Research Article

A Note on the Multiple Twisted Carlitz’s Type $q$-Bernoulli Polynomials

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We give the twisted Carlitz’s type $q$-Bernoulli polynomials and numbers associated with $p$-adic $q$-inetrals and discuss their properties. Furthermore, we define the multiple twisted Carlitz’s type $q$-Bernoulli polynomials and numbers and obtain the distribution relation for them.

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1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, be the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1 - q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$

(1.1)

(cf. [1–20]) for all $x \in \mathbb{Z}_p$. For a fixed odd positive integer $d$ with $(p, d) = 1$, let

$$X = X_d = \mathop{\lim}_{n \to \infty} \frac{\mathbb{Z}}{dp^n\mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),$$

$$a + dp^n\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \},$$

(1.2)
where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^n \). For any \( n \in \mathbb{N} \),

\[
\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q}
\]  \hspace{1cm} (1.3)

is known to be a distribution on \( X \) (cf. [1–20]).

We say that \( f \) is uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by \( f \in \text{UD}(\mathbb{Z}_p) \), if the difference quotients

\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]  \hspace{1cm} (1.4)

have a limit \( l = f'(a) \) as \( (x, y) \to (a, a) \) (cf. [10–13]). The \( p \)-adic \( q \)-integral of a function \( f \in \text{UD}(\mathbb{Z}_p) \) was defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{x = 0}^{p^n-1} f(x) q^x.
\]  \hspace{1cm} (1.5)

By using \( p \)-adic \( q \)-integrals on \( \mathbb{Z}_p \), it is well known that

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_1(x) \frac{t^n}{n!},
\]  \hspace{1cm} (1.6)

where \( \mu_1(x + p^n\mathbb{Z}_p) = 1/p^n \). Then, we note that the Bernoulli numbers \( B_n \) were defined as

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},
\]  \hspace{1cm} (1.7)

and hence, we have

\[
B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x)
\]  \hspace{1cm} (1.8)

for all \( n \in \mathbb{N} \cup \{0\} \). For \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{0\} \), the multiple Bernoulli polynomials \( B_n^{(k)}(x) \) were defined as

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}
\]  \hspace{1cm} (1.9)

(cf. [2]). We note that

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} \prod_{k=1}^{k} \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_k)^n d\mu_1(x_1) \cdots d\mu_1(x_k).
\]  \hspace{1cm} (1.10)
In this section, we assume that $T$  twisted Carlitz’s type multiple twisted Carlitz’s type associated with numbers. By we derive In this case, we have We obtain the multiple twisted Carlitz’s type $q$-Bernoulli polynomials and numbers. We also obtain the distribution relation for them.

2. Twisted Carlitz’s type $q$-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q_p| < p^{-1/(p - 1)}$. By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we derive

$$I_q(f_1) = \frac{1}{q} I_q(f) + \left( \frac{q - 1}{\log q} f'(0) + (q - 1) f(0) \right),$$

(cf. [8]), where $f_1(x) = f(x + 1)$. From (1.5), we can derive

$$q^n I_q(f_n) = I_q(f) + \frac{q(q - 1)}{\log q} \left( \sum_{i=0}^{n-1} f'(i)q^i + \log q \sum_{i=0}^{n-1} f(i)q^i \right),$$

(cf. [8]), where $n \in \mathbb{N}$ and $f_n(x) = f(x + n)$.

Let $T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \{ w \mid w^{p^n} = 1 \}$ is the cyclic group of order $p^n$. For $w \in T_p$, we denote the locally constant function by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$, $x \to w^x$. If we take $f(x) = \phi_w(x) = w^x$, then we have

$$\int_{\mathbb{Z}_p} e^{iz} \phi_w(x) d\mu_q(x) = \left( \frac{1}{\log q + t} \right) \frac{q(q - 1)}{q \log q} \equiv F_{w}^q(t).$$
Now we define the twisted $q$-Bernoulli polynomials as follows:

$$F_{n,w}^q(x, t) = \left( \frac{\log q + t}{q e^t - 1} \right) q(q-1)e^{xt} = \sum_{n=0}^{\infty} B_{n,w}^q(x) \frac{t^n}{n!},$$

(2.4)

We note that $B_{n,w}^q(0) = B_n^q$ are called the twisted $q$-Bernoulli numbers and by substituting $\omega = 1$, $\lim_{q \to 1} B_{n,1}^q = B_n$ are the familiar Bernoulli numbers. By (2.3), we obtain the following Witt’s type formula for the twisted $q$-Bernoulli polynomials and numbers.

**Theorem 2.1.** For $n \in \mathbb{N}$ and $\omega \in T_p$, one has

$$\int_{\mathbb{Z}_p} (t + x)^n \omega^t d\mu_q(t) = B_{n,w}^q(x).$$

(2.5)

From (2.5), we consider the twisted Carlitz’s type $q$-Bernoulli polynomials by using $p$-adic $q$-integrals. For $\omega \in T_p$, we define the twisted Carlitz’s type $q$-Bernoulli polynomials as follows:

$$\beta_{n,w}^q(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{\infty} \binom{n}{i} q^i (-1)^i \frac{1}{1 - q^{i+1} \omega}.$$

(2.6)

When $x = 0$, we write $\beta_{n,w}^q(0) = \beta_{n,w}^q$ which are called twisted Carlitz’s type $q$-Bernoulli numbers. Note that if $\omega = 1$, then $\lim_{q \to 1}\beta_{n,1}^q = B_n$. From (2.6), we can see that

$$\beta_{n,w}^q(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{\infty} \binom{n}{i} q^i (-1)^i \frac{1}{1 - q^{i+1} \omega}.$$

(2.7)

From (2.7), we can derive the generating function for the twisted Carlitz’s type $q$-Bernoulli polynomials as follows:

$$G_{n,w}^q(x, t) = \sum_{n=0}^{\infty} \beta_{n,w}^q(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^i (-1)^i \frac{1}{1 - q^{i+1} \omega} \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^i (-1)^i \sum_{l=0}^{\infty} q^{(i+1)l} \omega^l \right) \frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{q^l \omega^l}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{(x+1)i} (-1)^i \right) \frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty} q^l \omega^l e^{t(x+1)} t^n \frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty} q^l \omega^l e^{t(x+1)} t^n.$$  (2.8)

Then it is easily to see that

$$G_{n,w}^q(x, t) = \int_{\mathbb{Z}_p} e^{t(x+1)} \omega^t d\mu_q(t).$$

(2.9)
By the $k$th differentiation on both sides of (2.8) at $t = 0$, we also have
\[
\beta^q_{n,w}(x) = \left. \frac{d^k}{dt^k} G^q_{n,w}(x,t) \right|_{t=0} = \sum_{l=0}^{\infty} q^l \omega^l [x + l]^n_q
\] (2.10)
for $n \in \mathbb{N} \cup \{0\}$. We note that
\[
\beta^q_{n,w} = \beta^q_{n,w}(0) = \sum_{l=0}^{\infty} q^l \omega^l [l]^n_q.
\] (2.11)

In view of (2.10), we define twisted Carlitz’s type $q$-zeta function as follows:
\[
\zeta^q_{w}(s,x) = \sum_{l=0}^{\infty} q^l \omega^l \left[ x + l \right]^s_q
\] (2.12)
for all $s \in \mathbb{C}$ and $\text{Re}(x) > 0$. We note that $\zeta^q_{w}(s,x)$ is analytic function in the whole complex $s$-plane. We also have the following theorem in which twisted Carlitz’s type $q$-zeta functions interpolate twisted Carlitz’s type $q$-Bernoulli numbers and polynomials.

**Theorem 2.2.** For $k \in \mathbb{N} \cup \{0\}$ and $w \in T_p$, one has
\[
\zeta^q_{w}(-k,x) = \beta^q_{k,w}(x),
\] (2.13)
\[
\zeta^q_{w}(-k,0) = \beta^q_{k,w}.
\]

From (2.11), we obtain the following distribution relation for the twisted $q$-Bernoulli polynomials.

**Theorem 2.3.** For $r \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, and $w \in T_p$, one has
\[
\beta^q_{n,w}(x) = [r]^n_q \sum_{i=0}^{r-1} \omega^i q^i \beta^q_{n,w} \left( \frac{i + x}{r} \right).
\] (2.14)

**Proof.** If we put $i + rl = j$ and $i = 1 \cdots r$ and $l = 0, 1, \ldots$, then by (2.11), we have
\[
\beta^q_{n,w}(x) = \sum_{j=0}^{\infty} \omega^j q^j \left[ x + j \right]^n_q
\]
\[
= \sum_{i=0}^{\infty} \sum_{l=0}^{r-1} \omega^{i+rl} q^{i+rl} \left[ x + i + rl \right]^n_q
\]
\[
= \sum_{i=0}^{\infty} \omega^i q^i \sum_{l=0}^{\infty} \omega^l q^l \left( 1 - q^{r(i+x)/r+l} \right)^n
\]
\[
= [r]^n_q \sum_{i=0}^{r-1} \omega^i q^i \beta^q_{n,w} \left( \frac{i + x}{r} \right).
\] (2.15)
3. Multiple twisted Carlitz’s type \( q \)-Bernoulli polynomials

In this section, we consider the multiple twisted Carlitz’s type \( q \)-Bernoulli polynomials as follows:

\[
\beta_{k,w}^{(h,q)}(x) = \lim_{q \to \infty} \frac{1}{[r]_q^h} \sum_{j_1, \ldots, j_h=0}^{r-1} [x + x_1 + \cdots + x_h]^n_q w^{x_1 + \cdots + x_h} d\mu_q(x_1) \cdots d\mu_q(x_h)
\]

(3.1)

where \( h \in \mathbb{N} \), \( k \in \mathbb{N} \cup \{0\} \), and \( w \in T_p \). We note that \( \beta_{n,w}^{(h,q)}(0) = \beta_{n,w}^{(h,q)} \) are called the multiple twisted Carlitz’s type \( q \)-Bernoulli numbers. We also obtain the generating function of the multiple twisted Carlitz’s type \( q \)-Bernoulli polynomials as follows:

\[
G_{w}^{(h,q)}(x,t) = \sum_{j_1, \ldots, j_h=0}^{r-1} [x + x_1 + \cdots + x_h]^n_q t^{x_1 + \cdots + x_h} d\mu_q(x_1) \cdots d\mu_q(x_h) = \sum_{l=0}^{\infty} \beta_{l,w}^{(h,q)}(x) \frac{t^l}{l!}.
\]

(3.2)

Finally, we have the following distribution relation for the multiple twisted \( q \)-Bernoulli polynomials.

**Theorem 3.1.** For each \( w \in T_p \), \( h, r \in \mathbb{N} \), \( n \in \mathbb{N} \cup \{0\} \), and \( w \in T_p \),

\[
\beta_{n,w}^{(h,q)}(x) = [r]_q^{n-h} \sum_{j_1, \ldots, j_h=0}^{r-1} [x + x_1 + \cdots + x_h]^n_q t^{x_1 + \cdots + x_h} d\mu_q(x_1) \cdots d\mu_q(x_h) \]

(3.3)

Proof. If we put \( j_k + rl_k = x_k \), \( j_k = 0, 1, \ldots, r - 1 \), and \( k = 1 \cdots h \), then by (3.1), we have

\[
\beta_{k,w}^{(h,q)}(x) = \lim_{q \to \infty} \frac{1}{[r]_q^h} \sum_{j_1, \ldots, j_h=0}^{r-1} [x + x_1 + \cdots + x_h]^n_q w^{x_1 + \cdots + x_h} q^{x_1 + \cdots + x_h}
\]

(3.4)
\[ \left[ r \right]_{q}^{n-h} \sum_{j_{1},...,j_{h}=0}^{r-1} w^{h+j_{1}+\cdots+j_{h}} q^{j_{1}+\cdots+j_{h}} \lim_{\nu \to \infty} \frac{1}{[p^{\nu}]_{q}^{r}} \sum_{h_{1},...,h_{r}=0}^{p^{\nu}-1} \left[ \frac{x+j_{1}+\cdots+j_{h}}{r} + l_{1} + \cdots + l_{h} \right]^{n} \left[ q^{r(h_{1}+\cdots+h_{r})} \right]_{q}^{n} \]

\[ = \left[ r \right]_{q}^{n-h} \sum_{j_{1},...,j_{h}=0}^{r-1} w^{h+j_{1}+\cdots+j_{h}} q^{j_{1}+\cdots+j_{h}} \left( \frac{x+j_{1}+\cdots+j_{h}}{r} \right). \]  

(3.4)

**Question 1.** Are there the analytic multiple twisted Carlitz’s type \( q \)-zeta functions which interpolate multiple twisted Carlitz’s type \( q \)-Bernoulli polynomials?

**References**


