Research Article

On the Stability of Quadratic Functional Equations

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Let $X, Y$ be vector spaces and $k$ a fixed positive integer. It is shown that a mapping $f(kx + y) + f(kx - y) = 2k^2f(x) + 2f(y)$ for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Furthermore, the Hyers-Ulam-Rassias stability of the above functional equation in Banach spaces is proven.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Aoki [3] for additive mapping and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [6], following the same approach as in [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6] as well as by Rassias and Šemrl [7] that one cannot prove a Th.M. Rassias’ type theorem when $p = 1$. J. M. Rassias [8], following the spirit of the innovative approach of Th. M. Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional
Proof. for all 

Throughout this paper, assume that \( k \) is a fixed positive integer.

In this paper, we solve the functional equation

\[
f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y)
\]

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in Banach spaces.

2. Hyers-Ulam-Rassias stability of the quadratic functional equation

Proposition 2.1. Let \( X \) and \( Y \) be vector spaces. A mapping \( f : X \to Y \) satisfies

\[
f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y)
\]

for all \( x, y \in X \) if and only if the mapping \( f : X \to Y \) satisfies

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

for all \( x, y \in X \).

Proof. Assume that \( f : X \to Y \) satisfies (2.1).

Letting \( x = y = 0 \) in (2.1), we get \( f(0) = 0 \).

Letting \( y = 0 \) in (2.1), we get \( f(kx) = k^2 f(x) \) for all \( x \in X \).

Letting \( x = 0 \) in (2.1), we get \( f(-y) = f(y) \) for all \( y \in X \).

It follows from (2.1) that

\[
f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y) = 2f(kx) + 2f(y)
\]

for all \( x, y \in X \). So the mapping \( f : X \to Y \) satisfies

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

for all \( x, y \in X \).

Assume that \( f : X \to Y \) satisfies \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) for all \( x, y \in X \).

We prove (2.1) for \( k = j \) by induction on \( j \).

For the case \( j = 1 \), (2.1) holds by the assumption.

For the case \( j = 2 \), since

\[
f(2x + y) + f(2x - y) = f(x + y + x) + f(x - y + x)
\]

\[
= 2f(x + y) + 2f(x) - f(y) + 2f(x - y) + 2f(x) - f(-y)
\]

\[
= 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y)
\]

\[
= 4f(x) + 4f(y) + 4f(x) - 2f(y)
\]

\[
= 8f(x) + 2f(y)
\]

for all \( x, y \in X \), then (2.1) holds.
Assume that (2.1) holds for \( j = n - 2 \) and \( j = n - 1 \) \( (2 < n \leq k) \). By the assumption,

\[
f(nx + y) + f(nx - y) = f((n - 1)x + y + x) + f((n - 1)x - y + x)
= 2f((n - 1)x + y) + 2f(x) - f((n - 2)x + y)
+ 2f((n - 1)x - y) + 2f(x) - f((n - 2)x - y)
= 4(n - 1)^2 f(x) + 4f(y) + 4f(x) - 2(n - 2)^2 f(x) - 2f(y)
= 2n^2 f(x) + 2f(y)
\]

for all \( x, y \in X \), (2.1) holds for \( j = n \). Hence the mapping \( f : X \rightarrow Y \) satisfies (2.1) for \( j = k \). \( \square \)

From now on, assume that \( X \) is a normed vector space with norm \( \| \cdot \| \) and that \( Y \) is a Banach space with norm \( \| \cdot \| \).

For a given mapping \( f : X \rightarrow Y \), we define

\[
Df(x, y) := f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y)
\]

for all \( x, y \in X \).

Now we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation \( Df(x, y) = 0 \).

**Theorem 2.2.** Let \( f : X \rightarrow Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \rightarrow [0, \infty) \) such that

\[
\varphi(x, y) := \sum_{j=0}^{\infty} \frac{1}{k^{2j}} \varphi(k^j x, k^j y) < \infty,
\]

\[
\|Df(x, y)\| \leq \varphi(x, y)
\]

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{2k^2} \varphi(x, 0)
\]

for all \( x \in X \).

**Proof.** Letting \( y = 0 \) in (2.9), we get

\[
\|2f(kx) - 2k^2 f(x)\| \leq \varphi(x, 0)
\]

for all \( x \in X \). So

\[
\left\| f(x) - \frac{1}{k^2} f(kx) \right\| \leq \frac{1}{2k^2} \varphi(x, 0)
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{k^{2j}} f(k^j x) - \frac{1}{k^{2m}} f(k^m x) \right\| \leq \sum_{j=1}^{m-1} \frac{1}{2k^{2j+2}} \varphi(k^j x, 0)
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.13) that the sequence \( \{(1/k^{2n})f(k^n x)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{(1/k^{2n})f(k^n x)\} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{1}{k^{2n}} f(k^n x)
\]

(2.14)

for all \( x \in X \).

By (2.8),

\[
\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{k^{2n}} \|Df(k^n x, k^n y)\| \leq \lim_{n \to \infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y) = 0
\]

(2.15)

for all \( x, y \in X \). So \( DQ(x, y) = 0 \). By Proposition 2.1, the mapping \( Q : X \to Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.13), we get (2.10).

Now, let \( T : X \to Y \) be another quadratic mapping satisfying (2.1) and (2.10). Then we have

\[
\|Q(x) - T(x)\| = \frac{1}{k^{2n}} \|Q(k^n x) - T(k^n x)\|
\]

\[
\leq \frac{1}{k^{2n}} \left( \|Q(k^n x) - f(k^n x)\| + \|T(k^n x) - f(k^n x)\| \right)
\]

\[
\leq \frac{1}{k^{2n+2}} \varphi(k^n x, 0),
\]

which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( Q(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( Q \). So there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (2.10).

\[\]

**Corollary 2.3.** Let \( p < 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping such that

\[
\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]

(2.17)

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{\theta}{8 - 2^{p+1}} \|x\|^p
\]

(2.18)

for all \( x \in X \).

**Proof.** The proof follows from Theorem 2.2 by taking

\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]

(2.19)

for all \( x, y \in A \). \( \square \)

**Theorem 2.4.** Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \to [0, \infty) \) satisfying (2.9) such that

\[
\varphi(x, y) := \sum_{j=0}^{\infty} k^{2j} \varphi\left(\frac{x}{k^j}, \frac{y}{k^j}\right) < \infty
\]

(2.20)

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{2\varphi}\left(\frac{x}{k}, 0\right)
\]

(2.21)

for all \( x \in X \).
Proof. It follows from (2.11) that
\[ \left\| f(x) - k^2 f \left( \frac{x}{k} \right) \right\| \leq \frac{1}{2} \varphi \left( \frac{x}{k}, 0 \right) \] (2.22)
for all \( x \in X \). Hence
\[ \left\| k^{2i} f \left( \frac{x}{k^i} \right) - k^{2m} f \left( \frac{x}{k^m} \right) \right\| \leq \sum_{j=i}^{m-1} \frac{k^{2j}}{2} \varphi \left( \frac{x}{k^{j+1}}, 0 \right) \] (2.23)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.23) that the sequence \( \{k^{2n} f(x/k^n)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{k^{2n} f(x/k^n)\} \) converges. So one can define the mapping \( Q : X \rightarrow Y \) by
\[ Q(x) := \lim_{n \to \infty} k^{2n} f \left( \frac{x}{k^n} \right) \] (2.24)
for all \( x \in X \).

By (2.20),
\[ \left\| DQ(x, y) \right\| = \lim_{n \to \infty} k^{2n} \left\| Df \left( \frac{x}{k^n}, \frac{y}{k^n} \right) \right\| \leq \lim_{n \to \infty} k^{2n} \varphi \left( \frac{x}{k^n}, \frac{y}{k^n} \right) = 0 \] (2.25)
for all \( x, y \in X \). So \( DQ(x, y) = 0 \). By Proposition 2.1, the mapping \( Q : X \rightarrow Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.23), we get (2.21).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Corollary 2.5. Let \( p > 2 \) and \( \theta \) be positive real numbers, and let \( f : X \rightarrow Y \) be a mapping satisfying (2.17). Then there exists a unique quadratic mapping \( Q : X \rightarrow Y \) such that
\[ \left\| f(x) - Q(x) \right\| \leq \frac{\theta}{2^{p+1} - 8} \|x\|^p \] (2.26)
for all \( x \in X \).

Proof. The proof follows from Theorem 2.4 by taking
\[ \varphi(x, y) := \theta (\|x\|^p + \|y\|^p) \] (2.27)
for all \( x, y \in A \). \( \square \)

From now on, assume that \( k = 2 \).

Theorem 2.6. Let \( f : X \rightarrow Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \rightarrow [0, \infty) \) satisfying (2.9) such that
\[ \tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(3^j x, 3^j y) < \infty \] (2.28)
for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \rightarrow Y \) such that
\[ \left\| f(x) - Q(x) \right\| \leq \frac{1}{9} \tilde{\varphi}(x, x) \] (2.29)
for all \( x \in X \).
Proof. Letting $y = x$ in (2.9), we get

$$\|f(3x) - 9f(x)\| \leq \varphi(x,x)$$  \hspace{1cm} (2.30)

for all $x \in X$. So

$$\|f(x) - \frac{1}{9}f(3x)\| \leq \frac{1}{9}\varphi(x,x)$$  \hspace{1cm} (2.31)

for all $x \in X$. Hence

$$\left\| \frac{1}{9}f(3^lx) - \frac{1}{9^m}f(3^mx) \right\| \leq \sum_{i=l}^{m-1} \frac{1}{9^{i+1}}\varphi(3^lx,3^lx)$$  \hspace{1cm} (2.32)

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.32) that the sequence \{$(1/9^n)f(3^n x)$\} is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence \{$(1/9^n)f(3^n x)$\} converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{9^n}f(3^n x)$$  \hspace{1cm} (2.33)

for all $x \in X$.

By (2.28),

$$\|DQ(x,y)\| = \lim_{n \to \infty} \frac{1}{9^n} \|Df(3^n x,3^n y)\| \leq \lim_{n \to \infty} \frac{1}{9^n} \varphi(3^n x,3^n y) = 0$$  \hspace{1cm} (2.34)

for all $x,y \in X$. So $DQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.32), we get (2.29).

The rest of the proof is similar to the proof of Theorem 2.2.

Corollary 2.7. Let $p < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping such that

$$\|Df(x,y)\| \leq \theta \cdot \|x\|^p \cdot \|y\|^p$$  \hspace{1cm} (2.35)

for all $x,y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{9 - \theta p} \|x\|^{2p}$$  \hspace{1cm} (2.36)

for all $x \in X$.

Proof. The proof follows from Theorem 2.6 by taking

$$\varphi(x,y) := \theta \cdot \|x\|^p \cdot \|y\|^p$$  \hspace{1cm} (2.37)

for all $x,y \in A$. \square
Theorem 2.8. Let \( f : X \rightarrow Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \rightarrow [0, \infty) \) satisfying (2.9) such that
\[
\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} g^j \varphi \left( \frac{x}{3^j}, \frac{y}{3^j} \right) < \infty
\]  
for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \rightarrow Y \) such that
\[
\| f(x) - Q(x) \| \leq \varphi \left( \frac{x}{3}, \frac{x}{3} \right)
\]  
for all \( x \in X \).

Proof. It follows from (2.30) that
\[
\left\| f(x) - 9f \left( \frac{x}{3} \right) \right\| \leq \varphi \left( \frac{x}{3}, \frac{x}{3} \right)
\]  
for all \( x \in X \). Hence
\[
\left\| g^l f \left( \frac{x}{3^l} \right) - 9^m f \left( \frac{x}{3^m} \right) \right\| \leq \sum_{j=l}^{m} g^j \varphi \left( \frac{x}{3^{j+1}}, \frac{x}{3^{j+1}} \right)
\]  
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.41) that the sequence \( \{9^n f(x/3^n)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{9^n f(x/3^n)\} \) converges. So one can define the mapping \( Q : X \rightarrow Y \) by
\[
Q(x) := \lim_{n \to \infty} 9^n f \left( \frac{x}{3^n} \right)
\]  
for all \( x \in X \).

By (2.38),
\[
\| DQ(x, y) \| = \lim_{n \to \infty} \frac{1}{9^n} \| Df(3^n x, 3^n y) \| \leq \lim_{n \to \infty} \frac{1}{9^n} \varphi(3^n x, 3^n y) = 0
\]  
for all \( x, y \in X \). So \( DQ(x, y) = 0 \). By Proposition 2.1, the mapping \( Q : X \rightarrow Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.41), we get (2.39).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Corollary 2.9. Let \( p > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \rightarrow Y \) be a mapping satisfying (2.35). Then there exists a unique quadratic mapping \( Q : X \rightarrow Y \) such that
\[
\| f(x) - Q(x) \| \leq \frac{\theta}{9^p - 9} \| x \|^{2p}
\]  
for all \( x \in X \).

Proof. The proof follows from Theorem 2.8 by taking
\[
\varphi(x, y) := \theta \cdot \| x \|^p \cdot \| y \|^p
\]  
for all \( x, y \in A \). \( \square \)
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