda > 0$, there exists $v_1 \in W$ such that

$$\sup_{t \geq 0} I_\lambda(t v_1) < \frac{1}{N} g^{+\infty}_{(N-p)/p} S^{N/p}. \quad (4.14)$$

In particular, $\alpha_1^\lambda < (1/N)|g^{+\infty}_{(N-p)/p} S^{N/p}$ for all $\lambda \in (0, \Lambda_1)$ where $\Lambda_1$ is as in (1.5).

Proof. For convenience, we introduce the following notations:

$$I(u) = \int \Omega \left\{ \frac{|\nabla u|^p}{p} \right\} dx,$$

$$\chi_{B(0,2\rho_0)} = \begin{cases} 1 & \text{if } x \in B(0,2\rho_0), \\ 0 & \text{if } x \notin B(0,2\rho_0), \end{cases} \quad (4.15)$$

$$Q(u) = \frac{|\nabla u|^p}{\left(\sqrt{\frac{|\nabla \chi_{B(0,2\rho_0)}|}{p}}\right)^{1/p}}.$$

From (g3) to (g4), we know that there exists $\delta_0 \in (0, \rho_0)$ such that for all $x \in B(0, 2\delta_0)$,

$$g(x) = g(0) + o\left(|x|^{\beta}\right) \quad \text{for some } \beta > \frac{N}{p-1}. \quad (4.16)$$

Motivated by some ideas of selecting cut-off functions in [20, Lemma 4.1], we take such cut-off function $\eta(x)$ that satisfies $\eta(x) \in C^0(\overline{B(0,2\delta_0)})$, $\eta(x) = 1$ for $|x| < \delta_0$, $\eta(x) = 0$ for $|x| > 2\delta_0$, $0 \leq \eta \leq 1$, and $|\nabla \eta| \leq C$. Define, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-p)/p} \eta(x)}{\left(\varepsilon + |x|^{(p-1)}}\right)^{(N-p)/p}. \quad (4.17)$$

Step 1. Show that $\sup_{t \geq 0} I(t u_\varepsilon) \leq (1/N)|g^{+\infty}_{(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p})$.

On that purpose, we need to establish the following estimates (as $\varepsilon \to 0$):

$$\left|g^{+\infty}_{(N-p)/p} \chi_{B(0,2\rho_0)}\right|^p = \left|g^{+\infty}_{(N-p)/p} \Lambda_{(N-p)/p}\right| + O(\varepsilon^{N/p}), \quad (4.18)$$

$$|\nabla u_\varepsilon|^p = |\nabla u|^p + O(\varepsilon^{(N-p)/p}), \quad (4.19)$$
where \( U(x) = (1 + (x)_{1}^{p/(p-1)})^{-(N-p)/p} \in W^{1,p}(\mathbb{R}^{N}) \) is a minimizer of \( \{ ||\nabla u_{*}||_{p}^{p} \}_{u \in W^{1,p}(\mathbb{R}^{N})} \), that is,

\[
\frac{||\nabla U||_{p}(\mathbb{R}^{N})}{||U||_{p}(\mathbb{R}^{N})} = S = \inf_{u \in W^{1,p}(\mathbb{R}^{N})} \frac{||\nabla u||_{p}(\mathbb{R}^{N})}{||u||_{p}^{p}(\mathbb{R}^{N})},
\]

and \( \omega_{N} = 2\pi^{N/2}/N! (N/2) \) which is the volume of the unit ball \( B(0, 1) \) in \( \mathbb{R}^{N} \). We only show that equality (4.18) is valid; proofs of (4.19) are very similar to [20]. In view of (4.17), we get that

\[
\left| (g_{B(0,2\delta)})^{1/p’} u_{e} \right|_{p’} = \int_{B(0,\delta)} g(x)|u_{e}|^{p’} dx = \int_{\mathbb{R}^{N}} \frac{\varepsilon^{N/p} \eta^{p’}(x)g(x)}{(\varepsilon + |x|^{p/(p-1)})^{N}} dx.
\]

On the other hand, let \( x = \varepsilon^{(p-1)/p} y \), we can deduce that

\[
\int_{\mathbb{R}^{N}} \frac{1}{\varepsilon + |x|^{p/(p-1)}}^{N} dx = \varepsilon^{-N/p} \int_{\mathbb{R}^{N}} \frac{1}{1 + |y|^{p/(p-1)}}^{N} dy = \varepsilon^{-N/p} ||U||_{L^{p^{*}}(\mathbb{R}^{N})}^{p^{*}}.
\]

Combining with \( g(0) = g^{*}_{\infty} \) and the equalities above, we have

\[
\varepsilon^{-N/p} ||g^{*}_{\infty}||_{\infty} ||U||_{L^{p^{*}}(\mathbb{R}^{N})} - \varepsilon^{-N/p} \left( g_{B(0,2\delta)} \right)^{1/p’} u_{e} ||_{p’}^{p’}
\]

\[
= \int_{\mathbb{R}^{N} \setminus B(0,\delta)} \frac{g(0)}{(\varepsilon + |x|^{p/(p-1)})^{N}} dx + \int_{B(0,\delta)} \frac{g(0) - g(x)}{(\varepsilon + |x|^{p/(p-1)})^{N}} dx,
\]

hence

\[
0 \leq \varepsilon^{-N/p} ||g^{*}_{\infty}||_{\infty} ||U||_{L^{p^{*}}(\mathbb{R}^{N})} - \varepsilon^{-N/p} \left( g_{B(0,2\delta)} \right)^{1/p’} u_{e} ||_{p’}^{p’}
\]

\[
\leq \int_{\mathbb{R}^{N} \setminus B(0,\delta)} \frac{g(0)}{|x|^{Np/(p-1)}} dx + \int_{B(0,\delta)} \frac{\alpha(|x|)}{|x|^{Np/(p-1)}} dx
\]

\[
\leq \int_{\mathbb{R}^{N} \setminus B(0,\delta)} \frac{g(0)}{|x|^{Np/(p-1)}} dx + \int_{B(0,\delta)} \frac{\alpha(|x|)}{|x|^{Np/(p-1)}} dx
\]

\[
= N\omega_{N} \int_{\delta_{0}}^{\infty} \frac{r^{N-1}g(0)}{r^{pN/(p-1)}} dr + \int_{\delta_{0}}^{\infty} \frac{\alpha(r)}{r^{pN/(p-1)}} dr
\]

\[
= (p-1)\omega_{N} \delta_{0}^{N/(p-1)} g(0) + \frac{\alpha(1)\delta_{0}^{N/(p-1)}}{\beta - (N/(p-1))} \leq C_{1} = \text{Const.}
\]
which leads to
\[
0 \leq 1 - |g^+|_{\infty}^{-1} \left( (g \chi_{B(0,2\rho)})^{1/p} u_{\epsilon} \right)_p^p |U|_L^p \leq C_1 |g^+|_{\infty}^{-1} |U|_{L^p} \leq 1, \quad (4.25)
\]
that is,
\[
1 - C_1 |g^+|_{\infty}^{-1} |U|_{L^p} \leq |g^+|_{\infty}^{-1} \left( (g \chi_{B(0,2\rho)})^{1/p} u_{\epsilon} \right)_p^p |U|_{L^p} \leq 1. \quad (4.26)
\]
Now, let \( \epsilon \) be small enough such that \( C_1 |g^+|_{\infty}^{-1} |U|_{L^p} \epsilon^{N/p} < 1 \), then from (4.26) we can deduce that
\[
1 - C_1 |g^+|_{\infty}^{-1} |U|_{L^p} \leq \left( 1 - C_1 |g^+|_{\infty}^{-1} |U|_{L^p} \epsilon^{N/p} \right)^{p/p'} \leq |g^+|_{\infty}^{-(N-p)/N} \left( (g \chi_{B(0,2\rho)})^{1/p} u_{\epsilon} \right)_p^p |U|_{L^p} \leq 1, \quad (4.27)
\]
which yields that
\[
|g^+|_{\infty}^{-(N-p)/N} |U|_{L^p} - C_1 |g^+|_{\infty}^{-(N-p)/N} |U|_{L^p} \leq \left( (g \chi_{B(0,2\rho)})^{1/p} u_{\epsilon} \right)_p^p \leq |g^+|_{\infty}^{-(N-p)/N} |U|_{L^p}, \quad (4.28)
\]
equivalently, equality (4.18) is valid.

Combining (4.18) and (4.19), we obtain that
\[
Q(u_{\epsilon}) = \frac{|\nabla U|_{L^p}^{p} + O(\epsilon^{(N-p)/p})}{|g^+|_{\infty}^{-(N-p)/N} |U|_{L^p} + O(\epsilon^{N/p})} \leq \frac{|\nabla U|_{L^p}^{p} + O(\epsilon^{(N-p)/p})}{|U|_{L^p} + O(\epsilon^{N/p})} \quad (4.29)
\]
Hence
\[
Q(u_{\epsilon}) - |g^+|_{\infty}^{-(N-p)/N} S = |g^+|_{\infty}^{-(N-p)/N} \left[ \frac{|\nabla U|_{L^p}^{p} + O(\epsilon^{(N-p)/p})}{|U|_{L^p} + O(\epsilon^{N/p})} - |U|_{L^p}^{p} \right] \leq \frac{|U|_{L^p}^{p} O(\epsilon^{(N-p)/p}) - |\nabla U|_{L^p}^{p} O(\epsilon^{N/p})}{|U|_{L^p}^{p} + O(\epsilon^{N/p})} \quad (4.30)
\]
\[
= O(\epsilon^{(N-p)/p}).
\]
Using the fact that

$$\max_{t \geq 0} \left( \frac{t^p}{p} - \frac{t^{p^*}}{p^*} b \right) = \frac{1}{N} \left( \frac{a}{b^p/p} \right)^{N/p}$$

for any \(a, b > 0\), we can deduce that

$$\sup_{t \geq 0} I(tu_\epsilon) = \frac{1}{N}(Q(u_\epsilon))^{N/p}. \quad (4.32)$$

From (4.30), we conclude that \(\sup_{t \geq 0} I(tu_\epsilon) \leq (1/N)|g^+|^{-(N-p)/p} S^{N/p} + O(\epsilon^{(N-p)/p})\).

**Step 2.** We claim that for any \(\lambda > 0\) there exists a constant \(\epsilon_1 > 0\) such that \(\sup_{t \geq 0} J_\lambda(tu_\epsilon) < (1/N)|g^+|^{-(N-p)/p} S^{N/p}\).

Using the definitions of \(J_\lambda, u_\epsilon\) and by (f2), (g3), we get

$$J_\lambda(tu_\epsilon) \leq \frac{t^p}{p} |\nabla u_\epsilon|_p^p, \quad \forall t \geq 0, \forall \lambda > 0. \quad (4.33)$$

Combining this with (4.19), let \(\epsilon \in (0, 1)\), then there exists \(t_0 \in (0, 1)\) independent of \(\epsilon\) such that

$$\sup_{0 \leq t \leq t_0} J_\lambda(tu_\epsilon) < \frac{1}{N}|g^+|^{-(N-p)/p} S^{N/p}, \quad \forall \lambda > 0, \forall \epsilon \in (0, 1). \quad (4.34)$$

Using the definitions of \(J_\lambda, u_\epsilon\), and by the results in Step 1 and (f2), we have

$$\sup_{t \geq 0} J_\lambda(tu_\epsilon) = \sup_{t \geq t_0} \left( I(tu_\epsilon) - \frac{t^q}{q} \lambda \int f(x)|u_\epsilon|^q dx \right) \leq \frac{1}{N}|g^+|^{-(N-p)/p} S^{N/p} + O(\epsilon^{(N-p)/p}) - \frac{t_0^q}{q} \int_{B(0, \delta_0)} |u_\epsilon|^q dx. \quad (4.35)$$

Let \(0 < \epsilon \leq \delta_0^{p/(p-1)}\), we have

$$\int_{B(0, \delta_0)} |u_\epsilon|^q dx = \int_{B(0, \delta_0)} \frac{\epsilon^q(N-p)/p^2}{(\epsilon + |x|^{p/(p-1)})^{((N-p)/p)q}} dx \geq \int_{B(0, \delta_0)} \frac{\epsilon^q(N-p)/p^2}{(2\delta_0^{p/(p-1)})^{((N-p)/p)q}} dx \quad (4.36)$$

$$= C_2(N, p, q, \delta_0) \epsilon^{q(N-p)/p^2}.$$
Combining (4.35) and (4.36), for all \( \varepsilon \in (0, \delta_0^{p/(p-1)}) \), we get
\[
\sup_{t \geq 0} J_\lambda(tu_\varepsilon) \leq \frac{1}{N} \|g^+\|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p}) - \frac{T_0}{q} \beta_0 C_2 \lambda e^{q(N-p)/p}. \tag{4.37}
\]

Hence, for any \( \lambda > 0 \), we can choose small positive constant \( \varepsilon_1 < \min\{1, \delta_0^{p/(p-1)}\} \) such that
\[
O(\varepsilon_1^{(N-p)/p}) - \frac{T_0}{q} \beta_0 C_2 \lambda e^{q(N-p)/p} < 0. \tag{4.38}
\]

From (4.34), (4.37), (4.38), we can deduce that for any \( \lambda > 0 \), there exists \( \varepsilon_1 > 0 \) such that
\[
\sup_{t \geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N} \|g^+\|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.39}
\]

**Step 3.** Prove that \( \alpha^-_1 < (1/N) S^{N/p} \) for all \( \lambda \in (0, \Lambda_1) \).

By (f2), (g2), and the definition of \( u_\varepsilon \), we have
\[
\int_\Omega f(x)|u_\varepsilon|^q\,dx > 0, \quad \int_\Omega g(x)|u_\varepsilon|^p\,dx > 0. \tag{4.40}
\]

Combining this with Lemma 2.7(ii), from the definition of \( \alpha^-_1 \) and the results in Step 2, for any \( \lambda \in (0, \Lambda_1) \), we obtain that there exists \( t_\varepsilon > 0 \) such that \( t_\varepsilon u_\varepsilon \in \mathcal{N}_\lambda^- \) and
\[
\alpha^-_1 \leq J_\lambda(t_\varepsilon u_\varepsilon) \leq \sup_{t \geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N} \|g^+\|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.41}
\]

This completes the proof. \( \square \)

Now, we establish the existence of a local minimum of \( J_\lambda \) on \( \mathcal{N}_\lambda^- \).

**Theorem 4.4.** If \( \lambda \in (0,(q/p)\Lambda_1) \), then \( J_\lambda \) satisfies the (PS)\( \alpha^-_1 \) condition. Moreover, \( J_\lambda \) has a minimizer \( U_\lambda \) in \( \mathcal{N}_\lambda^- \) and satisfies that
(i) \( J_\lambda(U_\lambda) = \alpha^-_1 \); 
(ii) \( U_\lambda \) is a positive solution of \( (E_{f,g}) \) in \( C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0,1) \), where \( \Lambda_1 \) is as in (1.5).

**Proof.** If \( \lambda \in (0,(q/p)\Lambda_1) \), then by Theorem 2.6(ii), Proposition 3.3(ii), and Lemma 4.3, there exists a (PS)\( \alpha^-_1 \)-sequence \( \{u_n\} \subset \mathcal{N}_\lambda^- \) in \( W \) for \( J_\lambda \) with \( \alpha^-_1 \in (0,(1/N)\|g^+\|_\infty^{-(N-p)/p} S^{N/p}) \). From Lemma 4.2, there exists a subsequence still denoted by \( \{u_n\} \) and nontrivial solution \( U_\lambda \in W \) of \( (E_{f,g}) \) such that \( u_n \rightharpoonup U_\lambda \) weakly in \( W \). Now we prove that \( u_n \rightarrow U_\lambda \) strongly in \( W \) and \( J_\lambda(U_\lambda) = \alpha^-_1 \). By (3.29), if \( u \in \mathcal{N}_\lambda \), then
\[
J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^p - \frac{p^* - q}{p^* q} \int_\Omega f|u|^q\,dx. \tag{4.42}
\]
First, we prove that $U_\lambda \in \mathcal{N}_\lambda$. On the contrary, if $U_\lambda \not\in \mathcal{N}_\lambda$, then by Lemma 2.7, there exists a unique $t_\lambda^*$ such that $t_\lambda^* U_\lambda \in \mathcal{N}_\lambda$. Since $u_n \in \mathcal{N}_\lambda$, $J_\lambda(u_n) \geq J_\lambda(t u_n)$ for all $t \geq 0$ and by (4.42), we have
\[
\alpha_\lambda^* \leq J_\lambda(t_\lambda^* U_\lambda) < \lim_{n \to \infty} J_\lambda(t U_n) \leq \lim_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda^*,
\]
and this is contradiction.

In order to prove that $J_\lambda(U_\lambda) = \alpha_\lambda^*$, it suffices to recall that $u_n, U_\lambda \in \mathcal{N}_\lambda$ for all $n$, by (4.42), and applying Fatou’s lemma to get
\[
\alpha_\lambda^* \leq J_\lambda(U_\lambda) = \frac{p^* - p}{p^* p} \|U_\lambda\|^p - \frac{p^* - q}{p^* q} \int_\Omega f|U_\lambda|^q dx \leq \liminf_{n \to \infty} \left( \frac{p^* - p}{p^* p} \|u_n\|^p - \frac{p^* - q}{p^* q} \int_\Omega f|u_n|^q dx \right) \leq \liminf_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda^*.
\]
This implies that $J_\lambda(U_\lambda) = \alpha_\lambda^*$ and $\lim_{n \to \infty} \|u_n\|^p = \|U_\lambda\|^p$. Let $v_n = u_n - U_\lambda$, then Brézis and Lieb lemma [15] implies that
\[
\|v_n\|^p = \|u_n\|^p - \|U_\lambda\|^p + o_n(1).
\]
Therefore, $u_n \to U_\lambda$ strongly in $W$.

Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda$, by Lemma 2.3 we may assume that $U_\lambda$ is a nontrivial nonnegative solution of $(E_{f,g})$. Finally, by using the same arguments as in the proof of Theorem 3.4, for all $\lambda \in (0, (q/p)\Lambda_1)$, we have that $U_\lambda$ is a positive solution of $(E_{f,g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

Now, we complete the proof of Theorem 1.5. By Theorems 3.4 and 4.4, if $\lambda \in (0, (q/p)\Lambda_1)$, then we obtain $(E_{f,g})$ that has two positive solutions $u_1$ and $U_\lambda$ such that $u_1 \in \mathcal{N}_\lambda^+, U_\lambda \in \mathcal{N}_\lambda^-$, and $u_\lambda, U_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that $u_1$ and $U_\lambda$ are distinct.

References


