Research Article

Homomorphisms and Derivations in C*-Ternary Algebras

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In 2006, C. Park proved the stability of homomorphisms in C*-ternary algebras and of derivations on C*-ternary algebras for the following generalized Cauchy-Jensen additive mapping: 

\[ 2f\left(\frac{\sum_{i=1}^{p} x_i}{2} + \sum_{j=1}^{d} y_j\right) = \sum_{i=1}^{p} f(x_i) + 2\sum_{j=1}^{d} f(y_j). \]

In this note, we improve and generalize some results concerning this functional equation.

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1. Introduction and Preliminaries


**Theorem 1.1** (Th. M. Rassias). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \]  

(1.1)

for all \( x, y \in E \), where \( \varepsilon \) and \( p \) are constants with \( \varepsilon > 0 \) and \( p < 1 \). Then the limit

\[ L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]  

(1.2)
exists for all \( x \in E \), and \( L : E \to E' \) is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p
\]
for all \( x \in E \). If \( p < 0 \), then inequality (1.1) holds for \( x, y \neq 0 \) and (1.3) for \( x \neq 0 \). Also, if for each \( x \in E \) the mapping \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is linear.

It was shown by Gajda [5] as well as by Rassias and Šemrl [6] that one cannot prove a Rassias’s type theorem when \( p = 1 \). The counter examples of Gajda [5] as well as of Rassias and Šemrl [6] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings; compare Găvruța [7] and Jung [8], who among others studied the stability of functional equations. Theorem 1.1 provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (cf. the books of Czerwik [9], Hyers et al. [10]).

**Theorem 1.2** (Rassias [11–13]). Let \( X \) be a real normed linear space and \( Y \) a real Banach space. Assume that \( f : X \to Y \) is a mapping for which there exist constants \( \theta \geq 0 \) and \( p, q \in \mathbb{R} \) such that \( r = p + q \neq 1 \) and \( f \) satisfies the functional inequality (Cauchy-Găvruța-Rassias inequality)
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q
\]
for all \( x, y \in X \). Then there exists a unique additive mapping \( L : X \to Y \) satisfying
\[
\|f(x) - L(x)\| \leq \frac{\theta}{2^r - 2} \|x\|^r
\]
for all \( x \in X \). If, in addition, \( f : X \to Y \) is a mapping such that the transformation \( t \to f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( L \) is linear.

For the case \( r = 1 \), a counter example has been given by Găvruța [14]. The stability in Theorem 1.2 involving a product of different powers of norms is called *Ulam-Găvruța-Rassias stability* (see [15–17]). In 1994, a generalization of Theorems 1.1 and 1.2 was obtained by Găvruța [7], who replaced the bounds \( \varepsilon (\|x\|^p + \|y\|^p) \) and \( \theta \|x\|^p \|y\|^q \) by a general control function \( \varphi(x, y) \). During past few years several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see [16–44]).

Following the terminology of [45], a nonempty set \( G \) with a ternary operation \([\cdot, \cdot, \cdot] : G \times G \times G \to G\) is called a *ternary groupoid* and is denoted by \((G, [\cdot, \cdot, \cdot])\). The ternary groupoid \((G, [\cdot, \cdot, \cdot])\) is called *commutative* if \([x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]\) for all \( x_1, x_2, x_3 \in G \) and all permutations \( \sigma \) of \( \{1, 2, 3\} \).

If a binary operation \( \circ \) is defined on \( G \) such that \([x, y, z] = (x \circ y) \circ z\) for all \( x, y, z \in G \), then we say that \([\cdot, \cdot, \cdot]\) is derived from \( \circ \). We say that \((G, [\cdot, \cdot, \cdot])\) is a *ternary semigroup* if the operation \([\cdot, \cdot, \cdot]\) is *associative*, that is, if \([[[x, y, z], u, v], v] = [x, [y, z, u], v] = [x, y, [z, u, v]]\) holds for all \( x, y, z, u, v \in G \) (see [46]).

A \( C^* \)-ternary algebra is a complex Banach space \( A \), equipped with a ternary product \((x, y, z) \mapsto [x, y, z] \) of \( A^3 \) into \( A \), which are \( \mathbb{C} \)-linear in the outer variables, conjugate \( \mathbb{C} \)-linear
in the middle variable, and associative in the sense that 
\[ [x, y, [z, w, v]] = [x, [w, z, y], v] = 
[[x, y, z], w, v], \]
and satisfies 
\[ \|[[x, y, z]]\| \leq \|x\| \cdot \|y\| \cdot \|z\| \]
and 
\[ \|[x, x, x]\| = \|x\|^3 \] (see [45, 47]).

Every left Hilbert \( C \)-module is a \( C \)-ternary algebra via the ternary product \( [x, y, z] := (x, y)z \).

If a \( C \)-ternary algebra \( (A, [\cdot, \cdot, \cdot]) \) has an identity, that is, an element \( e \in A \) such that \( x = [x, e, e] = [e, e, x] \) for all \( x \in A \), then it is routine to verify that \( A \), endowed with \( x \circ y := [x, e, y] \) and \( x^* := [e, x, e] \), is a unital \( C \)-algebra. Conversely, if \( (A, \circ) \) is a unital \( C \)-algebra, then \( [x, y, z] := x \circ y \circ z \) makes \( A \) into a \( C \)-ternary algebra.

A \( C \)-linear mapping \( H : A \rightarrow B \) is called a \( C \)-ternary algebra homomorphism if
\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]
for all \( x, y, z \in A \). If, in addition, the mapping \( H \) is bijective, then the mapping \( H : A \rightarrow B \) is called a \( C \)-ternary algebra isomorphism. A \( C \)-linear mapping \( \delta : A \rightarrow A \) is called a \( C \)-ternary derivation if
\[
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]
\]
for all \( x, y, z \in A \) (see [23, 45, 48]).

Let \( (A, \circ) \) be a \( C \)-algebra and \( [x, y, z] := x \circ y \circ z \) for all \( x, y, z \in A \). The mapping \( H : A \rightarrow A \) defined by \( H(x) = -ix \) is a \( C \)-ternary algebra isomorphism. Let \( a \in A \) with \( a^* = a \). The mapping \( \delta_a : A \rightarrow A \) defined by \( \delta_a(x) = i(ax - xa) \) is a \( C \)-ternary derivation.

There are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation (cf. [49–51]).

Throughout this paper, assume that \( p, d \) are nonnegative integers with \( p + d \geq 3 \), and that \( A \) and \( B \) are \( C \)-ternary algebras.

### 2. Stability of Homomorphisms in \( C \)-Ternary Algebras

The stability of homomorphisms in \( C \)-ternary algebras has been investigated in [31] (see also [37]). In this note, we improve some results in [31]. For a given mapping \( f : A \rightarrow B \), we define
\[
C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) := \frac{2f}{\frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j} - \sum_{j=1}^p \mu f(x_j) - 2\sum_{j=1}^d \mu f(y_j)
\]
for all \( \mu \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and all \( x_1, \ldots, x_p, y_1, \ldots, y_d \in A \).

One can easily show that a mapping \( f : A \rightarrow B \) satisfies
\[
C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0
\]
for all $\mu \in T^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

(2.3)

for all $\mu, \lambda \in T^1$ and all $x, y \in A$.

We will use the following lemmas in this paper.

**Lemma 2.1** (see [30]). Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in T^1$. Then the mapping $f$ is $C$-linear.

**Lemma 2.2.** Let $\{x_n\}_n, \{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in $A$. Then the sequence $\{[x_n, y_n, z_n]\}_n$ is convergent in $A$.

**Proof.** Let $x, y, z \in A$ such that

$$\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.$$  

(2.4)

Since

$$[x_n, y_n, z_n] - [x, y, z] = [x_n - x, y_n - y, z_n - z] + [x_n - x, y_n, z_n] + [x, y_n - y, z_n] + [x_n, y, z_n - z]$$

(2.5)

for all $n$, we get

$$\| [x_n, y_n, z_n] - [x, y, z] \| \leq \| x_n - x \| \| y_n - y \| \| z_n - z \| + \| x_n - x \| \| y_n \| \| z \|$$

$$+ \| x \| \| y_n - y \| \| z_n \| + \| x_n \| \| y \| \| z_n - z \|$$

(2.6)

for all $n$. So

$$\lim_{n \to \infty} [x_n, y_n, z_n] = [x, y, z].$$

(2.7)

This completes the proof. □

**Theorem 2.3** (see [31]). Let $r$ and $\theta$ be nonnegative real numbers such that $r \not\in [1,3]$, and let $f : A \to B$ be a mapping such that

$$\| C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) \|_B \leq \theta \left( \sum_{j=1}^p \| x_j \|_A^r + \sum_{j=1}^d \| y_j \|_A^r \right),$$

(2.8)

$$\| f([x, y, z]) - [f(x), f(y), f(z)] \|_B \leq \theta (\| x \|_A^r + \| y \|_A^r + \| z \|_A^r)$$

(2.9)
for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A \). Then there exists a unique \( C^* \)-ternary algebra homomorphism \( H : A \to B \) such that

\[
\| f(x) - H(x) \|_B \leq \frac{2^r(p + d)\theta}{2(p + 2d)^{r} - (p + 2d)2^r} \| x \|_A^r
\]

(2.10)

for all \( x \in A \).

In the following theorem we have an alternative result of Theorem 2.3.

**Theorem 2.4.** Let \( r, s, \) and \( \theta \) be nonnegative real numbers such that \( 0 < r < 1 \), \( 0 < s < 3 \) (resp., \( r > 1 \), \( s > 3 \)), and let \( d \geq 2 \). Suppose that \( f : A \to B \) is a mapping with \( f(0) = 0 \), satisfying (2.8) and

\[
\| f([x, y, z]) - [f(x), f(y), f(z)] \|_B \leq \theta (\| x \|_A^s + \| y \|_A^s + \| z \|_A^s)
\]

(2.11)

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \). Then there exists a unique \( C^* \)-ternary algebra homomorphism \( H : A \to B \) such that

\[
\| f(x) - H(x) \|_B \leq \frac{d\theta}{2|d - d|} \| x \|_A^r
\]

(2.12)

for all \( x \in A \).

**Proof.** We prove the theorem in two cases.

**Case 1.** \( 0 < r < 1 \) and \( 0 < s < 3 \).

Letting \( \mu = 1, x_1 = \cdots = x_p = 0 \) and \( y_1 = \cdots = y_d = x \) in (2.8), we get

\[
\| f(dx) - df(x) \|_B \leq \frac{d\theta}{2} \| x \|_A^r
\]

(2.13)

for all \( x \in A \). If we replace \( x \) by \( d^n x \) in (2.13) and divide both sides of (2.13) to \( d^{n+1} \), we get

\[
\left\| \frac{1}{d^{n+1}} f \left( d^{n+1} x \right) - \frac{1}{d^n} f (d^n x) \right\|_B \leq \frac{\theta}{2} d^{r(n-1)} \| x \|_A^r
\]

(2.14)

for all \( x \in A \) and all nonnegative integers \( n \). Therefore,

\[
\left\| \frac{1}{d^{n+1}} f \left( d^{n+1} x \right) - \frac{1}{d^m} f (d^m x) \right\|_B \leq \frac{\theta}{2} \sum_{i=m}^{n} d^{r(i-1)} \| x \|_A^r
\]

(2.15)
for all \( x \in A \) and all nonnegative integers \( n \geq m \). From this it follows that the sequence \( \{(1/d^n)f(d^n x)\} \) is Cauchy for all \( x \in A \). Since \( B \) is complete, the sequence \( \{(1/d^n)f(d^n x)\} \) converges. Thus one can define the mapping \( H : A \to B \) by

\[
H(x) := \lim_{n \to \infty} \frac{1}{d^n} f(d^n x)
\]

for all \( x \in A \). Moreover, letting \( m = 0 \) and passing the limit \( n \to \infty \) in (2.15), we get (2.12). It follows from (2.8) that

\[
\left\| 2H \left( \frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j \right) - \sum_{j=1}^{p} \mu H(x_j) - 2 \sum_{j=1}^{d} \mu H(y_j) \right\|_B
\]

\[
= \lim_{n \to \infty} \frac{1}{d^n} \left\| 2f \left( \frac{d^n \sum_{j=1}^{p} \mu x_j}{2} + d^n \sum_{j=1}^{d} \mu y_j \right) - \sum_{j=1}^{p} \mu f(d^n x_j) - 2 \sum_{j=1}^{d} \mu f(d^n y_j) \right\|_B
\]

\[
\leq \lim_{n \to \infty} \frac{d^n \theta}{d^n} \left( \sum_{j=1}^{p} \|x_j\|^s_A + \sum_{j=1}^{d} \|y_j\|^s_A \right) = 0
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x_1, \ldots, x_p, y_1, \ldots, y_d \in A \). Hence

\[
2H \left( \frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j \right) = \sum_{j=1}^{p} \mu H(x_j) + 2 \sum_{j=1}^{d} \mu H(y_j)
\]

(2.18)

for all \( \mu \in \mathbb{T}^1 \) and all \( x_1, \ldots, x_p, y_1, \ldots, y_d \in A \). So \( H(\lambda x + \mu y) = \lambda H(x) + \mu H(y) \) for all \( \lambda, \mu \in \mathbb{T}^1 \) and all \( x, y \in A \). Therefore by Lemma 2.1 the mapping \( H : A \to B \) is \( \mathbb{C} \)-linear.

It follows from Lemma 2.2 and (2.11) that

\[
\left\| H([x, y, z]) - [H(x), H(y), H(z)] \right\|_B
\]

\[
= \lim_{n \to \infty} \frac{1}{d^{3n}} \left\| f(\{d^n x, d^n y, d^n z\}) - [f(d^n x), f(d^n y), f(d^n z)] \right\|_B
\]

\[
= \theta \lim_{n \to \infty} \frac{d^{3n}}{d^{3n}} \left( \|x\|^s_A + \|y\|^s_A + \|z\|^s_A \right) = 0
\]

for all \( x, y, z \in A \). Thus

\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]

(2.20)

for all \( x, y, z \in A \). Therefore the mapping \( H \) is a \( \mathbb{C} \)-ternary algebra homomorphism.
Now let $T : A \rightarrow B$ be another $C^*$-ternary algebra homomorphism satisfying (2.12). Then we have

$$\|H(x) - T(x)\|_B = \lim_{n \to \infty} \frac{1}{d^n} \|f(d^n x) - T(d^n x)\|_B \leq \frac{d \theta}{2|d - d^n|} \lim_{n \to \infty} \frac{d^n}{d^n} \|x\|_A^r = 0$$

(2.21)

for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of $H$. Thus the mapping $H : A \rightarrow B$ is a unique $C^*$-ternary algebra homomorphism satisfying (2.12), as desired.

Case 2. $r > 1$ and $s > 3$.

Similar to the proof of Case 1, we conclude that the sequence $\{d^n f(d^{-n}x)\}$ is a Cauchy sequence in $B$. So we can define the mapping $H : A \rightarrow B$ by

$$H(x) := \lim_{n \to \infty} d^n f(d^{-n}x)$$

(2.22)

for all $x \in A$. The rest of the proof is similar to the proof of Case 1.

Theorem 2.5 (see [31]). Let $r$ and $\theta$ be nonnegative real numbers such that $r \notin [1/(p + d), 1]$, and let $f : A \rightarrow B$ be a mapping such that

$$\|C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_B \leq \theta \prod_{j=1}^{p} \|x_j\|_A^r \prod_{j=1}^{d} \|y_j\|_A^r$$

(2.23)

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta \|x\|_A^r \|y\|_A^r \|z\|_A^r$$

(2.24)

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \rightarrow B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2^{(p+d)r} \theta}{2(p + 2d)^{(p+d)r} - 2^{(p+d)r} (p + 2d)} \|x\|_A^{(p+d)r}$$

(2.25)

for all $x \in A$.

The following theorem shows that the mapping $f : A \rightarrow B$ in Theorem 2.5 is a $C^*$-ternary algebra homomorphism when $r > 0$. 

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Abstract and Applied Analysis

Now let $T : A \rightarrow B$ be another $C^*$-ternary algebra homomorphism satisfying (2.12). Then we have

$$\|H(x) - T(x)\|_B = \lim_{n \to \infty} \frac{1}{d^n} \|f(d^n x) - T(d^n x)\|_B \leq \frac{d \theta}{2|d - d^n|} \lim_{n \to \infty} \frac{d^n}{d^n} \|x\|_A^r = 0$$

(2.21)

for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of $H$. Thus the mapping $H : A \rightarrow B$ is a unique $C^*$-ternary algebra homomorphism satisfying (2.12), as desired.

Case 2. $r > 1$ and $s > 3$.

Similar to the proof of Case 1, we conclude that the sequence $\{d^n f(d^{-n}x)\}$ is a Cauchy sequence in $B$. So we can define the mapping $H : A \rightarrow B$ by

$$H(x) := \lim_{n \to \infty} d^n f(d^{-n}x)$$

(2.22)

for all $x \in A$. The rest of the proof is similar to the proof of Case 1.

Theorem 2.5 (see [31]). Let $r$ and $\theta$ be nonnegative real numbers such that $r \notin [1/(p + d), 1]$, and let $f : A \rightarrow B$ be a mapping such that

$$\|C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_B \leq \theta \prod_{j=1}^{p} \|x_j\|_A^r \prod_{j=1}^{d} \|y_j\|_A^r$$

(2.23)

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta \|x\|_A^r \|y\|_A^r \|z\|_A^r$$

(2.24)

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \rightarrow B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2^{(p+d)r} \theta}{2(p + 2d)^{(p+d)r} - 2^{(p+d)r} (p + 2d)} \|x\|_A^{(p+d)r}$$

(2.25)

for all $x \in A$.

The following theorem shows that the mapping $f : A \rightarrow B$ in Theorem 2.5 is a $C^*$-ternary algebra homomorphism when $r > 0$. 

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Theorem 2.6. Let \( r, s, q, r_1, \ldots, r_p, s_1, \ldots, s_d \) and \( \theta \) be nonnegative real numbers such that \( r + s + q \neq 3 \) and \( r_k > 0 \) (\( s_k > 0 \)) for some \( 1 \leq k \leq p, p \geq 2 \) (\( 1 \leq k \leq d, d \geq 2 \)).

Let \( f : A \to B \) be a mapping satisfying

\[
\begin{align*}
&\|C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_B \leq \theta \prod_{j=1}^{p} \|x_j\|_A^\mu \cdot \prod_{j=1}^{d} \|y_j\|_A^\mu, \\
&\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta \|x\|_A\|y\|_A\|z\|_A
\end{align*}
\]  

(2.26)

(2.27)

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A \). Then the mapping \( f : A \to B \) is a C*-ternary algebra homomorphism. (We put \( \|\cdot\|_0^0 = 1 \).)

Proof. Let \( r_k > 0 \) for some \( 1 \leq k \leq p \) (we have similar proof when \( s_k > 0 \) for some \( 1 \leq k \leq d \)). We now assume, without loss of generality, that \( r_1 > 0 \). Letting \( x_1 = \cdots = x_p = y_1 = \cdots = y_d = 0 \) in (2.26), we get that \( f(0) = 0 \). Letting \( x_2 = 2x \) and \( x_1 = x_3 = \cdots = x_p = y_1 = \cdots = y_d = 0 \) in (2.26), we get

\[
\mu f(2x) = 2f(\mu x)
\]  

(2.28)

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \). Setting \( \mu = 1 \) in (2.28), we get that \( f(2x) = 2f(x) \) for all \( x \in A \). Therefore,

\[
f(\mu x) = \mu f(x), \quad f(2\mu x) = 2\mu f(x)
\]  

(2.29)

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \). If we put \( x_2 = 2x \) and \( y_1 = y \) and \( x_1 = x_3 = \cdots = x_p = y_2 = \cdots = y_d = 0 \) in (2.26), we get

\[
2f(\mu x + \mu y) = \mu f(2x) + 2\mu f(y)
\]  

(2.30)

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \). It follows from (2.29) and (2.30) that

\[
f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)
\]  

(2.31)

for all \( \lambda, \mu \in \mathbb{T}^1 \) and all \( x, y \in A \). Therefore, by Lemma 2.1 the mapping \( f : A \to B \) is \( \mathbb{C} \)-linear. Let \( r + s + q > 3 \). Then it follows from (2.27) that

\[
\begin{align*}
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \\
= \lim_{n \to \infty} 8^n \|f \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) - [f \left( \frac{x}{2^n} \right), f \left( \frac{y}{2^n} \right), f \left( \frac{z}{2^n} \right)] \|_B \\
\leq \theta \|x\|_A \|y\|_A \|z\|_A \lim_{n \to \infty} \left( \frac{8}{2^{r+s+q}} \right)^n = 0
\end{align*}
\]  

(2.32)
for all \( x, y, z \in A \). Therefore,
\[
f([x, y, z]) = [f(x), f(y), f(z)]
\] (2.33)
for all \( x, y, z \in A \). Similarly, for \( r + s + q < 3 \), we get (2.33). \( \square \)

In the rest of this section, assume that \( A \) is a unital \( C^* \)-ternary algebra with norm \( \| \cdot \|_A \) and unit \( e \), and that \( B \) is a unital \( C^* \)-ternary algebra with norm \( \| \cdot \|_B \) and unit \( e' \).

We investigate homomorphisms in \( C^* \)-ternary algebras associated with the functional equation \( C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0 \).

**Theorem 2.7** (see [31]). Let \( r > 1 \) \((r < 1)\) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (2.8) such that
\[
f([x, y, z]) = [f(x), f(y), f(z)]
\] (2.34)
for all \( x, y, z \in A \). If \( \lim_{n \to \infty}((p + 2d)^n/2^n)f(2^n e/(p + 2d)^n) = e' (\lim_{n \to \infty}(2^n/(p + 2d)^n)f((p + 2d)^n/2^n)e = e') \), then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra isomorphism.

In the following theorems we have alternative results of Theorem 2.7.

**Theorem 2.8.** Let \( r < 1 \), \( s < 2 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping satisfying (2.8) and (2.11). If there exist a real number \( \lambda > 1 \) \((0 < \lambda < 1)\) and an element \( x_0 \in A \) such that \( \lim_{n \to \infty}(1/\lambda^n)f(\lambda^n x_0) = e' (\lim_{n \to \infty}\lambda^n f(x_0/\lambda^n) = e') \), then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

**Proof.** By using the proof of Theorem 2.4, there exists a unique \( C^* \)-ternary algebra homomorphism \( H : A \to B \) satisfying (2.12). It follows from (2.12) that
\[
H(x) = \lim_{n \to \infty} \frac{1}{\lambda^n} f(\lambda^n x), \quad \left( H(x) = \lim_{n \to \infty} \lambda^n f\left(\frac{x}{\lambda^n}\right) \right)
\] (2.35)
for all \( x \in A \) and all real numbers \( \lambda > 1 \) \((0 < \lambda < 1)\). Therefore, by the assumption we get that \( H(x_0) = e' \). Let \( \lambda > 1 \) and \( \lim_{n \to \infty}(1/\lambda^n)f(\lambda^n x_0) = e' \). It follows from (2.11) that
\[
\|[H(x), H(y), H(z)] - [H(x), H(y), f(z)]\|_B
= \|[H(x, y, z)] - [H(x), H(y), f(z)]\|_B
= \lim_{n \to \infty} \frac{1}{\lambda^{2n}} \|[f(\lambda^n x, \lambda^n y, z)] - [f(\lambda^n x), f(\lambda^n y), f(z)]\|_B
\leq \theta \lim_{n \to \infty} \frac{1}{\lambda^{2n}} (\lambda^{2n}\|x\|_A^p + \lambda^{2n}\|y\|_A^p + \|z\|_A^p) = 0
\] (2.36)
for all \( x \in A \). So \([H(x), H(y), H(z)] = [H(x), H(y), f(z)]\) for all \( x, y, z \in A \). Letting \( x = y = x_0 \) in the last equality, we get \( f(z) = H(z) \) for all \( z \in A \). Similarly, one can shows that \( H(x) = f(x) \) for all \( x \in A \) when \( 0 < \lambda < 1 \) and \( \lim_{n \to \infty}\lambda^n f(x_0/\lambda^n) = e' \). Therefore, the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism. \( \square \)
3. Derivations on $C^*$-Ternary Algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\| \cdot \|_A$.

Park [31] proved the Hyers-Ulam-Rassias stability and Ulam-Găvruţa-Rassias stability of derivations on $C^*$-ternary algebras for the following functional equation:

$$C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0. \quad (3.1)$$

For a given mapping $f : A \to A$, let

$$Df(x, y, z) = f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \quad (3.2)$$

for all $x, y, z \in A$.

**Theorem 3.1** (see [31]). Let $r$ and $\theta$ be nonnegative real numbers such that $r \notin [1, 3]$, and let $f : A \to A$ a mapping satisfying (2.8) and

$$\|Df(x, y, z)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) \quad (3.3)$$

for all $x, y, z \in A$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{2^r(p + d)}{2(p + 2d)^r - (p + 2d)2^r} \theta \|x\|_A^r \quad (3.4)$$

for all $x \in A$.

**Theorem 3.2** (see [31]). Let $r$ and $\theta$ be nonnegative real numbers such that $r \notin [1/(p + d), 1]$, and let $f : A \to A$ be a mapping satisfying (2.23) and

$$\|Df(x, y, z)\|_A \leq \theta\|x\|_A^r \|y\|_A^r \|z\|_A^r \quad (3.5)$$

for all $x, y, z \in A$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{2^{(p + d)r}}{2(p + 2d)^{(p + d)r} - (p + 2d)2^{(p + d)r}} \theta \|x\|_A^{(p + d)r} \quad (3.6)$$

for all $x \in A$.

In the following theorems we generalize and improve the results in Theorems 3.1 and 3.2.
Theorem 3.3. Let \( \varphi : A^{p+d} \to [0, \infty) \) and \( \psi : A^3 \to [0, \infty) \) be functions such that

\[
\bar{\varphi}(x) := \sum_{n=0}^{\infty} \gamma^n \varphi(\gamma^n x, \ldots, \gamma^n x) < \infty, \tag{3.7}
\]

\[
\lim_{n \to \infty} \gamma^n \varphi(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d) = 0, \tag{3.8}
\]

\[
\lim_{n \to \infty} \gamma^{-3n} \varphi(\gamma^n x, \gamma^n y, \gamma^n z) = 0, \quad \lim_{n \to \infty} \gamma^{-2n} \varphi(\gamma^n x, \gamma^n y, z) = 0 \tag{3.9}
\]

for all \( x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A \) where \( \gamma = (p+2d)/2 \). Suppose that \( f : A \to A \) is a mapping satisfying

\[
\|C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_A \leq \varphi(x_1, \ldots, x_p, y_1, \ldots, y_d), \tag{3.10}
\]

\[
\|Df(x, y, z)\|_A \leq \varphi(x, y, z) \tag{3.11}
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A \). Then the mapping \( f : A \to A \) is a C*-ternary derivation.

Proof. Let us assume \( \mu = 1 \) and \( x_1 = \cdots = x_p = y_1 = \cdots = y_d = x \) in (3.10). Then we get

\[
\left\| 2f\left(\frac{p+2d}{2}x\right) - (p+2d)f(x) \right\|_A \leq \varphi(x, \ldots, x) \tag{3.12}
\]

for all \( x \in A \). If we replace \( x \) in (3.12) by \( \gamma^n x \) and divide both sides of (3.12) by \( \gamma^{n+1} \), then we get

\[
\left\| \frac{1}{\gamma^{n+1}} f(\gamma^{n+1} x) - \frac{1}{\gamma^n} f(\gamma^n x) \right\|_A \leq \frac{1}{2\gamma^{n+1}} \varphi(\gamma^n x, \ldots, \gamma^n x) \tag{3.13}
\]

for all \( x \in A \) and all integers \( n \geq 0 \). Hence

\[
\left\| \frac{1}{\gamma^{n+1}} f(\gamma^{n+1} x) - \frac{1}{\gamma^m} f(\gamma^m x) \right\|_A \leq \frac{1}{2\gamma^{n+1}} \sum_{i=0}^{n-1} \varphi(\gamma^i x, \ldots, \gamma^i x) \tag{3.14}
\]

for all \( x \in A \) and all integers \( n \geq m \geq 0 \). From this it follows that the sequence \( \{(1/\gamma^n)f(\gamma^n x)\} \) is Cauchy for all \( x \in A \). Since \( A \) is complete, the sequence \( \{(1/\gamma^n)f(\gamma^n x)\} \) converges. Thus we can define the mapping \( \delta : A \to A \) by

\[
\delta(x) := \lim_{n \to \infty} \frac{1}{\gamma^n} f(\gamma^n x) \tag{3.15}
\]

for all \( x \in A \). Moreover, letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.14), we get

\[
\left\| \delta(x) - f(x) \right\|_A \leq \frac{1}{2\gamma^n} \varphi(x) \tag{3.16}
\]
for all $x \in A$. It follows from (3.8) and (3.10) that

$$
\| C_\mu \delta(x_1, \ldots, x_p, y_1, \ldots, y_d) \|_A
\leq \lim_{n \to \infty} \frac{1}{\gamma^n} \| C_\mu f(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d) \|_A
$$

(3.17)

for all $\mu \in T^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Hence

$$
2\delta \left( \frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j \right) = \sum_{j=1}^p \mu \delta(x_j) + 2 \sum_{j=1}^d \mu \delta(y_j)
$$

(3.18)

for all $\mu \in T^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. So $\delta(\lambda x + \mu y) = \lambda \delta(x) + \mu \delta(y)$ for all $\lambda, \mu \in T^1$ and all $x, y \in A$. Therefore, by Lemma 2.1 the mapping $\delta : A \to A$ is $C$-linear.

It follows from (3.9) and (3.11) that

$$
\| D\delta(x, y, z) \|_A = \lim_{n \to \infty} \frac{1}{\gamma^{3n}} \| D f(\gamma^n x, \gamma^n y, \gamma^n z) \|_A \leq \lim_{n \to \infty} \frac{1}{\gamma^{3n}} \varphi(\gamma^n x, \gamma^n y, \gamma^n z) = 0
$$

(3.19)

for all $x, y, z \in A$. Hence

$$
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]
$$

(3.20)

for all $x, y, z \in A$. So the mapping $\delta : A \to A$ is a $C^*$-ternary derivation.

It follows from (3.9) and (3.11)

$$
\| [\delta(x, y, z) - [\delta(x), y, z] - [x, \delta(y), z] - [x, y, f(z)] \|_A
\leq \lim_{n \to \infty} \frac{1}{\gamma^{2n}} \| f(\gamma^n x, \gamma^n y, z) - [f(\gamma^n x), \gamma^n y, z]
\|_A

(3.21)

and

for all $x, y, z \in A$. Thus

$$
\delta(x, y, z) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, f(z)]
$$

(3.22)

for all $x, y, z \in A$. Hence we get from (3.20) and (3.22) that

$$
[x, y, \delta(z)] = [x, y, f(z)]
$$

(3.23)
for all $x, y, z \in A$. Letting $x = y = f(z) - \delta(z)$ in (3.23), we get

$$\|f(z) - \delta(z)\|_A^3 = \|[f(z) - \delta(z), f(z) - \delta(z), f(z) - \delta(z)]\|_A = 0$$  \hfill (3.24)$$

for all $z \in A$. Hence $f(z) = \delta(z)$ for all $z \in A$. So the mapping $f : A \to A$ is a $C^*$-ternary derivation, as desired.

**Corollary 3.4.** Let $r < 1$, $s < 2$, and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.8) and

$$\|D_f(x, y, z)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$  \hfill (3.25)$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a $C^*$-ternary derivation.

**Proof.** Define

$$\varphi(x_1, \ldots, x_p, y_1, \ldots, y_d) = \theta \left( \sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r \right),$$

$$\varphi(x, y, z) = \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$, and apply Theorem 3.3. \hfill $\square$

**Corollary 3.5.** Let $r, s$, and $\theta$ be nonnegative real numbers such that $s, r(p + d) < 1$, and let $f : A \to A$ be a mapping satisfying (2.23) and

$$\|D_f(x, y, z)\|_A \leq \theta\|x\|_A^s \|y\|_A^s \|z\|_A^s$$  \hfill (3.27)$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a $C^*$-ternary derivation.

**Proof.** Define

$$\varphi(x_1, \ldots, x_p, y_1, \ldots, y_d) = \theta \left( \prod_{j=1}^{p} \|x_j\|_A^r \prod_{j=1}^{d} \|y_j\|_A^r \right),$$

$$\varphi(x, y, z) = \theta\|x\|_A^s \|y\|_A^s \|z\|_A^s$$

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$, and apply Theorem 3.3. \hfill $\square$
Theorem 3.6. Let $\varphi : A^{p+d} \to [0, \infty)$ and $\psi : A^3 \to [0, \infty)$ be functions such that

\[
\hat{\varphi}(x) := \sum_{n=1}^{\infty} y^n \varphi\left( \frac{x}{y^n}, \ldots, \frac{x}{y^n} \right) < \infty,
\]

\[
\lim_{n \to \infty} y^n \varphi\left( \frac{x_1}{y^n}, \ldots, \frac{x_p}{y^n}, \frac{y_1}{y^n}, \ldots, \frac{y_d}{y^n} \right) = 0,
\]

\[
\lim_{n \to \infty} y^{3n} \varphi\left( \frac{x}{y^n}, \frac{y}{y^n}, \frac{z}{y^n} \right) = 0, \quad \lim_{n \to \infty} y^{2n} \varphi\left( \frac{x}{y^n}, \frac{y}{y^n}, \frac{z}{y^n} \right) = 0
\]

(3.29)

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$ where $\gamma = (p + 2d)/2$. Suppose that $f : A \to A$ is a mapping satisfying (3.10) and (3.11). Then the mapping $f : A \to A$ is a $C^*$-ternary derivation.

Proof. If we replace $x$ in (3.12) by $x/y^{n+1}$ and multiply both sides of (3.12) by $y^n$, then we get

\[
\|y^{n+1}f\left( \frac{x}{y^{n+1}} \right) - y^n f\left( \frac{x}{y^n} \right) \|_A \leq \frac{y^n}{2} \varphi\left( \frac{x}{y^{n+1}}, \ldots, \frac{x}{y^{n+1}} \right)
\]

(3.30)

for all $x \in A$ and all integers $n \geq 0$. Hence

\[
\|y^{n+1}f\left( \frac{x}{y^{n+1}} \right) - y^m f\left( \frac{x}{y^m} \right) \|_A \leq \frac{1}{2y} \sum_{i=m+1}^{n+1} y^i \varphi\left( \frac{x}{y^i}, \ldots, \frac{x}{y^i} \right)
\]

(3.31)

for all $x \in A$ and all integers $n \geq m \geq 0$. From this it follows that the sequence $\{y^n f(x/y^n)\}$ is Cauchy for all $x \in A$. Since $A$ is complete, the sequence $\{y^n f(x/y^n)\}$ converges. Thus we can define the mapping $\delta : A \to A$ by

\[
\delta(x) := \lim_{n \to \infty} y^n f\left( \frac{x}{y^n} \right)
\]

(3.32)

for all $x \in A$. The rest of the proof is similar to the proof of Theorem 3.3, and we omit it. \qed

Corollary 3.7. Let $r, s$, and $\theta$ be nonnegative real numbers such that $s, r(p + d) > 1$, and let $f : A \to A$ be a mapping satisfying (2.23) and (3.27). Then the mapping $f : A \to A$ is a $C^*$-ternary derivation.

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Abstract and Applied Analysis


