Research Article

Some New Wilker-Type Inequalities for Circular and Hyperbolic Functions

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, China

Correspondence should be addressed to Ling Zhu, zhuling0571@163.com

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In this paper, we give some new Wilker-type inequalities for circular and hyperbolic functions in exponential form by using generalizations of Cusa-Huygens inequality and Cusa-Huygens-type inequality.

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1. Introduction

Wilker [1] proposed two open questions, the first of which was the following statement.

Problem 1. Let $0 < x < \pi/2$. Then

$$\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > 2$$

(1.1)

holds.

Sumner et al. [2] proved inequality (1.1). Guo et al. [3] gave a new proof of inequality (1.1). Zhu [4, 5] showed two new simple proofs of Wilker’s inequality above, respectively. Recently, Wu and Srivastava [6] obtained Wilker-type inequality as follows:

$$\left( \frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2}.$$  

(1.2)

Baricz and Sandor [7] found that inequality (1.2) can be proved by using inequality (1.1).
On the other hand, in the form of inequality (1.1), Zhu [5] obtained the following Wilker type inequality:

\[
\left( \frac{\sinh x}{x} \right)^2 + \frac{\tanh x}{x} > 2, \quad x > 0.
\] (1.3)

In fact, we can obtain further results:

\[
\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > \left( \frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2},
\] (1.4)

\[
\left( \frac{\sinh x}{x} \right)^2 + \frac{\tanh x}{x} > \left( \frac{x}{\sinh x} \right)^2 + \frac{x}{\tanh x} > 2, \quad x > 0.
\]

In this note, we establish the following four new Wilker type inequalities in exponential form for circular and hyperbolic functions.

**Theorem 1.1.** Let \(0 < x < \pi/2\), \(\alpha \in \mathbb{R}\) and \(\alpha \neq 0\). Then

(i) when \(\alpha > 0\), the inequality

\[
\left( \frac{\sin x}{x} \right)^{2\alpha} + \left( \frac{\tan x}{x} \right)^{\alpha} > \left( \frac{x}{\sin x} \right)^{2\alpha} + \left( \frac{x}{\tan x} \right)^{\alpha}
\] (1.5)

holds;

(ii) when \(\alpha < 0\), inequality (1.5) is revered.

**Theorem 1.2.** Let \(0 < x < \pi/2\) and \(\alpha \geq 1\). Then the inequality

\[
\left( \frac{\sin x}{x} \right)^{2\alpha} + \left( \frac{\tan x}{x} \right)^{\alpha} > \left( \frac{x}{\sin x} \right)^{2\alpha} + \left( \frac{x}{\tan x} \right)^{\alpha} > 2
\] (1.6)

holds.

**Theorem 1.3.** Let \(x > 0\), \(\alpha \in \mathbb{R}\) and \(\alpha \neq 0\). Then

(i) when \(\alpha > 0\), the inequality

\[
\left( \frac{\sinh x}{x} \right)^{2\alpha} + \left( \frac{\tanh x}{x} \right)^{\alpha} > \left( \frac{x}{\sinh x} \right)^{2\alpha} + \left( \frac{x}{\tanh x} \right)^{\alpha}
\] (1.7)

holds;

(ii) when \(\alpha < 0\), inequality (1.7) is revered.
Theorem 1.4. Let \( x > 0 \) and \( \alpha \geq 1 \). Then the inequality
\[
\left( \frac{\sinh x}{x} \right)^{2\alpha} + \left( \frac{\tanh x}{x} \right)^{\alpha} > \left( \frac{x}{\sinh x} \right)^{2\alpha} + \left( \frac{x}{\tanh x} \right)^{\alpha} > 2
\] (1.8)
holds.

2. Lemmas

Lemma 2.1 (see [8–24]). Let \( f, g : [a, b] \to \mathbb{R} \) be two continuous functions which are differentiable on \((a, b)\). Further, let \( g' \neq 0 \) on \((a, b)\). If \( f'/g' \) is increasing (or decreasing) on \((a, b)\), then the functions \((f(x) - f(b))/(g(x) - g(b))\) and \((f(x) - f(a))/(g(x) - g(a))\) are also increasing (or decreasing) on \((a, b)\).

Lemma 2.2 (see [25–27]). Let \( a_n \) and \( b_n \) \((n = 0, 1, 2, \ldots)\) be real numbers, and let the power series \( A(t) = \sum_{n=0}^{\infty} a_n t^n \) and \( B(t) = \sum_{n=0}^{\infty} b_n t^n \) be convergent for \(|t| < R\). If \( b_n > 0 \) for \( n = 0, 1, 2, \ldots \), and if \( a_n/b_n \) is strictly increasing (or decreasing) for \( n = 0, 1, 2, \ldots \), then the function \( A(t)/B(t) \) is strictly increasing (or decreasing) on \((0, R)\).

Lemma 2.3 (see [28, 29]). Let \( |x| < \pi \), then the inequality
\[
\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n}
\] (2.1)
holds.

Lemma 2.4. Let \( |x| < \pi \), then the inequality
\[
\frac{1}{\sin^2 x} = \csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n - 1) x^{2n-2}
\] (2.2)
holds.

Proof. The following power series expansion can be found in [30, 1.3.1.4 (3)]
\[
\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi.
\] (2.3)

Then
\[
\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n - 1) x^{2n-2}, \quad |x| < \pi.
\] (2.4)
Lemma 2.5 (see [5, 31]). Let $0 < x < \pi/2$. Then the inequality

$$
\left( \frac{\sin x}{x} \right)^3 > \cos x \tag{2.5}
$$

holds.

Lemma 2.6 (see [5, 31, 32]). Let $x > 0$. Then the inequality

$$
\left( \frac{\sinh x}{x} \right)^3 > \cosh x \tag{2.6}
$$

holds.

Lemma 2.7. Let $0 < x < \pi/2$. Then the function $G(\alpha) = ((\sin x/x)^{2\alpha} + (\tan x/x)^{\alpha})/((x/\sin x)^{2\alpha} + (x/\tan x)^{\alpha})$ increases as $\alpha$ increases on $(-\infty, +\infty)$.

Lemma 2.8. Let $x > 0$. Then the function $H(\alpha) = ((\sinh x/x)^{2\alpha} + (\tanh x/x)^{\alpha})/((x/\sinh x)^{2\alpha} + (x/\tanh x)^{\alpha})$ increases as $\alpha$ increases on $(-\infty, +\infty)$.

Lemma 2.9 (a generalization of Cusa-Huygens inequality). Let $0 < x < \pi/2$ and $\alpha \geq 1$. Then the inequality

$$
2 \left( \frac{x}{\sin x} \right)^{\alpha} + \left( \frac{x}{\tan x} \right)^{\alpha} > 3 \tag{2.7}
$$

or

$$
\left( \frac{\sin x}{x} \right)^{\alpha} < \frac{2}{3} + \frac{1}{3} (\cos x)^{\alpha} \tag{2.8}
$$

holds.

Lemma 2.10 (a generalization of Cusa-Huygens type inequality). Let $x > 0$ and $\alpha \geq 1$. Then the inequality

$$
2 \left( \frac{x}{\sinh x} \right)^{\alpha} + \left( \frac{x}{\tanh x} \right)^{\alpha} > 3 \tag{2.9}
$$

or

$$
\left( \frac{\sinh x}{x} \right)^{\alpha} < \frac{2}{3} + \frac{1}{3} (\cosh x)^{\alpha} \tag{2.10}
$$

holds.
3. Proofs of Lemma 2.7 and Theorem 1.1

Proof of Lemma 2.7. Direct calculation yields \( G'(\alpha) = J(\alpha)/[(x/\sin x)^{2\alpha} + (x/\tan x)^{\alpha}]^2 \), where

\[
J(\alpha) = \left[ \left( \frac{\tan x}{x} \right)^{\alpha} \left( \frac{x}{\sin x} \right)^{2\alpha} - \left( \frac{x}{\tan x} \right)^{\alpha} \left( \frac{\sin x}{x} \right)^{2\alpha} + 2 \right] \log \frac{\tan x}{x}
+ 2 \left[ \left( \frac{\tan x}{x} \right)^{\alpha} \left( \frac{x}{\sin x} \right)^{2\alpha} - \left( \frac{x}{\tan x} \right)^{\alpha} \left( \frac{\sin x}{x} \right)^{2\alpha} - 2 \right] \log \frac{x}{\sin x}
= \left[ \left( \frac{2x}{\sin 2x} \right)^{\alpha} - \left( \frac{\sin 2x}{2x} \right)^{\alpha} + 2 \right] \log \frac{\tan x}{x} + 2 \left[ \left( \frac{2x}{\sin 2x} \right)^{\alpha} - \left( \frac{\sin 2x}{2x} \right)^{\alpha} - 2 \right] \log \frac{x}{\sin x}
= \log \left[ \left( \frac{\tan x}{x} \right)^{(2x/\sin 2x)^{\alpha} - (\sin 2x/2x)^{\alpha} + 2} \left( \frac{\sin 2x}{\sin x} \right)^{(\sin x/x)^{3} - 1} \left( \frac{1}{\cos x} \right)^{2} \right] \tag{3.1}
\]

First, we have \([ (\sin x/x)^{3} (1/\cos x) ]^{2} > 1 \) by Lemma 2.5. Second, when letting \( 2x/\sin 2x = t \) for \( 0 < x < \pi/2 \), we have \( t > 1 \), and \( t^{\alpha} - t^{-\alpha} > 0 \) for \( \alpha > 0 \), so \( t^{\alpha} - t^{-\alpha} > 1 \) and \( (2x/\sin 2x)^{(2x/\sin 2x)^{\alpha} - (\sin 2x/2x)^{\alpha}} \) \([ (\sin x/x)^{3} (1/\cos x) ]^{2} > 1 \). Thus \( J(\alpha) > 0 \) and \( G'(\alpha) > 0 \). The proof of Lemma 2.7 is complete.

\( \square \)

Proof of Theorem 1.1. From Lemma 2.7 we have \( G(\alpha) > G(0) = 1 \) for \( \alpha > 0 \). That is, (1.5) holds. At the same time, we have \( G(\alpha) < G(0) = 1 \) for \( \alpha < 0 \). That is, (1.5) is revered.

\( \square \)

4. Proofs of Lemma 2.9 and Theorem 1.2

Proof of Lemma 2.9. Let \( F(x) = ((\sin x/x)^{\alpha} - 1)/((\cos x)^{\alpha} - 1) =: f(x)/g(x) \), where \( f(x) = (\sin x/x)^{\alpha} - 1 \), and \( g(x) = (\cos x)^{\alpha} - 1 \). Then

\[
k(x) = \frac{f'(x)}{g'(x)} = \left( \frac{\sin x}{x \cos x} \right)^{\alpha - 1} \sin x - x \cos x \frac{x^2 \sin x}{x^2 \sin x}, \quad k'(x) = \left( \frac{\sin x}{x \cos x} \right)^{\alpha - 2} \frac{u(x)}{x^4 \sin x \cos^2 x} \tag{4.1}
\]

where

\[
u(x) = (\alpha - 1)(x - \sin x \cos x)(\sin x - x \cos x) + \cos x \left( x^2 - 2 \sin^2 x + x \sin x \cos x \right) - \left( x \sin x + \sin^2 x \cos x - 2 x^2 \cos x \right)
= \left( x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x \right) \alpha
= \left( x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x \right) (\alpha - G(x)), \tag{4.2}
\]
where \( G(x) = (x \sin x + \sin^2 x \cos x - 2x^2 \cos x) / (x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x) \). Then
\[
G(x) = \frac{2x/ \sin 2x + 1 - 2x^2/ \sin^2 x}{2x/ \sin 2x - 1 - (x/ \sin x)^2 + x \cot x} := \frac{A(x)}{B(x)},
\]
where \( A(x) = 2x/ \sin 2x + 1 - (2x^2/ \sin^2 x), \) and \( B(x) = 2x/ \sin 2x - 1 - (x/ \sin x)^2 + x \cot x \). By (2.1), (2.2), and (2.3), we have
\[
A(x) = 1 + \sum_{n=1}^{\infty} \frac{2^{2n-2}}{(2n)!} |B_{2n}|(2x)^{2n} + 1 - 2 \left( 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}|(2n-1)x^{2n} \right) = \sum_{n=1}^{\infty} \frac{(2^{2n} - 4n)2^{2n}}{(2n)!} |B_{2n}|x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n},
\]
\[
B(x) = 1 + \sum_{n=1}^{\infty} \frac{2^{2n-2}}{(2n)!} |B_{2n}|(2x)^{2n} - 1 - \left( 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}|(2n-1)x^{2n} \right) + 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}|x^{2n} = \sum_{n=1}^{\infty} b_n x^{2n},
\]
where \( a_n = ((2^{2n} - 4n)2^{2n}/(2n)!)|B_{2n}| \) and \( b_n = ((2^{2n} - 2n - 2)2^{2n}/(2n)!)|B_{2n}| > 0 \).

When setting \( c_n = a_n/b_n, \) we have that \( c_n = (2^{2n} - 4n)/(2^{2n} - 2n - 2) \) is increasing for \( n = 2, 3, \ldots, \) \( A(x)/B(x) \) is increasing from \((0, \pi/2)\) onto \((4/5, 1)\) by Lemma 2.2. When \( \alpha \geq 1, \) we have \( u(x) \geq 0. \) So \( k(x) \) is increasing on \((0, \pi/2)\). This leads to that \( f'(x)/g'(x) \) is increasing on \((0, \pi/2)\). Thus \( F(x) = f(x)/g(x) = (f(x) - f(0^+))/(g(x) - g(0^+)) \) is increasing on \((0, \pi/2)\) by Lemma 2.1. At the same time, \( \lim_{x \to 0^+} F(x) = 1/3 \). So the proof of Lemma 2.9 is complete.

Proof of Theorem 1.2. From Theorem 1.1, when \( \alpha \geq 1 \) we have
\[
\left( \frac{\sin x}{x} \right)^{2\alpha} + \left( \frac{\tan x}{x} \right)^\alpha > \left( \frac{x}{\sin x} \right)^{2\alpha} + \left( \frac{x}{\tan x} \right)^\alpha.
\]
On the other hand, when \( \alpha \geq 1 \) we can obtain
\[
1 + \left( \frac{x}{\sin x} \right)^{2\alpha} + \left( \frac{x}{\tan x} \right)^\alpha \geq 2 \left( \frac{x}{\sin x} \right)^\alpha + \left( \frac{x}{\tan x} \right)^\alpha > 3
\]
by the arithmetic mean-geometric mean inequality and Lemma 2.9. So
\[
\left( \frac{x}{\sin x} \right)^{2\alpha} + \left( \frac{x}{\tan x} \right)^\alpha > 2
\]
holds.

Combining (4.5) and (4.7) gives (1.6).
5. Proofs of Lemma 2.8 and Theorem 1.3

Proof of Lemma 2.8. Direct calculation yields \( H'(x) = I(a)/[(x/\sinh x)^{2a} + (x/\tanh x)^{2a}]^2 \), where

\[
I(a) = \left[ \left( \frac{x}{\sinh x} \right)^a + \left( \frac{x}{\tanh x} \right)^a \right] + 2 \left[ \left( \frac{x}{\sinh x} \right)^a + \left( \frac{x}{\tanh x} \right)^a \right] + 2 \log \frac{x}{\sinh x}
\]

Adding this to the previous line gives

\[
I(a) = \left( \frac{x}{\sinh x} \right)^a + \left( \frac{x}{\tanh x} \right)^a \left[ \left( \frac{x}{\sinh x} \right)^a + \left( \frac{x}{\tanh x} \right)^a \right] + 2 \log \left( \frac{x}{\sinh x} \right)^3 \frac{1}{\cosh x} \right].
\]

First, \((\tanh x/x)^a(x/\sinh x)^{2a} + (x/\tanh x)^a(x/\sinh x)^{2a} + 2 > 0 \) for \( x > 0 \). Second, we have \( \log[(\sinh x/x)^a(1/\cosh x)] > 0 \) by Lemma 2.6. Thus \( I(a) > 0 \) and \( H'(x) > 0 \). The proof of Lemma 2.8 is complete.

Proof of Theorem 1.3. From Lemma 2.8 we have \( H(x) > H(0) = 1 \) for \( a > 0 \). That is, (1.7) holds. At the same time, we have \( H(x) < H(0) = 1 \) for \( \alpha < 0 \). That is, (1.7) is revered.

6. Proofs of Lemma 2.10 and Theorem 1.4

Proof of Lemma 2.10. Let \( Q(x) = ((\sinh x/x)^a - 1)/((\cosh x)^a - 1) =: f(x)/g(x) \), where \( f(x) = (\sinh x/x)^a - 1 \) and \( g(x) = (\cosh x)^a - 1 \). Then

\[
k(x) = \frac{f(x)}{g'(x)} = \frac{\sinh x}{l \cosh x} \left( \frac{x}{l \cosh x} \right)^{a-1} x \cosh x - \sinh x = 2 x \sinh x \frac{\sinh x}{l \cosh x} = 2 x \sinh x \frac{a_1}{B(x)},
\]

where \( A(x) = x \cosh x - \sinh x \) and \( B(x) = x^2 \sinh x \). Since

\[
A(x) = x \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(2n)x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} a_n x^{2n+3},
\]

\[
B(x) = x \sum_{n=0}^{\infty} \frac{x^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty} b_n x^{2n+3},
\]

where \( a_n = (2n+2)/(2n+3)! \) and \( b_n = 1/(2n+1)! \).

When setting \( n = a_n/b_n \), we have \( c_n = 1/(2n+3) \) is decreasing for \( n = 0, 1, 2, \ldots \), \( A(x)/B(x) \) is decreasing on \((0, +\infty)\) by Lemma 2.2. At the same time, the function \((\tanh x/x)^{a-1}\) is decreasing on \((0, +\infty)\) when \( a \geq 1 \). By (6.1), we obtain that \( k(x) = f'(x)/g'(x) \) is decreasing on \((0, +\infty)\). Thus \( Q(x) = f(x)/g(x) = (f(x)-(f(0^+)))/(g(x)-(g(0^+))) \) is decreasing on \((0, +\infty)\) by Lemma 2.1. At the same time, \( \lim_{x \to 0} Q(x) = 1/3 \). So the proof of Lemma 2.10 is complete. □
Proof of Theorem 1.4. By the same way as Theorem 1.2, we can prove Theorem 1.4. □

7. Open Problem

In this section, we pose the following open problem: find the respective largest range of $\alpha$ such that the inequalities (1.6) and (1.8) hold.

References


