Research Article

Solution and Stability of a Mixed Type Cubic and Quartic Functional Equation in Quasi-Banach Spaces

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We obtain the general solution and the generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation

\[ f(x+2y)+f(x-2y) = 4(f(x+y)+f(x-y)) - 24f(y) - 6f(x) + 3f(2y) \]

in quasi-Banach spaces.

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1. Introduction

We recall some basic facts concerning quasiBanach space. A quasinorm is a real-valued function on X satisfying the following.

(1) \(\|x\| \geq 0 \) for all \(x \in X\) and \(\|x\| = 0 \) if and only if \(x = 0\).

(2) \(\|\lambda \cdot x\| = |\lambda| \cdot \|x\|\) for all \(\lambda \in \mathbb{R}\) and all \(x \in X\).

(3) There is a constant \(K \geq 1\) such that \(\|x + y\| \leq K(\|x\| + \|y\|)\) for all \(x, y \in X\).

The pair \((X, \| \cdot \|)\) is called a quasinormed space if \(\| \cdot \|\) is a quasinorm on \(X\). A quasiBanach space is a complete quasinormed space. A quasinorm \(\| \cdot \|\) is called a \(p\)-norm \((0 < p \leq 1)\) if

\[\|x + y\|^p \leq \|x\|^p + \|y\|^p\]  \hspace{1cm} (1.1)
for all \( x, y \in X \). In this case, a quasiBanach space is called a \( p \)-Banach space. Given a \( p \)-norm, the formula \( d(x, y) := \|x - y\|_p \) gives us a translation invariant metric on \( X \). By the Aoki-Rolowicz theorem [1] (see also [2]), each quasinorm is equivalent to some \( p \)-norm. Since it is much easier to work with \( p \)-norms, henceforth we restrict our attention mainly to \( p \)-norms.

The stability problem of functional equations originated from a question of Ulam [3] in 1940, concerning the stability of group homomorphisms. Let \( (G_1, \cdot) \) be a group and let \( (G_2, \ast) \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \), such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(x \ast y), h(x) \ast h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [4] gave the first affirmative answer to the question of Ulam for Banach spaces. Let \( f : E \to E' \) be a mapping between Banach spaces such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]  

for all \( x, y \in E \), and for some \( \delta > 0 \). Then there exists a unique additive mapping \( T : E \to E' \) such that

\[
\|f(x) - T(x)\| \leq \delta
\]

for all \( x \in E \). Moreover if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in E \), then \( T \) is linear. Rassias [5] succeeded in extending the result of Hyers’ Theorem by weakening the condition for the Cauchy difference controlled by \( (\|x\|^p + \|y\|^p) \), \( p \in [0, 1) \) to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Rassias [6], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 7–18]).

The following cubic functional equation, which is the oldest cubic functional equation, was introduced by the third author of this paper, Rassias [6] (in 2001):

\[
f(x + 2y) + 3f(x) = 3f(x + y) + f(x - y) + 6f(y).
\]

Jun and Kim [19] introduced the following cubic functional equation:

\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),
\]

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function \( f(x) = x^3 \) satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function \( f \) between real
vector spaces $X$ and $Y$ is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and $C$ is symmetric for each fixed one variable and is additive for fixed two variables (see also [20]).

The quartic functional equation (1.6) was introduced by Rassias [21] (in 2000) and then (in 2005) was employed by Park and Bae [22] and others, such that:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x).$$ \hspace{1cm} (1.6)

In fact they proved that a function $f$ between real vector spaces $X$ and $Y$ is a solution of (1.6) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all $x$ (see also [21-29]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. In this paper we deal with the following functional equation:

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) - 24f(y) - 6f(x) + 3f(2y)$$ \hspace{1cm} (1.7)

in quasiBanach spaces. It is easy to see that the function $f(x) = ax^3 + bx^4$ is a solution of the functional equation (1.7). In the present paper we investigate the general solution of functional equation (1.7) when $f$ is a mapping between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7) whenever $f$ is a mapping between two quasiBanach spaces. We only mention here the papers [30, 31] concerning the stability of the mixed type functional equations.

### 2. General Solution

Throughout this section, $X$ and $Y$ will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we shall need the following two lemmas.

**Lemma 2.1.** If an even function $f : X \rightarrow Y$ satisfies (1.7), then $f$ is quartic.

**Proof.** Putting $x = y = 0$ in (1.7), we get $f(0) = 0$. Setting $x = 0$ in (1.7), by evenness of $f$ we obtain

$$f(2y) = 16f(y)$$ \hspace{1cm} (2.1)

for all $y \in X$. Hence (1.7) can be written as

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) + 24f(y) - 6f(x).$$ \hspace{1cm} (2.2)

This means that $f$ is quartic function, which completes the proof of the lemma. \square
Lemma 2.2. If an odd function \( f : X \to Y \) satisfies (1.7), then \( f \) is a cubic function.

Proof. Setting \( x = y = 0 \) in (1.7) gives \( f(0) = 0 \). Putting \( x = 0 \) in (1.7), then by oddness of \( f \), we have

\[
    f(2y) = 8f(y). \tag{2.3}
\]

Hence (1.7) can be written as

\[
    f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \tag{2.4}
\]

Replacing \( x \) by \( x + y \) in (2.4), we obtain

\[
    f(x + 3y) + f(x - y) = 4f(x + 2y) - 6f(x + y) + 4f(x). \tag{2.5}
\]

Substituting \(-y\) for \( y \) in (2.5) gives

\[
    f(x - 3y) + f(x + y) = 4f(x - 2y) - 6f(x - y) + 4f(x). \tag{2.6}
\]

If we subtract (2.5) from (2.6), we obtain

\[
    f(x + 3y) - f(x - 3y) = 4f(x + 2y) - 4f(x - 2y) - 6f(x + y) + 5f(x - y). \tag{2.7}
\]

Let us interchange \( x \) and \( y \) in (2.7). Then we see that

\[
    f(3x + y) + f(3x - y) = 4f(2x + y) + 4f(2x - y) - 5f(x + y) - 5f(x - y). \tag{2.8}
\]

With the substitution \( y := x + y \) in (2.4), we have

\[
    f(3x + 2y) - f(x + 2y) = 4f(2x + y) - 4f(y) - 6f(x). \tag{2.9}
\]

From the substitution \( y := -y \) in (2.9) it follows that

\[
    f(3x - 2y) - f(x - 2y) = 4f(2x - y) + 4f(y) - 6f(x). \tag{2.10}
\]

If we add (2.9) to (2.10), we have

\[
    f(3x + 2y) + f(3x - 2y) = 4f(2x + y) + 4f(2x - y) + f(x + 2y) + f(x - 2y) - 12f(x). \tag{2.11}
\]

Replacing \( x \) by \( 2x \) in (2.7) and using (2.3), we obtain

\[
    f(2x + 3y) - f(2x - 3y) = 32f(x + y) - 32f(x - y) - 5f(2x + y) + 5f(2x - y). \tag{2.12}
\]
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Interchanging $x$ with $y$ in (2.12) gives the equation

$$f(3x + 2y) + f(3x - 2y) = 32f(x + y) + 32f(x - y) - 5f(x + 2y) - 5f(x - 2y).$$

(2.13)

If we compare (2.11) and (2.13) and employ (2.4), we conclude that

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

(2.14)

This means that $f$ is cubic function. This completes the proof of Lemma.

\[ \square \]

**Theorem 2.3.** A function $f : X \rightarrow Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$, and that $C$ is symmetric for each fixed one variable and is additive for fixed two variables.

*Proof.* Let $f$ satisfy (1.7). We decompose $f$ into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x))$$

(2.15)

for all $x \in X$. By (1.7), we have

$$f_e(x + 2y) + f_e(x - 2y) = \frac{1}{2}[f(x + 2y) + f(-x - 2y) + f(x - 2y) + f(-x + 2y)]$$

(2.16)

$$= 4f_e(x + y) + f_e(x - y) - 24f_e(y) - 6f_e(x) + 3f_e(2y)$$

for all $x, y \in X$. This means that $f_e$ satisfies in (1.7). Similarly we can show that $f_o$ satisfies (1.7). By Lemmas 2.1 and 2.2, $f_e$ and $f_o$ are quartic and cubic, respectively. Thus there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f_e(x) = Q(x, x, x, x)$ and that $f_o(x) = C(x, x, x)$ for all $x \in X$, and $C$ is symmetric for each fixed one variable and is additive for fixed two variables. Thus $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$. The proof of the converse is trivial.

\[ \square \]

### 3. Stability

Throughout this section, $X$ and $Y$ will be a uniquely two-divisible abelian group and a quasiBanach spaces respectively, and $p$ will be a fixed real number in $[0, 1]$. We need the following lemma in the main theorems. Now before taking up the main subject, given $f : X \rightarrow Y$, we define the difference operator $D_f : X \times X \rightarrow Y$ by

$$D_f(x, y) = f(x + 2y) + f(x - 2y) - 4[f(x + y) + f(x - y)] - 3f(2y) + 24f(y) + 6f(x)$$

(3.1)
for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x, y)\| \leq \phi(x, y)$$

(3.2)

for an upper bound $\phi : X \times X \to [0, \infty)$.

**Lemma 3.1.** Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Then

$$\left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p.$$  

(3.3)

**Theorem 3.2.** Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \to \mathbb{R}^+$ be a function such that

$$\lim_{n \to \infty} 16 \ln \frac{x}{2^n} \varphi\left( \frac{y}{2^n} \right) = 0$$

(3.4)

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 16^{l i p} \varphi^p \left( 0, \frac{y}{2^n} \right) < \infty$$

(3.5)

for all $y \in X$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\| \leq \varphi(x, y),$$

(3.6)

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n \to \infty} 16 \ln f \left( \frac{x}{2^n} \right)$$

(3.7)

exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\| \leq K \frac{16}{16} \left[ \tilde{\varphi}_e(x) \right]^{1/p},$$

(3.8)

where

$$\tilde{\varphi}_e(x) := \sum_{i=|l|+1/2}^{\infty} 16^{l i p} \varphi^p \left( 0, \frac{x}{2^n} \right)$$

(3.9)

for all $x \in X$. 
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Proof. Let $l = 1$. By putting $x = 0$ in (3.6), we get

$$\|f(2y) - 16f(y)\|_Y \leq \varphi(0, y)$$

(3.10)

for all $y \in X$. Replacing $y$ by $x$ in (3.10) yields

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(0, x)$$

(3.11)

for all $x \in X$. Let $\varphi(x) = \varphi(0, x)$ for all $x \in X$, then by (3.11), we get

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(x)$$

(3.12)

for all $x \in X$. Interchanging $x$ with $x/2^{n+1}$ in (3.12), and multiplying by $16^n$ it follows that

$$\left\|16^n f \left( \frac{x}{2^{n+1}} \right) - 16^n f \left( \frac{x}{2^n} \right) \right\|_Y \leq K16^n \varphi \left( \frac{x}{2^{n+1}} \right)$$

(3.13)

for all $x \in X$ and all nonnegative integers $n$. Since $Y$ is $p$-Banach space, then by (3.13) we have

$$\left\|16^n f \left( \frac{x}{2^{n+1}} \right) - 16^n f \left( \frac{x}{2^n} \right) \right\|_Y \leq \sum_{i=m}^{n} \left\|16^{i+1} f \left( \frac{x}{2^{i+1}} \right) - 16^i f \left( \frac{x}{2^i} \right) \right\|_Y$$

(3.14)

$$\leq Kp \sum_{i=m}^{n} 16^i p \varphi \left( \frac{x}{2^{i+1}} \right)$$

for all nonnegative integers $n$ and $m$ with $n \geq m$ and all $x \in X$. Since $\varphi^p(x) = \varphi^p(0, x)$ for all $x \in X$. Therefore by (3.5) we have

$$\sum_{i=1}^{\infty} 16^i p \varphi \left( \frac{x}{2^i} \right) < \infty$$

(3.15)

for all $x \in X$. Therefore we conclude from (3.14) and (3.15) that the sequence $\{16^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, it follows that the sequence $\{16^n f(x/2^n)\}$ converges for all $x \in X$. We define the mapping $Q : X \rightarrow Y$ by (3.7) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.14), we get

$$\|f(x) - Q(x)\|_Y \leq Kp \sum_{i=0}^{\infty} 16^i p \varphi \left( \frac{x}{2^{i+1}} \right) = \frac{Kp}{16^0} \sum_{i=1}^{\infty} 16^i p \varphi \left( \frac{x}{2^i} \right)$$

(3.16)

for all $x \in X$. Therefore (3.8) follows from (3.9) and (3.16). Now we show that $Q$ is quartic. It follows from (3.4), (3.6) and (3.7)

$$\|D_Q(x, y)\|_Y = \lim_{n \rightarrow \infty} 16^n \|D_f \left( \frac{x}{2^n}, \frac{y}{2^n} \right)\|_Y \leq \lim_{n \rightarrow \infty} 16^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0$$

(3.17)
for all \( x, y \in X \). Therefore the mapping \( Q : X \to Y \) satisfies (1.7). Since \( Q(0) = 0 \), then by Lemma 2.1 we get that the mapping \( Q : X \to Y \) is quartic. To prove the uniqueness of \( Q \), let \( T : X \to Y \) be another quartic mapping satisfies (3.8). Since

\[
\lim_{n \to \infty} 16^{np} \sum_{i=1}^{\infty} 16^{np} \left( \frac{x}{2^{1+n}} \right)^2 \left( \frac{y}{2^{1+n}} \right)^2 = \lim_{n \to \infty} 16^{np} \sum_{i=n+1}^{\infty} 16^{np} \left( \frac{x}{2^{1+n}} \right)^2 \left( \frac{y}{2^{1+n}} \right)^2 = 0 \tag{3.18}
\]

for all \( y \in X \) and all \( x \in \{0\} \), then

\[
\lim_{n \to \infty} 16^{np} q_e \left( \frac{x}{2^n} \right) = 0 \tag{3.19}
\]

for all \( x \in X \). It follows from (3.8), (3.19)

\[
\|Q(x) - T(x)\|_Y^p = \lim_{n \to \infty} 16^{np} \left\| f \left( \frac{x}{2^n} \right) - T \left( \frac{x}{2^n} \right) \right\|_Y^p \leq \frac{Kp}{16p} \lim_{n \to \infty} 16^{np} \left( \frac{x}{2^n} \right) = 0 \tag{3.20}
\]

for all \( x \in X \). Hence \( Q = T \). For \( l = -1 \), we obtain

\[
\left\| \frac{f(2^n x)}{16^n} - f(x) \right\|_Y^p \leq \frac{Kp}{16^n} \sum_{i=0}^{\infty} \left( \frac{0.2^i}{16^n} \right), \tag{3.21}
\]

from which one can prove the result by a similar technique.

\[ \square \]

Corollary 3.3. Let \( \theta, r, s, u, v \) be nonnegative real numbers such that \( s \neq 4 \neq u + v \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|Df(x, y)\|_Y \leq \theta(\|x\|_X^r \|y\|_X^s + \|x\|_X^r + \|y\|_X^s) \tag{3.22}
\]

for all \( x, y \in X \). Then there exists a unique quartic function \( Q : X \to Y \) satisfying

\[
\|f(x) - Q(x)\|_Y \leq K \theta \left\{ \frac{1}{\|16^n - 2^n\|} \right\}^{1/p} \|x\|_X^r \tag{3.23}
\]

for all \( x \in X \).

Proof. It follows from Theorem 3.2 that \( \varphi(x, y) := \theta(\|x\|_X^r \|y\|_X^s + \|x\|_X + \|y\|_X) \) for all \( x, y \in X \). \[ \square \]

Theorem 3.4. Let \( l \in \{1, -1\} \) be fixed and let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} 8^{ln} \varphi \left( \frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) = 0 \tag{3.24}
\]
for all \( x, y \in X \) and
\[
\sum_{i=1}^{\infty} 8^{i^p} \phi^p \left( 0, \frac{y}{2^i} \right) < \infty
\] (3.25)
for all \( y \in X \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality
\[
\| D_f(x, y) \|_Y \leq \varphi(x, y),
\] (3.26)
for all \( x, y \in X \). Then the limit
\[
C(x) := \lim_{n \to \infty} 8^{ln} f \left( \frac{x}{2^m} \right)
\] (3.27)
exists for all \( x \in X \) and \( C : X \to Y \) is a unique cubic function satisfying
\[
\| f(x) - C(x) \|_Y \leq \frac{K}{24} \left[ \tilde{\phi}_o(x) \right]^{1/p}
\] (3.28)
for all \( x \in X \), where
\[
\tilde{\phi}_o(x) := \sum_{i=\lfloor l/2 \rfloor + 1}^{\infty} 8^{i^p} \phi^p \left( 0, \frac{x}{2^i} \right).
\] (3.29)

Proof. Let \( l = 1 \). Setting \( x = 0 \) in (3.26), we get
\[
\| 3f(2y) - 24f(y) \|_Y \leq \varphi(0, y)
\] (3.30)
for all \( y \in X \). If we replace \( y \) in (3.30) by \( x \) and divide both sides of (3.30) by 3, we get
\[
\| f(2x) - 8f(x) \|_Y \leq \frac{1}{3} \varphi(0, x)
\] (3.31)
for all \( x \in X \). Let \( \tilde{\phi}(x) = (1/3)\varphi(0, x) \) for all \( x \in X \), then by (3.31), we get
\[
\| f(2x) - 8f(x) \|_Y \leq \tilde{\phi}(x)
\] (3.32)
for all \( x \in X \). Multiply (3.32) by \( 8^n \) and replace \( x \) by \( x/2^{n+1} \), we obtain that
\[
\| 8^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 8^n f \left( \frac{x}{2^n} \right) \|_Y \leq K 8^n \phi \left( \frac{x}{2^{n+1}} \right)
\] (3.33)
for all \( x \in X \) and all nonnegative integers \( n \). Since \( Y \) is a \( p \)-Banach space, (3.33) follows that

\[
\left\| 8^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 8^mf\left(\frac{x}{2^m}\right) \right\|_Y^p \leq \sum_{i=m}^{n} \left\| 8^{i+1}f\left(\frac{x}{2^{i+1}}\right) - 8^i f\left(\frac{x}{2^i}\right) \right\|_Y^p
\]

\[
\leq K^p \sum_{i=m}^{n} 8^i \phi^p\left(\frac{x}{2^{i+1}}\right)
\]

(3.34)

for all nonnegative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in X \). Since \( \phi^p(x) = (1/3^p)\phi^p(0, x) \) for all \( x \in X \). Therefore it follows from (3.25) that

\[
\sum_{i=1}^{\infty} 8^i \phi^p\left(\frac{x}{2^{i+1}}\right) < \infty
\]

(3.35)

for all \( x \in X \), therefore we conclude from (3.34) and (3.35) that the sequence \( \{8^n f(x/2^n)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{8^n f(x/2^n)\} \) converges for all \( x \in X \). So one can define the mapping \( C : X \to Y \) by (3.27) for all \( x \in X \). Letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.34), we get

\[
\| f(x) - C(x) \|_Y^p \leq K^p \sum_{i=0}^{\infty} 8^i \phi^p\left(\frac{x}{2^{i+1}}\right) = K^p \frac{8^p}{\phi^p} \sum_{i=1}^{\infty} 8^i \phi^p\left(\frac{x}{2^{i}}\right)
\]

(3.36)

for all \( x \in X \). Therefore (3.28) follows from (3.29) and (3.36). Now we show that \( C \) is cubic. It follows from (3.24), (3.26) and (3.27)

\[
\| D_C(x, y) \|_Y = \lim_{n \to \infty} \| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \|_Y \leq \lim_{n \to \infty} 8^n \phi^p\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\]

(3.37)

for all \( x, y \in X \). Therefore the mapping \( C : X \to Y \) satisfies (1.7). Since \( f \) is an odd function, then (3.27) implies that the mapping odd. Therefore by Lemma 2.2 we get that the mapping \( C : X \to Y \) is cubic. The rest of proof is similar to the proof of Theorem 3.2.

\[ \square \]

**Corollary 3.5.** Let \( \theta \) be a nonnegative real number and \( r, s \) be real numbers such that \( s \neq 3 \neq u + v \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality

\[
\| D_f(x, y) \|_Y \leq \theta(|x|^u + |y|^v + \| x \|^u + \| y \|^v)
\]

(3.38)

for all \( x, y \in X \). Then there exists a unique cubic function \( C : X \to Y \) satisfying

\[
\| f(x) - C(x) \|_Y \leq \frac{K\theta}{3} \left\{ \frac{1}{|8^p - 2^p|} \right\}^{1/p} \| x \|^u
\]

(3.39)

for all \( x \in X \).

**Proof.** It follows from (3.38) and Theorem 3.4 that \( \varphi(x, y) := \theta(|x|^u + |y|^v + \| x \|^u + \| y \|^v) \) for all \( x, y \in X \). \[ \square \]
Theorem 3.6. Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function which satisfies

\[
\lim_{n \to \infty} \left\{ \left( \frac{|l|+l}{2} \right) 16^{\ln n} \varphi \left( \frac{x}{2^{l n}}, \frac{y}{2^{l n}} \right) + \left( \frac{|l|-l}{2} \right) 8^{\ln n} \varphi \left( \frac{x}{2^{l n}}, \frac{y}{2^{l n}} \right) \right\} = 0 \tag{3.40}
\]

for all $x, y \in X$ and

\[
\sum_{i=|l+1|/2}^{\infty} \left\{ \left( \frac{|l|+l}{2} \right) 16^{\ln i} \varphi^p \left( 0, \frac{y}{2^{l n}} \right) + \left( \frac{|l|-l}{2} \right) 8^{\ln i} \varphi^p \left( 0, \frac{y}{2^{l n}} \right) \right\} < \infty \tag{3.41}
\]

for all $y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

\[
\|D_f(x, y)\|_Y \leq \varphi(x, y) \tag{3.42}
\]

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying (1.7) and

\[
\|f(x) - Q(x) - C(x)\|_Y \leq \frac{K^3}{32} \left[ \bar{\varphi}_e(x) + \bar{\varphi}_e(-x) \right]^{1/p} + \frac{K^3}{48} \left[ \bar{\varphi}_o(x) + \bar{\varphi}_o(-x) \right]^{1/p} \tag{3.43}
\]

for all $x \in X$, where $\bar{\varphi}_e(x)$ and $\bar{\varphi}_o(x)$ that have been defined in (3.9) and (3.29), respectively.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and $\|D_{f_e}(x, y)\| \leq (K/2)[\varphi(x, y) + \varphi(-x, -y)]$ for all $x, y \in X$. Let

\[
\Phi(x, y) = \frac{K}{2} \left[ \varphi(x, y) + \varphi(-x, -y) \right] \tag{3.44}
\]

for all $x, y \in X$. So

\[
\lim_{n \to \infty} 16^{\ln n} \Phi \left( \frac{x}{2^{l n}}, \frac{y}{2^{l n}} \right) = 0 \tag{3.45}
\]

for all $x, y \in X$. Since

\[
\Phi^p(x, y) \leq \frac{K^p}{2^p} \left[ \varphi^p(x, y) + \varphi^p(-x, -y) \right] \tag{3.46}
\]

for all $x, y \in X$, then

\[
\sum_{i=1}^{\infty} 16^{\ln i} \Phi^p \left( \frac{x}{2^{l n}}, \frac{y}{2^{l n}} \right) < \infty \tag{3.47}
\]
for all $y \in X$ and all $x \in [0]$. Hence, in view of Theorem 3.2, there exists a unique quartic function $Q : X \to Y$ satisfying

$$
\|f_e(x) - Q(x)\|_Y \leq \frac{K}{16} \left[ \tilde{\Psi}_e(x) \right]^{1/p} \tag{3.48}
$$

for all $x \in X$, where

$$
\tilde{\Psi}_e(x) := \sum_{i=1}^{\infty} 16^{ip} \Phi^p \left( 0, \frac{x}{2^i} \right). \tag{3.49}
$$

We have

$$
\tilde{\Psi}_e(x) \leq \frac{K^p}{2^p} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] \tag{3.50}
$$

for all $x \in X$. Therefore it follows from (3.48) that,

$$
\|f_e(x) - Q(x)\|_Y \leq \frac{K^2}{32} \left[ \tilde{\psi}_e(x) + \tilde{\psi}_e(-x) \right]^{1/p} \tag{3.51}
$$

for all $x \in X$. Let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0, f_o(-x) = -f_o(x)$, and $\|Df_o(x,y)\| \leq \Phi(x,y)$ for all $x, y \in X$. From Theorem 3.4, it follows that there exists a unique cubic function $C : X \to Y$ satisfying

$$
\|f_o(x) - C(x)\|_Y \leq \frac{K}{24} \left[ \tilde{\Phi}_o(x) \right]^{1/p} \tag{3.52}
$$

for all $x \in X$, where

$$
\tilde{\Phi}_o(x) := \sum_{i=1}^{\infty} 8^{ip} \Phi^p \left( 0, \frac{x}{2^i} \right). \tag{3.53}
$$

Since

$$
\tilde{\Phi}_o(x) \leq \frac{K^p}{2^p} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)] \tag{3.54}
$$

for all $x \in X$, it follows from (3.52) that,

$$
\|f_o(x) - C(x)\|_Y \leq \frac{K^2}{48} \left[ \tilde{\phi}_o(x) + \tilde{\phi}_o(-x) \right]^{1/p} \tag{3.55}
$$

for all $x \in X$. Hence (3.43) follows from (3.51) and (3.55). \qed
Corollary 3.7. Let \( \theta, r, s \) be nonnegative real numbers such that \( u + v, s \in (4, \infty) \cup (-\infty, 3) \). Suppose that a function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|D_f(x, y)\|_Y \leq \theta(\|x\|_X^p \|y\|_Y^p + \|x\|_X^p + \|y\|_Y^p) \tag{3.56}
\]

for all \( x, y \in X \). Then there exists a unique quartic function \( Q : X \to Y \) and a unique cubic function \( C : X \to Y \) satisfying (1.7) and

\[
\|f(x) - Q(x) - C(x)\|_Y \leq K^{3}\theta\left( \frac{1}{16^p - 2^p} \right)^{1/p} \left( \frac{1}{18^p - 2^p} \right)^{1/p} \|x\|_X^p \tag{3.57}
\]

for all \( x \in X \).

Proof. It follows from Theorem 3.6 that

\[
\varphi(x, y) := \theta(\|x\|_X^p \|y\|_Y^p + \|x\|_X^p + \|y\|_Y^p) \tag{3.58}
\]

for all \( x, y \in X \). \( \square \)

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**References**


