Research Article

Fuzzy Stability of Jensen-Type Quadratic Functional Equations

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We prove the generalized Hyers-Ulam stability of the following quadratic functional equations

\[ 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y) \quad \text{and} \quad f(ax + ay) + (ax - ay) = 2a^2 f(x) + 2a^2 f(y) \]

in fuzzy Banach spaces for a nonzero real number \( a \) with \( a \neq \pm 1/2 \).

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1. Introduction and Preliminaries


J. M. Rassias [6] proved a similar stability theorem in which he replaced the factor \( ||x||^p + ||y||^p \) by \( ||x||^p \cdot ||y||^q \) for \( p, q \in \mathbb{R} \) with \( p + q \neq 1 \) (see also [7, 8] for a number of other new results). The papers of J. M. Rassias [6–8] introduced the Ulam-Gavruta-Rassias stability of functional equations. See also [9–11].

The functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.1) \]
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [12] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [13] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. In [14], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.


\[
 f(rx + sy) + f(sx - ry) = \left( r^2 + s^2 \right) [f(x) + f(y)],
\]  

motivated from the following pertinent algebraic equation

\[
 |ax + by|^2 + |bx - ay|^2 = \left( a^2 + b^2 \right) (|x|^2 + |y|^2).
\]  

The solution of the functional equation (1.2) is called an Euler-Lagrange-type quadratic mapping. J. M. Rassias [16, 17] introduced and investigated the relative functional equations. In addition, J. M. Rassias [18] generalized the algebraic equation (1.3) to the following equation

\[
 mn|ax + by|^2 + |nbx - may|^2 = \left( ma^2 + nb^2 \right) \left( n|x|^2 + m|y|^2 \right),
\]  

and introduced and investigated the general pertinent Euler-Lagrange quadratic mappings. Analogous quadratic mappings were introduced and investigated in [19, 20].

These Euler-Lagrange mappings are named Euler-Lagrange-Rassias mappings and the corresponding Euler-Lagrange equations are called Euler-Lagrange-Rassias equations. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler-Lagrange partial differential equations are known in calculus of variations. Therefore, we think that J. M. Rassias’ introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler-Lagrange mappings.


\[
 \sum_{i=1}^{n} r_i Q \left( \sum_{j=1}^{n} r_j (x_i - x_j) \right) + \left( \sum_{i=1}^{n} r_i \right) Q \left( \sum_{i=1}^{n} r_i x_i \right) = \left( \sum_{i=1}^{n} r_i \right)^2 \sum_{i=1}^{n} r_i Q(x_i),
\]  

whose solution is said to be a generalized quadratic mapping of Euler-Lagrange-Rassias type.

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9, 23–26]).

Katsaras [27] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on
a vector space from various points of view [28–30]. In particular, Bag and Samanta [31], following Cheng and Mordeson [32], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michálek type [33]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [34].

We use the definition of fuzzy normed spaces given in [31] and [35–38] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the quadratic functional equations

\begin{equation}
2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y),
\end{equation}

\begin{equation}
f(ax + ay) + f(ax - ay) = 2a^2f(x) + 2a^2f(y)
\end{equation}

in the fuzzy normed vector space setting.

**Definition 1.1 (see [31, 35–38])**. Let \( X \) be a real vector space. A function \( N : X \times \mathbb{R} \to [0, 1] \) is called a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

- \((N_1)\) \( N(x, t) = 0 \) for \( t \leq 0 \);
- \((N_2)\) \( x = 0 \) if and only if \( N(x, t) = 1 \) for all \( t > 0 \);
- \((N_3)\) \( N(cx, t) = N(x, t/|c|) \) if \( c \neq 0 \);
- \((N_4)\) \( N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\} \);
- \((N_5)\) \( N(x, \cdot) \) is a non-decreasing function of \( \mathbb{R} \) and \( \lim_{t \to \infty} N(x, t) = 1 \);
- \((N_6)\) for \( x \neq 0 \), \( N(x, \cdot) \) is continuous on \( \mathbb{R} \).

The pair \((X, N)\) is called a fuzzy normed vector space.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [35–38].

**Definition 1.2 (see [31, 35–38])**. Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is said to be convergent or converge if there exists an \( x \in X \) such that \( \lim_{n \to \infty} N(x_n - x, t) = 1 \) for all \( t > 0 \). In this case, \( x \) is called the limit of the sequence \( \{x_n\} \) and we denote it by \( N\text{-}\lim_{n \to \infty} x_n = x \).

**Definition 1.3 (see [31, 35–38])**. Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \) there exists an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and all \( p > 0 \), we have \( N(x_{n+p} - x_n, t) > 1 - \varepsilon \).

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping \( f : X \to Y \) between fuzzy normed vector spaces \( X \) and \( Y \) is continuous at a point \( x_0 \in X \) if for each sequence \( \{x_n\} \) converging to \( x_0 \) in \( X \), then the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \). If \( f : X \to Y \) is continuous at each \( x \in X \), then \( f : X \to Y \) is said to be continuous on \( X \) (see [34]).

In this paper, we prove the generalized Hyers-Ulam stability of the quadratic functional equations (1.6) and (1.7) in fuzzy Banach spaces.

Throughout this paper, assume that \( X \) is a vector space and that \((Y, N)\) is a fuzzy Banach space. Let \( a \) be a nonzero real number with \( a \neq (\pm 1/2) \).
2. Fuzzy Stability of Quadratic Functional Equations

We prove the fuzzy stability of the quadratic functional equation (1.6).

**Theorem 2.1.** Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \). Suppose that \( \varphi \) is a mapping from \( X \) to a fuzzy normed space \((Z, N')\) such that

\[
N \left( 2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y), t + s \right) \geq \min \left\{ N' \left( \varphi(x), t \right), N' \left( \varphi(y), s \right) \right\}
\]

(2.1)

for all \( x, y \in X \setminus \{0\} \) and all positive real numbers \( t, s \). If \( \varphi(3x) = \alpha \varphi(x) \) for some positive real number \( \alpha < 9 \), then there is a unique quadratic mapping \( Q : X \to Y \) such that \( Q(x) = \lim_{n \to \infty} f(3^n x) / 9^n \) and

\[
N(\varphi(3x), t) \geq M \left( x, \frac{(9 - \alpha)t}{18} \right),
\]

(2.2)

where

\[
M(x, t) := \min \left\{ N' \left( \varphi(x), \frac{3}{2} t \right), N' \left( \varphi(2x), \frac{3}{2} t \right), N' \left( \varphi(3x), \frac{3}{2} t \right), N' \left( \varphi(0), \frac{3}{2} t \right) \right\}.
\]

(2.3)

**Proof.** Putting \( y = 3x \) and \( s = t \) in (2.1), we get

\[
N \left( 2f(2x) + 2f(-x) - f(x) - f(3x), 2t \right) \geq \min \left\{ N' \left( \varphi(x), t \right), N' \left( \varphi(3x), t \right) \right\}
\]

(2.4)

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( 2x \), \( y \) by \( 0 \), and \( s \) by \( t \) in (2.1), we obtain

\[
N \left( 4f(x) - f(2x), 2t \right) \geq \min \left\{ N' \left( \varphi(2x), t \right), N' \left( \varphi(0), t \right) \right\}.
\]

(2.5)

Thus

\[
N \left( 9f(x) - f(3x), 6t \right) \geq \min \left\{ N' \left( \varphi(x), t \right), N' \left( \varphi(2x), t \right), N' \left( \varphi(3x), t \right), N' \left( \varphi(0), t \right) \right\},
\]

(2.6)

and so

\[
N \left( f(x) - \frac{f(3x)}{9}, t \right) \geq \min \left\{ N' \left( \varphi(x), \frac{3}{2} t \right), N' \left( \varphi(2x), \frac{3}{2} t \right), N' \left( \varphi(3x), \frac{3}{2} t \right), N' \left( \varphi(0), \frac{3}{2} t \right) \right\}.
\]

(2.7)

Then by the assumption,

\[
M(3x, t) = M \left( x, \frac{t}{\alpha} \right).
\]

(2.8)
Replacing $x$ by $3^n x$ in (2.7) and applying (2.8), we get

$$N \left( \frac{f(3^n x)}{9^n} - \frac{f(3^{n+1} x)}{9^{n+1}}, \frac{\alpha^n t}{9^n} \right) = N \left( \frac{f(3^n x)}{9^n} - \frac{f(3^{n+1} x)}{9^{n+1}}, \frac{\alpha^n t}{9^n} \right)$$

$$\geq M(3^n x, \alpha^n t)$$

$$= M(x, t).$$

Thus for each $n > m$ we have

$$N \left( \frac{f(3^n x)}{9^n} - \frac{f(3^n x)}{9^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{9^k} \right)$$

$$= N \left( \sum_{k=m}^{n-1} \left( \frac{f(3^k x)}{9^k} - \frac{f(3^{k+1} x)}{9^{k+1}} \right), \sum_{k=m}^{n-1} \frac{\alpha^k t}{9^k} \right)$$

$$\geq \min \left\{ \bigcup_{k=m}^{n-1} \left\{ N \left( \frac{f(3^k x)}{9^k} - \frac{f(3^{k+1} x)}{9^{k+1}}, \frac{\alpha^k t}{9^k} \right) \right\} \right\}$$

$$\geq M(x, t).$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \to \infty} M(x, t) = 1$, there is some $t_0 > 0$ such that $M(x, t_0) > 1 - \epsilon$. Since $\sum_{k=0}^\infty \alpha^k t_0/9^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n_1} \alpha^k t_0/9^k < \delta$ for $n > m \geq n_0$. It follows that

$$N \left( \frac{f(3^n x)}{9^n} - \frac{f(3^n x)}{9^n}, \delta \right) \geq N \left( \frac{f(3^n x)}{9^n} - \frac{f(3^n x)}{9^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{9^k} \right)$$

$$\geq M(x, t_0)$$

$$\geq 1 - \epsilon$$

for all $t \geq t_0$. This shows that the sequence $\{ f(3^n x)/9^n \}$ is Cauchy in $(Y, N)$. Since $(Y, N)$ is complete, $\{ f(3^n x)/9^n \}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N - \lim_{t \to \infty} f(3^n x)/9^n$. Moreover, if we put $m = 0$ in (2.10), then we observe that

$$N \left( \frac{f(3^n x)}{9^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{9^k} \right) \geq M(x, t).$$

Thus

$$N \left( \frac{f(3^n x)}{9^n} - f(x), t \right) \geq M \left( x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/9)^k} \right).$$

(2.13)
Next we show that $Q$ is quadratic. Let $x, y \in X$. Then we have

$$N\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right)$$

$$\geq \min \left\{ N\left(2Q\left(\frac{x+y}{2}\right) - \frac{2f(3^n(x+y)/2)}{9n}, \frac{t}{5}\right), \right.$$

$$N\left(2Q\left(\frac{x-y}{2}\right) - \frac{2f(3^n(x-y)/2)}{9n}, \frac{t}{5}\right), \right.$$

$$N\left(\frac{f(3^n x)}{9n} - Q(x), \frac{t}{5}\right), \left. N\left(\frac{f(3^n y)}{9n} - Q(y), \frac{t}{5}\right)\right\},$$

$$N\left(\frac{2f(3^n(x+y)/2)}{9n} + \frac{2f(3^n(x-y)/2)}{9n} - \frac{f(3^n x)}{9n} - \frac{f(3^n y)}{9n}, \frac{t}{5}\right)\right\}. \quad (2.14)$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ and the fifth term, by (2.1), is greater than or equal to

$$\min \left\{ N\left(\varphi(3^n x), \frac{9nt}{10}\right), N\left(\varphi(3^n y), \frac{9nt}{10}\right)\right\} = \min \left\{ N\left(\varphi(x), \frac{9n t}{10}\right), \left(\varphi(y), \frac{9n t}{10}\right)\right\},$$

$$\text{which tends to 1 as } n \to \infty. \text{ Hence}$$

$$N\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right) = 1 \quad (2.16)$$

for all $x, y \in X$ and all $t > 0$. This means that $Q$ satisfies the Jensen quadratic functional equation and so it is quadratic.

Next, we approximate the difference between $f$ and $Q$ in a fuzzy sense. For every $x \in X$ and $t > 0$, by (2.13), for large enough $n$, we have

$$N(Q(x) - f(x), t) \geq \min \left\{ N\left(Q(x) - \frac{f(3^n y)}{9n}, \frac{t}{2}\right), N\left(\frac{f(3^n y)}{9n} - f(x), \frac{t}{2}\right)\right\}$$

$$\geq M\left(x, \frac{t}{2 \sum_{k=0}^{\infty} (a/9)^k}\right) \quad (2.17)$$

$$= M\left(x, \frac{(9-a)t}{18}\right).$$
Abstract and Applied Analysis

The uniqueness assertion can be proved by a standard fashion; cf. [36]: Let $Q'$ be another quadratic mapping from $X$ into $Y$, which satisfies the required inequality. Then for each $x \in X$ and $t > 0$,

$$N(Q(x) - Q'(x), t) \geq \min \left\{ N(Q(x) - f(x), \frac{t}{2}), N(Q'(x) - f(x), \frac{t}{2}) \right\} \geq M \left( x, \frac{(9 - \alpha)t}{36} \right).$$

(2.18)

Since $Q$ and $Q'$ are quadratic,

$$N(Q(x) - Q'(x), t) = N(Q(3^n x) - Q'(3^n x), 9^n t) \geq M \left( x, \frac{(9/\alpha)^n(9 - \alpha)t}{36} \right).$$

(2.19)

for all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Since $0 < \alpha < 9$, $\lim_{n \to \infty}(9/\alpha)^n = \infty$. Hence the right-hand side of the above inequality tends to 1 as $n \to \infty$. It follows that $Q(x) = Q'(x)$ for all $x \in X$.

**Theorem 2.2.** Let $f : X \to Y$ be an even mapping with $f(0) = 0$. Suppose that $\varphi$ is a mapping from $X$ to a fuzzy normed space $(Z, N')$ satisfying (2.1). If $\varphi(3x) = \alpha\varphi(x)$ for some real number $\alpha$ with $\alpha > 9$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\lim_{n \to \infty} 3^n f(x/3^n)$ and

$$N(Q(x) - f(x), t) \geq M \left( x, \frac{(\alpha - 9)t}{2\alpha} \right),$$

(2.20)

where

$$M(x, t) := \min \left\{ N'(\varphi \left( \frac{x}{3}, \frac{t}{6} \right)), N'(\varphi \left( \frac{2x}{3}, \frac{t}{6} \right)), N'(\varphi(x), \frac{t}{6}), N'(\varphi(0), \frac{t}{6}) \right\}. \quad (2.21)$$

**Proof.** It follows from (2.7) that

$$N(f(x) - 9f \left( \frac{x}{3}, \frac{t}{6} \right), t) \geq \min \left\{ N' \left( \varphi \left( \frac{x}{3}, \frac{t}{6} \right), N' \left( \varphi \left( \frac{2x}{3}, \frac{t}{6} \right), N' \left( \varphi(x), \frac{t}{6} \right), N' \left( \varphi(0), \frac{t}{6} \right) \right) \right\}. \quad (2.22)$$

Then by the assumption,

$$M \left( \frac{x}{3}, \frac{t}{6} \right) = M(x, at). \quad (2.23)$$
Replacing $x$ by $x/3^n$ in (2.22) and applying (2.23), we get

$$N\left(9^n f\left(\frac{x}{3^n}\right) - 9^{n+1} f\left(\frac{x}{3^{n+1}}\right), \sum_{k=m}^{n-1} \frac{g^k t}{a^k} \right) = N\left( f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^{n+1}}\right), \frac{t}{a^n} \right)$$

$$\geq M\left(\frac{x}{3^n}, \frac{t}{a^n}\right)$$

Thus for each $n > m$ we have

$$N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \sum_{k=m}^{n-1} \frac{g^k t}{a^k} \right)$$

$$= N\left(\sum_{k=m}^{n-1} \left(g^k f\left(\frac{x}{3^k}\right) - g^{k+1} f\left(\frac{x}{3^{k+1}}\right)\right), \sum_{k=m}^{n-1} \frac{g^k t}{a^k} \right)$$

$$\geq \min\left\{ \sum_{k=m}^{n-1} \left(N\left(g^k f\left(\frac{x}{3^k}\right) - g^{k+1} f\left(\frac{x}{3^{k+1}}\right), \frac{g^k t}{a^k}\right) \right) \right\}$$

$$\geq M(x,t).$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \to \infty} M(x,t) = 1$, there is some $t_0 > 0$ such that $M(x,t_0) > 1 - \epsilon$. Since $\sum_{k=0}^{\infty} g^k t_0 / a^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0}^{\infty} g^k t_0 / a^k < \delta$ for $n > m \geq n_0$. It follows that

$$N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \delta \right) \geq N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \sum_{k=m}^{n-1} \frac{g^k t}{a^k} \right)$$

$$\geq M(x,t_0)$$

$$\geq 1 - \epsilon$$

for all $t \geq t_0$. This shows that the sequence $\{9^n f(x/3^n)\}$ is Cauchy in $(Y,N)$. Since $(Y,N)$ is complete, $\{9^n f(x/3^n)\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\lim_{t \to \infty} 9^n f(x/3^n)$. Moreover, if we put $m = 0$ in (2.8), then we observe that

$$N\left(9^n f\left(\frac{x}{3^n}\right) - f(x), \sum_{k=0}^{n-1} \frac{g^k t}{a^k} \right) \geq M(x,t).$$

(2.27)

Thus

$$N\left(9^n f\left(\frac{x}{3^n}\right) - f(x), t \right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1} (9/\alpha)^k} \right).$$

(2.28)

The rest of the proof is similar to the proof of Theorem 2.1. \qed
Theorem 2.3. Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \). Suppose that \( \varphi \) is a mapping from \( X \) to a fuzzy normed space \((Z,N')\) satisfying (2.1). If \( \varphi(2x) = \alpha \varphi(x) \) for some positive real number \( \alpha \) with \( \alpha < 4 \), then there is a unique quadratic mapping \( Q : X \to Y \) such that \( Q(x) = N\lim_{n \to \infty} f(2^n x)/4^n \) and

\[
N(Q(x) - f(x), t) \geq M\left(x, \frac{(4 - \alpha)t}{8}\right) \tag{2.29}
\]

where \( M(x,t) = \min\{N'(\varphi(2x), 2t), N'(\varphi(0), 2t)\} \).

Proof. Letting \( y = 0 \) and replacing \( x \) by \( 2x \) and \( s \) by \( t \) in (2.1), we obtain

\[
N(4f(x) - f(2x), 2t) \geq \min\left\{ N'(\varphi(2x), t), N'(\varphi(0), t) \right\}. \tag{2.30}
\]

Thus

\[
N\left(f(x) - \frac{f(2x)}{4}, t\right) \geq \min\left\{ N'(\varphi(2x), 2t), N'(\varphi(0), 2t) \right\}. \tag{2.31}
\]

Then by the assumption,

\[
M(2x, t) = M\left(x, \frac{t}{\alpha}\right). \tag{2.32}
\]

Replacing \( x \) by \( 2^n x \) in (2.31) and applying (2.32), we get

\[
N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1} x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) = N\left(f(2^n x) - \frac{f(4^n x)}{4^n}, \alpha^n t\right)
\]

\[
\geq M(2^n x, \alpha^n t)
\]

\[
= M(x, t). \tag{2.33}
\]

Thus for each \( n > m \) we have

\[
N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right)
\]

\[
= N\left(\sum_{k=m}^{n-1} \left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}\right), \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right)
\]

\[
\geq \min\left\{ \bigcup_{k=m}^{n-1} N\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \alpha^k t\right) \right\}
\]

\[
\geq M(x, t). \tag{2.34}
\]
Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \to \infty} M(x, t) = 1$, there is some $t_0 > 0$ such that $M(x, t_0) > 1 - \varepsilon$. Since $\sum_{k=0}^{\infty} a_k t_0^k / 4^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} a_k t_0^k / 4^k < \delta$ for $n > m \geq n_0$. It follows that

$$N \left( \frac{f(2^m x) - f(2^n x)}{4^m} - \frac{f(2^m x) - f(2^n x)}{4^n}, \delta \right) \geq N \left( \frac{f(2^m x) - f(2^n x)}{4^m} - \frac{f(2^m x) - f(2^n x)}{4^n}, \sum_{k=m}^{n-1} a_k t_0^k / 4^k \right)$$

$$\geq M(x, t_0)$$

$$\geq 1 - \varepsilon$$

for all $t \geq t_0$. This shows that the sequence $\{f(2^n x) / 4^n\}$ is Cauchy in $(Y, N)$. Since $(Y, N)$ is complete, $\{f(2^n x) / 4^n\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\lim_{t \to \infty} f(2^m x) / 4^n$. Moreover, if we put $m = 0$ in (2.34), then we observe that

$$N \left( \frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1} a_k t / 4^k \right) \geq M(x, t).$$

Thus

$$N \left( \frac{f(2^n x)}{4^n} - f(x), t \right) \geq M \left( x, \frac{t}{\sum_{k=0}^{n-1} (\alpha / 4)^k} \right).$$

The rest of the proof is similar to the proof of Theorem 2.1. 

**Theorem 2.4.** Let $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that $\varphi$ is a mapping from $X$ to a fuzzy normed space $(Z, N')$ satisfying (2.1). If $\varphi(2x) = a\varphi(x)$ for some real number $\alpha$ with $\alpha > 4$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\lim_{t \to \infty} A^n f(x / 2^n)$ and

$$N(Q(x) - f(x), t) \geq M \left( x, \frac{(\alpha - 4) t}{2\alpha} \right),$$

where $M(x, t) := \min \{ N'(\varphi(x), t / 2), N'(\varphi(0), t / 2) \}$.

**Proof.** It follows from (2.31) that

$$N \left( f(x) - 4f \left( \frac{x}{2} \right), t \right) \geq \min \left\{ N' \left( \varphi(x), \frac{t}{2} \right), N' \left( \varphi(0), \frac{t}{2} \right) \right\}.$$ 

Then by the assumption,

$$M \left( \frac{x}{2}, t \right) = M(x, \alpha t).$$
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Replacing \( x \) by \( x/2^n \) in (2.39) and applying (2.40), we get

\[
N\left(4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right) = N\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^{n+1}}\right)\right) \\
\geq M\left(\frac{x}{2^n}, \frac{t}{a^n}\right) \\
= M(x,t).
\]  

(2.41)

Thus for each \( n > m \) we have

\[
N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \sum_{k=m}^{n-1} 4^k t i\right) \\
= N\left(\sum_{k=m}^{n-1} \left(4^k f\left(\frac{x}{2^k}\right) - 4^{k+1} f\left(\frac{x}{2^{k+1}}\right)\right), \sum_{k=m}^{n-1} 4^k t i\right) \\
\geq \min\left\{ \sum_{k=m}^{n-1} \left[N\left(4^k f\left(\frac{x}{2^k}\right) - 4^{k+1} f\left(\frac{x}{2^{k+1}}\right), 4^k t i\right)\right] \right\} \\
\geq M(x,t).
\]  

(2.42)

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be given. Since \( \lim_{t \to \infty} M(x,t) = 1 \), there is some \( t_0 > 0 \) such that \( M(x,t_0) > 1 - \varepsilon \). Since \( \sum_{k=0}^{\infty} 4^k t_0 / \alpha^k \leq \infty \), there is some \( n_0 \in \mathbb{N} \) such that \( \sum_{k=m}^{n_1} 4^k t_0 / \alpha^k < \delta \) for \( n > m \geq n_0 \). It follows that

\[
N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \delta\right) \geq N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \sum_{k=m}^{n-1} 4^k t i\right) \\
\geq M(x,t_0) \\
\geq 1 - \varepsilon
\]  

(2.43)

for all \( t \geq t_0 \). This shows that the sequence \( \left\{4^n f\left(x/2^n\right)\right\} \) is Cauchy in \((Y,N)\). Since \((Y,N)\) is complete, \( \left\{4^n f\left(x/2^n\right)\right\} \) converges to some \( Q(x) \in Y \). Thus we can define a mapping \( Q : X \to Y \) by \( Q(x) := \lim_{n \to \infty} 4^n f\left(x/2^n\right) \). Moreover, if we put \( m = 0 \) in (2.42), then we observe that

\[
N\left(4^n f\left(\frac{x}{2^n}\right) - f(x), \sum_{k=0}^{n-1} 4^k t i\right) \geq M(x,t).
\]  

(2.44)

Thus

\[
N\left(4^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1} (4/\alpha)^k}\right).
\]  

(2.45)

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)
Now we prove the fuzzy stability of the quadratic functional equation (1.7) for the case \( a \neq (\pm 1/2) \).

**Theorem 2.5.** Let \(|2a| > 1\) and \( f : X \rightarrow Y \) a mapping with \( f(0) = 0 \). Suppose that \( \varphi \) is a mapping from \( X \) to a fuzzy normed space \((Z, N')\) such that

\[
N\left( f(ax + ay) + f(ax - ay) - 2a^2 f(x) - 2a^2 f(y), t + s \right) \geq \min\left\{ N'(\varphi(x), t), N'(\varphi(y), s) \right\}
\]  
(2.46)

for all \( x, y \in X \setminus \{0\} \) and all positive real numbers \( t, s \). If \( \varphi(2ax) = a\varphi(x) \) for some positive real number \( a \) with \( 0 < a < 4a^2 \), then there is a unique quadratic mapping \( Q : X \rightarrow Y \) such that \( Q(x) = N\lim_{n \to \infty} f((2a)^n x)/{(2a)^{2n}} \) and

\[
N(Q(x) - f(x), t) \geq N'\left( \varphi(x), \frac{4a^2 - a}{4} t \right)
\]  
(2.47)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Putting \( y = x \) and \( s = t \) in (2.46), we get

\[
N\left( f(2ax) - 4a^2 f(x), 2t \right) \geq N'(\varphi(x), t)
\]  
(2.48)

for all \( x \in X \) and all \( t > 0 \). Thus

\[
N\left( f(x) - \frac{f(2ax)}{4a^2}, \frac{t}{2a^2} \right) \geq N'(\varphi(x), t)
\]  
(2.49)

and so

\[
N\left( f(x) - \frac{f(2ax)}{4a^2}, t \right) \geq N'\left( \varphi(x), 2a^2 t \right).
\]  
(2.50)

Replacing \( x \) by \((2a)^n x\) in (2.50), we get

\[
N\left( \frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f((2a)^{n+1} x)}{(2a)^{2n+2}}, \frac{a^n t}{(2a)^{2n}} \right) = N\left( f((2a)^n x) - \frac{f((2a)^{n+1} x)}{4a^2}, a^n t \right)
\]  
(2.51)

\[\geq N'(\varphi(x), 2a^2 t).\]
Thus for each \( n > m \) we have

\[
N\left( \frac{f((2a)^{m}x)}{(2a)^{2m}} - \frac{f((2a)^{n}x)}{(2a)^{2n}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}} \right) = N\left( \sum_{k=m}^{n-1} \left( \frac{f((2a)^{k}x)}{(2a)^{2k}} - \frac{f((2a)^{k+1}x)}{(2a)^{2k+2}} \right), \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}} \right) \\
\geq \min \left\{ \sum_{k=m}^{n-1} \left\{ N\left( \frac{f((2a)^{k}x)}{(2a)^{2k}} - \frac{f((2a)^{k+1}x)}{(2a)^{2k+2}}, \frac{\alpha^k t}{(2a)^{2k}} \right) \right\} \right\} \\
\geq N\left( \varphi(x), 2a^2 t_0 \right) \\
\geq 1 - \varepsilon
\]

for all \( t \geq t_0 \). This shows that the sequence \( \{ f((2a)^{n}x)/(2a)^{2n} \} \) is Cauchy in \((Y, N)\). Since \((Y, N)\) is complete, \( \{ f((2a)^{n}x)/(2a)^{2n} \} \) converges to some \( Q(x) \in Y \). Thus we can define a mapping \( Q : X \to Y \) by \( Q(x) := N\lim_{t \to \infty} f((2a)^{n}x)/(2a)^{2n} \). Moreover, if we put \( m = 0 \) in (2.52), then we observe that

\[
N\left( \frac{f((2a)^{n}x)}{(2a)^{2n}} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{(2a)^{2k}} \right) \geq N\left( \varphi(x), 2a^2 t \right).
\]

Thus

\[
N\left( \frac{f((2a)^{n}x)}{(2a)^{2n}} - f(x), t \right) \geq N\left( \varphi(x), \frac{2a^2 t}{\sum_{k=0}^{n-1} (\alpha/(2a)^{2})^k} \right).
\]

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Theorem 2.6.** Let \( |2a| < 1 \) and \( f : X \to Y \) a mapping with \( f(0) = 0 \). Suppose that \( \varphi \) is a mapping from \( X \) to a fuzzy normed space \((Z, N')\) satisfying (2.46). If \( \varphi(2ax) = \alpha \varphi(x) \) for some real number
\( \alpha \) with \( \alpha > 4a^2 \), then there is a unique quadratic mapping \( Q : X \to Y \) such that \( Q(x) = N^{-}\lim_{n \to \infty} (2a)^{2n} f(x/(2a)^n) \) and

\[
N(Q(x) - f(x), t) \geq M \left( x, \frac{(\alpha - 4a^2)t}{4} \right) \tag{2.56}
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** It follows from (2.50) that

\[
N\left( f(x) - (2a)^2 f\left( \frac{x}{2a} \right), 2t \right) \geq N'\left( \varphi\left( \frac{x}{2a} \right), t \right) \tag{2.57}
\]

for all \( x \in X \) and all \( t > 0 \). Thus

\[
N\left( f(x) - 4a^2 f\left( \frac{x}{2a} \right), t \right) \geq N'\left( \varphi\left( \frac{x}{2a} \right), \frac{t}{2} \right) = N'\left( \varphi(x), \frac{\alpha t}{2} \right). \tag{2.58}
\]

Replacing \( x \) by \( x/(2a)^n \) in (2.58), we get

\[
N\left( (2a)^{2n} f\left( \frac{x}{(2a)^n} \right) - (2a)^{2n+2} f\left( \frac{x}{(2a)^{n+1}} \right), \frac{(2a)^{2n} t}{a^n} \right)
\]

\[
= N\left( f\left( \frac{x}{(2a)^n} \right) - 4a^2 f\left( \frac{x}{(2a)^{n+1}} \right), \alpha^n t \right) \tag{2.59}
\]

\[
\geq N'\left( \varphi(x), \frac{\alpha t}{2} \right).
\]

Thus for each \( n > m \) we have

\[
N\left( (2a)^{2m} f\left( \frac{x}{(2a)^m} \right) - (2a)^{2m} f\left( \frac{x}{(2a)^n} \right), \frac{\sum_{k=m}^{n-1} (2a)^{2k} t}{a^k} \right)
\]

\[
= N\left( \sum_{k=m}^{n-1} (2a)^{2k} f\left( \frac{x}{(2a)^k} \right) - (2a)^{2k+2} f\left( \frac{x}{(2a)^{k+1}} \right), \frac{\sum_{k=m}^{n-1} (2a)^{2k} t}{a^k} \right) \tag{2.60}
\]

\[
\geq \min \left\{ \bigcup_{k=m}^{n-1} \left[ N\left( (2a)^{2k} f\left( \frac{x}{(2a)^k} \right) - (2a)^{2k+2} f\left( \frac{x}{(2a)^{k+1}} \right), \frac{(2a)^{2k} t}{a^k} \right) \right] \right\}
\]

\[
\geq N'\left( \varphi(x), \frac{\alpha t}{2} \right).
\]
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Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \to \infty} N^t(\varphi(x), (a/2)t) = 1$, there is some $t_0 > 0$ such that $N^t(\varphi(x), (a/2)t_0) > 1 - \varepsilon$. Since $\sum_{k=0}^{\infty}(2a)^{2k}t_0/a^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n_0-1}(2a)^{2k}t_0/a^k < \delta$ for $n > m \geq n_0$. It follows that

$$N\left((2a)^{2m}f\left(\frac{x}{(2a)^m}\right) - (2a)^{2n}f\left(\frac{x}{(2a)^n}\right), \delta\right)$$

$$\geq N\left((2a)^{2m}f\left(\frac{x}{(2a)^m}\right) - (2a)^{2n}f\left(\frac{x}{(2a)^n}\right), \sum_{k=m}^{n_0-1}(2a)^{2k}t_0/a^k\right)$$

$$\geq N^t(\varphi(x), \frac{\alpha}{2}t_0)$$

$$\geq 1 - \varepsilon$$

for all $t \geq t_0$. This shows that the sequence $\{(2a)^{2n}f(x/(2a)^n)\}$ is Cauchy in $(Y,N)$. Since $(Y,N)$ is complete, $\{(2a)^{2n}f(x/(2a)^n)\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\lim_{t \to \infty}(2a)^{2n}f(x/(2a)^n)$. Moreover, if we put $m = 0$ in (2.60), then we observe that

$$N\left((2a)^{2n}f\left(\frac{x}{(2a)^n}\right) - f(x), \sum_{k=0}^{n-1}(2a)^{2k}t_0/a^k\right) \geq N^t(\varphi(x), \frac{\alpha}{2}t).$$

Thus

$$N\left((2a)^{2n}f\left(\frac{x}{(2a)^n}\right) - f(x), t\right) \geq N^t\left(\varphi(x), \frac{at}{2 \sum_{k=0}^{n-1}((2a)^2/a)^k}\right).$$

The rest of the proof is similar to the proof of Theorem 2.1. \qed

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**References**


