Some Computational Formulas for $D$-Nörlund Numbers

Guodong Liu
Department of Mathematics, Huizhou University, Huizhou, Guangdong 516015, China
Correspondence should be addressed to Guodong Liu, gdliu@pub.huizhou.gd.cn
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The author establishes some identities involving the $D$ numbers, Bernoulli numbers, and central factorial numbers of the first kind. A generating function and several computational formulas for $D$-Nörlund numbers are also presented.

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1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order $k$, for any integer $k$, may be defined by (see [1–5])

$$
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi.
$$

(1.1)

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order $k$, $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 6, 7]). By (1.1), we can get (see [4, page 145])

$$
\frac{d}{dx} B_n^{(k)}(x) = n B_n^{(k)}(x),
$$

(1.2)

$$
B_n^{(k+1)}(x) = \frac{k - n}{k} B_n^{(k)}(x) + (x - k) \frac{n}{k} B_n^{(k)}(x),
$$

(1.3)

$$
B_n^{(k+1)}(x + 1) = \frac{nx}{k} B_n^{(k)}(x) - \frac{n - k}{k} B_n^{(k)}(x),
$$

(1.4)

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.
The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [4, 8]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [4, page 150])

$$\frac{t}{(1 + t) \log(1 + t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \quad (1.5)$$

The $D$ numbers $D_{2n}^{(k)}$ may be defined by (see [4, 5])

$$(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \quad (1.6)$$

By (1.1), (1.6), and note that $\csc t = 2i/(e^{it} - e^{-it})$ (where $i^2 = -1$), we can get

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)} \left( \frac{k}{2} \right). \quad (1.7)$$

Taking $k = 1, 2$ in (1.7), and note that $B_{2n}^{(1)} (1/2) = (2^{1-2n} - 1)B_{2n}, B_{2n}^{(2)} (1) = (1 - 2n)B_{2n}$ (see [4, pages 22 and 145]), we have

$$D_{2n}^{(1)} = (2 - 2^{2n})B_{2n}, \quad D_{2n}^{(2)} = 4^n (1 - 2n)B_{2n}. \quad (1.8)$$

The numbers $D_{2n}^{(2n)}$ are called the $D$-Nörlund numbers. These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\int_0^{\pi/2} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \quad \int_0^{\pi/2} \frac{\sin t}{t} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)} \quad (1.9)$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \quad (1.10)$$

We now turn to the central factorial numbers $t(n, k)$ of the first kind, which are usually defined by (see [9–12])

$$x \left( x + \frac{n}{2} - 1 \right) \left( x + \frac{n}{2} - 2 \right) \cdots \left( x + \frac{n}{2} - n + 1 \right) = \sum_{k=0}^{n} t(n, k)x^k, \quad (1.11)$$

or by means of the following generating function:

$$\left( 2 \log \left( \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right) \right)^k = k! \sum_{n=k}^{\infty} t(n, k) \frac{x^n}{n!}. \quad (1.12)$$
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It follows from (1.11) or (1.12) that

\[ t(n, k) = t(n - 2, k - 2) - \frac{1}{4}(n - 2)^2 t(n - 2, k), \]  

(1.13)

and that

\[ t(n, 0) = \delta_{n,0} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \quad t(n, n) = 1 \quad (n \in \mathbb{N}), \]

\[ t(n, k) = 0 \quad (n + k \text{ odd}), \quad t(n, k) = 0 \quad (k > n \text{ or } k < 0), \]  

(1.14)

where \(\delta_{m,n}\) denotes the Kronecker symbol.

By (1.13), we have

\[ t(2n + 1, 1) = \frac{(-1)^n(n!)(2n)}{4^{2n}} \binom{2n}{n}, \quad t(2n + 2, 2) = (-1)^n(n!)^2 \quad (n \in \mathbb{N}_0), \]  

(1.15)

\[ t(2n + 2, 4) = (-1)^{n+1}(n!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \quad (n \in \mathbb{N}), \]  

(1.16)

\[ t(2n + 1, 3) = \frac{(-1)^{n-1}(2n)!}{4^{2n-1}} \binom{2n}{n} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\right) \quad (n \in \mathbb{N}). \]  

(1.17)

The main purpose of this paper is to prove some identities involving \(D\) numbers, Bernoulli numbers, and central factorial numbers of the first kind and obtain a generating function and several computational formulas for the \(D\)-Nörlund numbers. That is, we will prove the following main conclusion.

**Theorem 1.1.** Let \(n \in \mathbb{N}, k \in \mathbb{N} \setminus \{1\}\). Then

\[ D^{(k)}_{2n} = \frac{(2n - k + 2)(2n - k + 1)}{(k - 2)(k - 1)} D^{(k-2)}_{2n} - \frac{2n(2n - 1)(k - 2)}{k - 1} D^{(k-2)}_{2n-2}. \]  

(1.18)

**Remark 1.2.** By (1.18), we may immediately deduce the following (see [4, page 147]):

\[ D^{(2n+1)}_{2n} = \frac{(-1)^n(n!)(2n)}{4^n} \binom{2n}{n}, \quad D^{(2n+2)}_{2n} = \frac{(-1)^n 4^n}{2n + 1} (n!)^2. \]  

(1.19)

**Theorem 1.3.** Let \(n \geq k \quad (n, k \in \mathbb{N}_0)\). Then

\[ D^{(2n+1)}_{2n-2k} = \frac{4^{n-k}}{2n \binom{2n}{2k}} t(2n + 1, 2k + 1), \]  

(1.20)

\[ D^{(2n)}_{2n-2k} = \frac{4^{n-k}}{2n-1 \binom{2n-1}{2k-1}} t(2n, 2k) \quad (k \geq 1). \]  

(1.21)
Remark 1.4. By (1.20) and (1.17), we may immediately deduce the following:

\[
D_{2n}^{(2n+3)} = \frac{(-1)^n(2n)!}{2 \cdot 4^n} \left( \binom{2n+2}{n+1} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} \right) \right). \quad (1.22)
\]

Theorem 1.5. Let \( n \in \mathbb{N}_0 \). Then

\[
\sum_{j=0}^{n} (-1)^j \frac{4^j}{(2j+1)!} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n},
\]

so one finds \( D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \ldots \).

By (1.23), and note that

\[
\log(t + \sqrt{1 + t^2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)!} \binom{2n}{n} t^{2n+1} \quad (|t| < 1),
\]

\[
\frac{1}{\sqrt{1 + t^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n} t^{2n} \quad (|t| < 1),
\]

one may immediately deduce the following Corollary 1.6.

Corollary 1.6. Let \( n \in \mathbb{N}_0 \). Then

\[
\sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!} = \frac{t}{\sqrt{1 + t^2} \log(t + \sqrt{1 + t^2})} \quad (|t| < 1). \quad (1.25)
\]

Theorem 1.7. Let \( n \in \mathbb{N} \). Then

(i)

\[
D_{2n}^{(2n)} = \frac{(-1)^n(2n)!}{4^n} \binom{2n}{n} + n \cdot 4^n \sum_{j=1}^{n} \frac{1 - 2^{1-2j}}{j} t(2n, 2j) B_{2j}
\]

(ii)

\[
D_{2n}^{(2n)} = \frac{(-1)^n 4^n (n!)^2}{2n+1} + n \cdot 4^n \sum_{j=1}^{n} \frac{2j - 1}{j} t(2n, 2j) B_{2j}
\]

(1.26)
2. Proof of the Theorems

Proof of Theorem 1.1. By (1.4) and (1.3), we have

\[
B_{2n}^{(k)}(x + 1) = \frac{2nx}{k - 1} B_{2n-1}^{(k-1)}(x) - \frac{2n - k + 1}{k - 1} B_{2n}^{(k-1)}(x)
\]

\[
= \frac{2nx}{k - 1} \left( \frac{k - 2n - 1}{k - 2} B_{2n-1}^{(k-2)}(x) + (x - k + 2) \frac{2n - 1}{k - 2} B_{2n-2}^{(k-2)}(x) \right)
\]

\[
- \frac{2n - k + 1}{k - 1} \left( \frac{k - 2n - 2}{k - 2} B_{2n-1}^{(k-2)}(x) + (x - k + 2) \frac{2n}{k - 2} B_{2n-2}^{(k-2)}(x) \right)
\]

\[
= \frac{(2n - k + 1)(2n - k + 2)}{(k - 1)(k - 2)} B_{2n}^{(k-2)}(x) + \frac{2n(2n - 1)}{(k - 1)(k - 2)} x(x - k + 2) B_{2n-2}^{(k-2)}(x)
\]

\[
- \frac{2n(2n - k + 1)}{(k - 1)(k - 2)} (2x - k + 2) B_{2n-1}^{(k-2)}(x).
\]

Setting \(x = (k - 2)/2\) in (2.1), we get

\[
B_{2n}^{(k)} \left( \frac{k}{2} \right) = \frac{(2n - k + 1)(2n - k + 2)}{(k - 1)(k - 2)} B_{2n}^{(k-2)} \left( \frac{k - 2}{2} \right) - \frac{2n(2n - 1)(k - 2)}{4(k - 1)} B_{2n-1}^{(k-2)} \left( \frac{k - 2}{2} \right).
\]

(2.2)

By (2.2) and (1.7), we immediately obtain (1.18). This completes the proof of Theorem 1.1. □
Proof of Theorem 1.3. By the usage of Theorem 1.1 and (1.13).

Proof of Theorem 1.5. Note the identity (see [4, page 203])

\[
B_{2n+1}^{(k)}\left(x + \frac{k}{2}\right) = \sum_{j=0}^{n} \binom{2n+1}{2j+1} D_{2n-2j}^{(k-2j-1)} x \left(x^2 - \left(\frac{1}{2}\right)^2\right) \left(x^2 - \left(\frac{3}{2}\right)^2\right) \cdots \left(x^2 - \left(\frac{2j-1}{2}\right)^2\right),
\]

we have

\[
\lim_{x \to 0} \frac{B_{2n+1}^{(2n+1)}(x + (2n + 1)/2)}{x} = \frac{1}{4^n} \sum_{j=0}^{n} \binom{2n+1}{2j+1} D_{2n-2j}^{(2n-2j)} (-1)^j 1^2 3^2 \cdots (2j - 1)^2
\]

\[
= \frac{(2n + 1)!}{4^n} \sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j + 1)} \binom{2j}{j} D_{2n-2j}^{(2n-2j)} (2n - 2j)!. \tag{2.4}
\]

By (2.4) and (1.2), we have

\[
\lim_{x \to 0} (2n + 1)B_{2n}^{(2n+1)}\left(x + \frac{2n + 1}{2}\right) = \frac{(2n + 1)!}{4^n} \sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j + 1)} \binom{2j}{j} D_{2n-2j}^{(2n-2j)} (2n - 2j)!. \tag{2.5}
\]

that is,

\[
B_{2n}^{(2n+1)}\left(\frac{2n + 1}{2}\right) = \frac{(2n)!}{4^n} \sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j + 1)} \binom{2j}{j} D_{2n-2j}^{(2n-2j)} (2n - 2j)!. \tag{2.6}
\]

By (2.6) and (1.7), we have

\[
D_{2n}^{(2n+1)} = (2n)! \sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j + 1)} \binom{2j}{j} D_{2n-2j}^{(2n-2j)} (2n - 2j)!. \tag{2.7}
\]

By (2.7) and (1.19), we immediately obtain (1.23). This completes the proof of Theorem 1.5.

Proof of Theorem 1.7. By (1.6), we have

\[
D_{2n}^{(k)} = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(k-4j)} D_{2j}^{(l)}, \tag{2.8}
\]

where \(l\) is an integer.
Setting \( k = 2n + 1, l = 1 \) in (2.8), and note that \( D_0^{(1)} = 1 \), we have

\[
D_{2n}^{(2n+1)} = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)} = D_{2n}^{(2n)} + \sum_{j=1}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)}. \tag{2.9}
\]

By (2.9), (1.19), (1.8), and (1.21), we immediately obtain (1.26).

Setting \( k = 2n + 2, l = 2 \) in (2.8), and note that \( D_0^{(2)} = 1 \), we have

\[
D_{2n}^{(2n+2)} = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)} = D_{2n}^{(2n)} + \sum_{j=1}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)}. \tag{2.10}
\]

By (2.10), (1.19), (1.8), and (1.21), we immediately obtain (1.27).

Setting \( k = 2n, l = -1 \) in (2.8), and note that (1.20) and \( D_{2j}^{(-1)} = 1/(2j + 1) \), we immediately obtain (1.18). This completes the proof of Theorem 1.7.

**Proof of Theorem 1.8.** Setting \( k = 2n + 2, l = 1 \) in (2.8), and note (1.19), (1.20), and (1.8), we immediately obtain (1.29).

Setting \( k = 2n + 3, l = 2 \) in (2.8), and note (1.22), (1.20), and (1.8), we immediately obtain (1.30). This completes the proof of Theorem 1.8.

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**References**


