Research Article

On the Symmetries of the $q$-Bernoulli Polynomials

Taekyun Kim

Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, South Korea

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

Received 25 June 2008; Accepted 29 August 2008

Kupershmidt and Tuenter have introduced reflection symmetries for the $q$-Bernoulli numbers and the Bernoulli polynomials in (2005), (2001), respectively. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuenter derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the $q$-Bernoulli numbers and polynomials, which are different from Kupershmidt’s and Tuenter’s results. By using our symmetries for the $q$-Bernoulli polynomials, we can obtain some interesting relationships between $q$-Bernoulli numbers and polynomials.

Copyright © 2008 Taekyun Kim. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let $p$ be a fixed prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integer, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p < p^{-1}$. Let $q$ be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$. We say that $f$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we denote this property by $f \in \text{UD}(\mathbb{Z}_p)$ if the difference quotients,

$$F_f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{Z}_p \quad \text{by} \quad F_f(x,y) = \frac{f(x) - f(y)}{x - y},$$

have a limit $l = f'(a)$ as $(x,y) \rightarrow (a,a)$. The $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)$$

Abstract and Applied Analysis

From this integral, we derive several further interesting properties of symmetry for the $q$-Bernoulli numbers and polynomials in this paper. Kupershmidt [14] and Tuenter [20] have introduced reflection symmetries for the $q$-Bernoulli numbers and the Bernoulli polynomials. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuenter derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the $q$-Bernoulli numbers and polynomials, which are different from Kupershmidt’s and Tuenter’s results.

By using our symmetries for the $q$-Bernoulli polynomials, we can obtain some interesting relationships between $q$-Bernoulli numbers and polynomials.

2. On the symmetries of the $q$-Bernoulli polynomials

For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$ I(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \quad (2.1) $$

Let $f_1(x)$ be a translation with $f_1(x) = f(x + 1).$ Then, we have

$$ I(f_1) = I(f) + f'(0). \quad (2.2) $$

From (2.2), we can also derive

$$ I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \quad f'(i) = \frac{df(i)}{dx}. \quad (2.3) $$

Let $f(x) = q^x e^{tx}$, then we have

$$ \int_{\mathbb{Z}_p} q^x e^{tx} dx = \frac{t + \log q}{q^e - 1}. \quad (2.4) $$

It is known that the $q$-Bernoulli polynomials are defined as

$$ \frac{t + \log q}{q^e - 1} e^{tx} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \quad (2.5) $$

[17, 19]. Now we define an integral representation for the $q$-extension of Bernoulli numbers as follows:

$$ \int_{\mathbb{Z}_p} q^x e^{tx} dx = \frac{\log q + t}{q^e - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \quad (2.6) $$
From (2.3), (2.4), and (2.6), we can derive

$$\int_{z_p} q^n(x + y)^n dy = B_{n,q}(x), \quad \int_{z_p} q^n x^n dx = B_{n,q}. \quad \text{(2.7)}$$

By (2.3), we easily see that

$$\frac{1}{\log q + t} \left( \int_{z_p} q^{n+x} e^{(n+x)t} dx - \int_{z_p} q^x e^{xt} dx \right) = \frac{q^n e^{nt} - 1}{t + \log q} \int_{z_p} q^x e^{xt} dx = \frac{q^n e^{nt} - 1}{q e^t - 1} = \sum_{k=0}^{n-1} q^k e^t = \sum_{k=0}^{n-1} \left( \sum_{i=0}^{n-1} i^k q^i \right) \frac{t^k}{k!}. \quad \text{(2.8)}$$

In (2.2), it is not difficult to show that

$$\frac{1}{\log q + t} \left( \int_{z_p} q^{n+x} e^{(n+x)t} dx - \int_{z_p} q^x e^{xt} dx \right) = \frac{n! \int_{z_p} e^{xt} q^n dx}{\int_{z_p} e^{nt} q^n dx}. \quad \text{(2.9)}$$

For each integer $k \geq 0$, let

$$S_{k,q}(n) = 0^k + 1^k q + 2^k q^2 + \ldots + q^n n^k. \quad \text{(2.10)}$$

From (2.8) and (2.9), we derive

$$\frac{1}{\log q + t} \left( \int_{z_p} q^{n+x} e^{(n+x)t} dx - \int_{z_p} q^x e^{xt} dx \right) = \frac{n! \int_{z_p} e^{xt} q^n dx}{\int_{z_p} e^{nt} q^n dx} = \sum_{k=0}^{\infty} S_{k,q}(n-1) \frac{t^k}{k!}. \quad \text{(2.11)}$$

From (2.11), we note that

$$B_{k,q}(n) - B_{k,q} = k S_{k-1,q}(n-1) + \log q S_{k,q}(n-1), \quad \text{where } k, n \in \mathbb{N}. \quad \text{(2.12)}$$

Let $w_1, w_2 \in \mathbb{N}$, then we have

$$\frac{\int_{z_p} e^{(w_1 x_1 + w_2 x_2)t} q^{w_1 x_1 + w_2 x_2} dx_1 dx_2}{\int_{z_p} e^{w_1 x_2} q^{w_2 x_2} dx} = (t + \log q) \frac{q^{w_1 w_2} e^{w_1 w_2 t} - 1}{(q^{w_1 e^{w_1 t}} - 1)(q^{w_2 e^{w_2 t}} - 1)}. \quad \text{(2.13)}$$

By (2.11), we see that

$$\frac{w_1 \int_{z_p} e^{x_1} q^n dx}{\int_{z_p} e^{x_1} q^{w_1 x_1} dx} = \sum_{l=0}^{\infty} \left( \sum_{k=0}^{w_1-1} k^l q^k \right) \frac{t^l}{l!} = \sum_{l=0}^{\infty} S_{l,q}(w_1 - 1) \frac{t^l}{l!}. \quad \text{(2.14)}$$
Let
\[
T(w_1, w_2; x, t) = \frac{\int_{Z_p} q^{w_1 x_1 + w_2 x_2} e^{(w_1 x_1 + w_2 x_2) t} \, dx_1 \, dx_2}{\int_{Z_p} e^{w_1 x_3} q^{w_1 x_3} \, dx_3},
\]  
then we have
\[
T(w_1, w_2; x, t) = \frac{(t + \log q) e^{w_1 x_1} q^{w_1 x_1} (q^{w_1 x_1} - 1)}{(q^{w_1 x_1} - 1)(q^{w_2 x_2} - 1)}.
\]  
(2.15)

From (2.15) we derive
\[
T(w_1, w_2; x, t) = \left( \frac{1}{w_1} \int_{Z_p} e^{w_1 x_1 + w_2 x_2} q^{w_1 x_1} \, dx_1 \right) \left( \frac{w_1 \int_{Z_p} e^{w_1 x_1} q^{w_1 x_1} \, dx_1}{\int_{Z_p} e^{w_1 x_3} q^{w_1 x_3} \, dx_3} \right).
\]  
(2.16)

By (2.5), (2.14), and (2.17), we see that
\[
T(w_1, w_2; x, t) = \frac{1}{w_1} \left( \sum_{i=0}^{\infty} B_{i, q^{p^2}} (w_2 x) \frac{w_1^i t^i}{i!} \right) \left( \sum_{l=0}^{\infty} S_{l, q^{p^2}} (w_1 - 1) \frac{w_2^l t^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left( \binom{n}{i} B_{i, q^{p^2}} (w_2 x) S_{n-i, q^{p^2}} (w_1 - 1) w_1^{i-1} w_2^{n-i} \right) \frac{t^n}{n!}.
\]  
(2.18)

By the symmetry of p-adic invariant integral on \(Z_p\), we also see that
\[
T(w_1, w_2; x, t) = \frac{1}{w_2} \left( \sum_{i=0}^{\infty} B_{i, q^{p^2}} (w_1 x) \frac{w_2^i t^i}{i!} \right) \left( \sum_{l=0}^{\infty} S_{l, q^{p^2}} (w_2 - 1) \frac{w_1^l t^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left( \binom{n}{i} B_{i, q^{p^2}} (w_1 x) S_{n-i, q^{p^2}} (w_2 - 1) w_1^{i-1} w_2^{n-i} \right) \frac{t^n}{n!}.
\]  
(2.19)

By comparing the coefficients \(t^n/n!\) on the both sides of (2.18) and (2.19), we obtain the following theorem.

**Theorem 2.1.** For all \(w_1, w_2 (\in \mathbb{N})\), we have
\[
\sum_{i=0}^{n} \binom{n}{i} B_{i, q^{p^2}} (w_2) S_{n-i, q^{p^2}} (w_1 - 1) w_1^{i-1} w_2^{n-i} = \sum_{i=0}^{n} \binom{n}{i} B_{i, q^{p^2}} (w_1) S_{n-i, q^{p^2}} (w_2 - 1) w_2^{i-1} w_1^{n-i},
\]  
(2.20)

where \(\binom{n}{i}\) is the binomial coefficient.
If we take \( w_2 = 1 \) in Theorem 2.1, then we have

\[
B_{n,q}(w_1x) = \sum_{i=0}^{n} \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}.
\]  

(2.21)

Therefore, we obtain the following corollary.

**Corollary 2.2.** For \( n \geq 0 \), we have

\[
B_{n,q}(w_1x) = \sum_{i=0}^{n} \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}.
\]  

(2.22)

By (2.17), (2.18), and (2.19), we also see that

\[
T(w_1, w_2; x, t) = \left( \frac{e^{w_1 w_2 x t}}{w_1} \right) \left( \int_{Z_p} e^{w_1 x_1} q^{w_1 x_1} dx_1 \right) \left( \frac{w_1}{w_1} \int_{Z_p} e^{w_1 w_2 x} q^{w_1 w_2 x} dx \right)
\]

\[
= \left( \frac{e^{w_1 w_2 x t}}{w_1} \right) \left( \int_{Z_p} e^{w_1 x_1} q^{w_1 x_1} dx_1 \right) \left( \sum_{i=0}^{w_1-1} q^{w_1 i} e^{-w_1 i} \right)
\]

\[
= \frac{1}{w_1} \sum_{i=0}^{w_1-1} q^{w_1 i} \left( e^{(w_1 + w_2 x) w_2 x/w_1} \right) q^{x x_1} dx_1
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left( w_1 x + \frac{w_2}{w_1} i \right) q^{w_1 i} \right) t^n/n!.
\]

(2.23)

From the symmetry of \( T(w_1, w_2; x, t) \), we can also derive

\[
T(w_1, w_2; x, t) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left( w_1 x + \frac{w_1}{w_2} i \right) q^{w_1 i} \right) t^n/n!.
\]  

(2.24)

By comparing the coefficients \( t^n/n! \) on the both sides of (2.23) and (2.24), we obtain the following theorem.

**Theorem 2.3.** For \( n \in \mathbb{Z}_+ \), \( w_1, w_2 \in \mathbb{N} \), we have

\[
\sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left( w_1 x + \frac{w_2}{w_1} i \right) w_1^{n-1} q^{w_1 i} = \sum_{i=0}^{w_2-1} B_{n,q^{w_2}} \left( w_1 x + \frac{w_1}{w_2} i \right) w_2^{n-1} q^{w_1 i}.
\]  

(2.25)

**Remark 2.4.** Setting \( w_2 = 1 \) in Theorem 2.3, we get the multiplication theorem for the \( q \)-Bernoulli polynomials as follows:

\[
B_{n,q}(w_1 x) = w_1^{n-1} \sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left( x + \frac{i}{w_1} \right) q^i.
\]  

(2.26)
I cannot obtain the extended formulae of Theorems 2.1 and 2.3 related to the Carlitz’s $q$-Bernoulli numbers and polynomials. So, we suggest the following two questions.

**Question 1.** Find the extended formulae of Theorems 2.1 and 2.3, which are related to the Carlitz’s $q$-Bernoulli numbers and polynomials.

**Question 2.** Find the twisted formulae of Theorems 2.1 and 2.3, which are related to the twisted Carlitz’s $q$-Bernoulli polynomials.

**Remark 2.5.** In [12], $q$-Volkenborn integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \quad (2.27)$$

Thus, we note that Carlitz’s $q$-Bernoulli numbers can be written by

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]^n_q d\mu_q(x), \quad \text{Witt’s type formula.} \quad (2.28)$$

**Acknowledgments**

The author wishes to express his sincere gratitude to the referees for their valuable suggestions and comments. The present Research has been conducted by the research Grant of Kwangwoon University in 2008.

**References**


