Research Article

Permanence of Periodic Predator-Prey System with Functional Responses and Stage Structure for Prey

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A stage-structured three-species predator-prey model with Beddington-DeAngelis and Holling II functional response is introduced. Based on the comparison theorem, sufficient and necessary conditions which guarantee the predator and the prey species to be permanent are obtained. An example is also presented to illustrate our main results.

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1. Introduction

The aim of this paper is to investigate the permanence of the following periodic stage-structure predator-prey system with Beddington-DeAngelis and Holling II functional response:

\[ x_1'(t) = a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - \frac{p_1(t)x_1(t)}{k_1(t) + m(t)y_1(t) + n(t)x_1(t)}y_1(t), \]
\[ x_2'(t) = c(t)x_1(t) - f(t)x_2^2(t) - \frac{p_2(t)x_2(t)}{k_2(t) + x_2(t)}y_2(t), \]
\[ y_1'(t) = y_1(t) \left[ -g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t) + m(t)y_1(t) + n(t)x_1(t)} - q_1(t)y_1(t) \right], \]
\[ y_2'(t) = y_2(t) \left[ -g_2(t) + \frac{h_2(t)x_2(t)}{k_2(t) + x_2(t)} - q_2(t)y_2(t) \right], \]

(1.1)

where \( a(t), b(t), c(t), d(t), f(t), g_i(t), h_i(t), k_i(t), m(t), n(t), p_i(t), \) and \( q_i(t) \) \((i = 1, 2)\) are all continuous positive \( \omega \)-periodic functions. Here, \( x_1(t) \) and \( x_2(t) \) denote the density of immature and mature prey species at time \( t \), respectively, \( y_1(t) \) represents the density of the
The functional response of predator species $y_1(t)$ to immature prey species takes the Beddington-DeAngelis form, that is, $x_1(t)/(k_1(t) + m(t)y_1(t) + n(t)x_1(t))$. It was introduced by Beddington [1] and DeAngelis et al. [2] independently in 1975. It is similar to the well-known Holling type II functional response but has an extra term $m(t)y_1(t)$ in the denominator which models mutual interference between predators. The Beddington-DeAngelis form of functional response has some of the same qualitative features as the ratio-dependent models but avoids some of the same behaviors of ratio-dependent models at low densities which have been the source of controversy. The function $x_2(t)/(k_2(t) + x_2(t))$ represents the functional response of predator $y_2(t)$ to mature prey, which is called Holling type II function or Michaelis-Menten function. Holling type II is the second function that Holling proposed three kinds of functional response of the predator to prey based on numerous experiments for different species. The Holling type form of functional response is intuited prey-dependent model form. It is applied to almost invertebrate that is one of the most extensive applied functional responses.

Cui and Song [3] proposed the following predator-prey model with stage structure for prey:

$$
\begin{align*}
x'_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - p(t)x_1(t)y(t), \\
x'_2(t) &= c(t)x_1(t) - f(t)x_2(t), \\
y'(t) &= y(t)(-g(t) + h(t)x_1(t) - q(t)y(t)).
\end{align*}
$$

They obtained a set of sufficient and necessary conditions which guarantee the permanence of the system. For more back ground and the relevant work on system (1.2), one could refer to [3–6] and the references cited therein. Recently, Chen [7, 8] and Yang [9] consider the functional response of the predator to immature prey species. Lin and Hong [10] consider a biological model for two predators and one prey with periodic delays.

In reality, mature prey was also consumed by some predators. Different predator usually consumes prey in different stage structure. Some predators only prey on immature prey, and some predators only prey on mature prey. There is different functional response in different predator. So, we add a predator species which consumes mature prey to the model (1.2). By assuming that one predator consumes immature prey according to the Beddington-DeAngelis functional response while the other predator consumes mature prey according to Holling II functional response, we get model (1.1). In the resource limited environment,
Theorem 2.4. System has a positive \( \omega \)-periodic solution \( x(t) \) if and only if
\[
A_\omega \left( -g_1(t) + \frac{h_1(t)}{k_1(t) + n(t)} x_1^*(t) \right) > 0, \quad A_\omega \left( -g_2(t) + \frac{h_2(t)}{k_2(t) + x_2^*(t)} \right) > 0,
\]
where \( x_1^*(t), x_2^*(t) \) is the unique positive periodic solution of system (2.1) given by Lemma 2.2.

2. Main result

In this section, we introduce a definition and some lemmas which will be useful in subsequent sections and state the main result.

Definition 2.1. System (1.1) is said to be permanent if there exist positive constants \( m, M \) and \( T_0 \), such that each positive solution \( x_i(t), x_2(t), y_1(t), y_2(t) \) of the system (1.1) with any positive initial value \( \varphi \) fulfills \( m \leq x_i(t) \leq M, m \leq y_i(t) \leq M, i = 1, 2 \) for all \( t \geq T_0 \), where \( T_0 \) may depend on \( \varphi \).

Lemma 2.2 (see [12]). If \( a(t), b(t), c(t), d(t), \) and \( f(t) \) are all \( \omega \)-periodic, then system
\[
\begin{align*}
x_1'(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t), \\
x_2'(t) &= c(t)x_1(t) - f(t)\left( x_2^2(t) \right)
\end{align*}
\]
has a positive \( \omega \)-periodic solution \( (x_1^*(t), x_2^*(t)) \) which is globally asymptotically stable with respect to \( \mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0, x_2 > 0 \} \).

Lemma 2.3 (see [13]). If \( b(t) \) and \( a(t) \) are all \( \omega \)-periodic, and if \( A_\omega (b) > 0 \) and \( A_\omega (a) > 0 \) for all \( t \in \mathbb{R} \), then the system
\[
x' = x(b(t) - a(t)x)
\]
has a positive \( \omega \)-periodic solution which is globally asymptotically stable.

Now, we state the main result of this paper.

Theorem 2.4. System (1.1) is permanent if and only if
\[
A_\omega \left( -g_1(t) + \frac{h_1(t)}{k_1(t) + n(t)} x_1^*(t) \right) > 0, \quad A_\omega \left( -g_2(t) + \frac{h_2(t)}{k_2(t) + x_2^*(t)} \right) > 0,
\]
where \( (x_1^*(t), x_2^*(t)) \) is the unique positive periodic solution of system (2.1) given by Lemma 2.2.
3. Proof of the main result

We need the following propositions to prove Theorem 2.4. The hypothesis of the lemmas and theorem of the preceding section is assumed to hold in what follows.

**Proposition 3.1.** There exist positive constants $M_x$ and $M_y$ such that

$$
\lim_{t \to +\infty} \sup_{t} x_i(t) \leq M_x, \quad \lim_{t \to +\infty} \sup_{t} y_i(t) \leq M_y, \quad i = 1, 2
$$

(3.1)

for all solution of system (1.1) with positive initial values.

**Proof.** Obviously, $R^+_I = \{(x_1, x_2, y_1, y_2) \mid x_i > 0, y_i > 0\}$ is a positively invariant set of system (1.1). Given any solution $(x_1, x_2, y_1, y_2)$ of system (1.1), we have

$$
x_1'(t) \leq a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t),
$$

(3.2)

$$
x_2'(t) \leq c(t)x_1(t) - f(t)x_2^2(t).
$$

By Lemma 2.2, the following auxiliary equation:

$$
u_1'(t) = a(t)u_2(t) - b(t)u_1(t) - d(t)u_1^2(t),
$$

$$
u_2'(t) = c(t)u_1(t) - f(t)u_2^2(t)
$$

(3.3)

has a globally asymptotically stable positive $\omega$-periodic solution $(x_1^*, x_2^*)$. Let $(u_1(t), u_2(t))$ be the solution of (3.3) with $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$. By comparison theorem, we then have

$$
x_i(t) \leq u_i(t), \quad i = 1, 2,
$$

(3.4)

for $t \geq 0$. By (2.3), we can choose positive $\varepsilon > 0$ small enough such that

$$
A_\omega \left( -g(t) + \frac{h(t)(x_1^*(t) + \varepsilon)}{k(t)} \right) > 0.
$$

(3.5)

Thus, from the global attractivity of $(x_1^*(t), x_2^*(t))$, for the above given $\varepsilon > 0$, there exists a $T_0 > 0$, such that

$$
|u_i(t) - x_i^*(t)| < \varepsilon, \quad t \geq T_0.
$$

(3.6)

Inequality (3.4) combined with (3.6) leads to

$$
x_i(t) < x_i^*(t) + \varepsilon, \quad t > T_0.
$$

(3.7)
Proposition 3.2. There exist positive constants \( M_x = \max_{0 \leq t \leq \omega} \{ x^*_i(t) + \epsilon, \ i = 1, 2 \} \), we have

\[
\lim_{t \to +\infty} \sup_{i} x_i(t) \leq M_x. \tag{3.8}
\]

In addition, for \( t \geq T_0 \), from the third and fourth equations of (1.1) and (3.7) we get

\[
y'_i(t) \leq y_i(t) \left[ -g_i(t) + \frac{h_i(t)x_i(t)}{k_i(t)} - q_i(t)y_i(t) \right]
\leq y_i(t) \left[ -g_i(t) + \frac{h_i(t)(x^*_i(t) + \epsilon)}{k_i(t)} - q_i(t)y_i(t) \right]. \tag{3.9}
\]

Consider the following auxiliary equation:

\[
v'_i(t) = v_i(t) \left[ -g_i(t) + \frac{h_i(t)(x^*_i(t) + \epsilon)}{k_i(t)} - q_i(t)v_i(t) \right]. \tag{3.10}
\]

It follows from (3.5) and Lemma 2.3 that (3.10) has a unique positive \( \omega \)-periodic solution \( y^*_i(t) > 0 \) which is globally asymptotically stable. Similar to the above analysis, there exists a \( T_1 > T_0 \) such that for the above \( \epsilon \), one has

\[
y_i(t) < y^*_i(t) + \epsilon, \quad t \geq T_1. \tag{3.11}
\]

Let \( M_y = \max_{0 \leq t \leq \omega} \{ y^*_i(t) + \epsilon, \ i = 1, 2 \} \), then we have

\[
\lim_{t \to +\infty} \sup_{i} y_i(t) \leq M_y, \quad i = 1, 2. \tag{3.12}
\]

This completes the proof of Proposition 3.1. \( \square \)

Proposition 3.2. There exist positive constants \( \delta_{ix} < M_x, \ i = 1, 2 \), such that

\[
\lim_{t \to +\infty} \inf_{i} x_i(t) \geq \delta_{ix}, \quad i = 1, 2. \tag{3.13}
\]

Proof. By Proposition 3.1, there exists \( T_1 > 0 \) such that

\[
0 < x_i(t) \leq M_x, \quad 0 < y_i(t) \leq M_y, \quad t \geq T_1. \tag{3.14}
\]

Hence, from the first and second equations of system (1.1), we have

\[
x'_1(t) \geq a(t)x_2(t) - \left( b(t) + \frac{p_1(t)}{k_1(t)} M_y \right)x_1(t) - d(t)x_1^2(t),
x'_2(t) \geq c(t)x_1(t) - \left( f(t) + \frac{p_2(t)}{k_2(t)} M_y \right)x_2^2(t), \tag{3.15}
\]
for \( t \geq T_1 \). By Lemma 2.2, the following auxiliary equation:

\[
\begin{align*}
    u'_1(t) &= a(t)u_2(t) - \left( b(t) + \frac{p_1(t)}{k_1(t)} M_y \right) u_1(t) - d(t)u'_1(t), \\
    u'_2(t) &= c(t)u_1(t) - \left( f(t) + \frac{p_2(t)}{k_2(t)} M_y \right) u_2^2(t)
\end{align*}
\]

has a globally asymptotically stable positive \( \omega \)-periodic solution \((\bar{x}^*_1(t), \bar{x}^*_2(t))\). Let \((u_1(t), u_2(t))\) be the solution of (3.16) with \((u_1(T_1), u_2(T_2)) = (x_1(T_1), x_2(T_2))\), by comparison theorem, we have

\[
x_i(t) \geq u_i(t) \quad (i = 1, 2), \quad t > T_1.
\]

Thus, from the global attractivity of \((\bar{x}^*_1(t), \bar{x}^*_2(t))\), there exists a \( T_2 > T_1 \), such that

\[
|u_i(t) - \bar{x}^*_i(t)| < \frac{\bar{x}^*_i(t)}{2} \quad (i = 1, 2), \quad t > T_2.
\]

Inequality (3.18) combined with (3.17) leads to

\[
x_i(t) > \delta_{ix} = \min_{0 < s \leq \omega} \left\{ \frac{\bar{x}^*_i(t)}{2} \right\} \quad (i = 1, 2), \quad t > T_2.
\]

And so

\[
\lim_{t \to +\infty} \inf_{i=1,2} x_i(t) \geq \delta_{ix}, \quad i = 1, 2.
\]

The proof of Proposition 3.2 is complete. \(\square\)

**Proposition 3.3.** Suppose that (2.3) holds, then there exist positive constants \( \delta_{iy}, i = 1, 2 \), such that any solution \((x_1(t), x_2(t), y_1(t), y_2(t))\) of system (1.1) with positive initial value satisfies

\[
\lim_{t \to +\infty} \sup_{i=1,2} y_i(t) \geq \delta_{iy}, \quad i = 1, 2.
\]

**Proof.** By Assumption (2.3), we can choose constant \( \varepsilon_0 > 0 \) (without loss of generality, we may assume that \( \varepsilon_0 < (1/2)\min_{0 < s \leq \omega} \{x^*_i(t)\} \), where \((x^*_1(t), x^*_2(t))\) is the unique positive periodic solution of system (2.1)) such that

\[
A_\omega(q_{\varepsilon_0}(t)) > 0, \quad A_\omega'(q_{\varepsilon_0}(t)) > 0,
\]

where
Consider the following equations with a parameter $\beta > 0$:

\[
\begin{align*}
    x_1'(t) &= a(t)x_2(t) - \left( b(t) + \frac{2\beta p_1(t)}{k_1(t)} \right)x_1(t) - d(t)x_1^2(t), \\
    x_2'(t) &= c(t)x_1(t) - \left( f(t) + \frac{2\beta p_2(t)}{k_2(t)} \right)x_2^2(t).
\end{align*}
\]  

(3.24)

By Lemma 2.2, the system (3.24) has a unique positive $\omega$-periodic solution $(x^*_1(t), x^*_2(t))$, which is globally attractive. Let $(\bar{x}_{i\beta}(t), \bar{x}_{2\beta}(t))$ be the solution of (3.24) with initial condition $\bar{x}_{i\beta}(0) = x^*_i(0)$, $i = 1, 2$. Hence, for above $\varepsilon_0$, there exists a sufficiently large $T_3 > T_2$ such that

\[
    |\bar{x}_{i\beta}(t) - x^*_i(t)| < \frac{\varepsilon_0}{4} \quad (i = 1, 2), \quad t > T_3.
\]  

(3.25)

By the continuity of the solution in the parameter, we have $\bar{x}_{i\beta}(t) \to x^*_i(t)$ uniformly in $[T_3, T_3 + \omega]$ as $\beta \to 0$. Hence, for $\varepsilon_0 > 0$, there exists a $\beta_0 = \beta_0(\varepsilon_0) > 0$ such that

\[
    |\bar{x}_{i\beta}(t) - x^*_i(t)| < \frac{\varepsilon_0}{4} \quad (i = 1, 2), \quad t \in [T_3, T_3 + \omega], \quad 0 < \beta < \beta_0.
\]  

(3.26)

So, we have

\[
    |x^*_{i\beta}(t) - x^*_i(t)| \leq |\bar{x}_{i\beta}(t) - x^*_i(t)| + |\bar{x}_{i\beta}(t) - x^*_i(t)| < \frac{\varepsilon_0}{2}, \quad t \in [T_3, T_3 + \omega].
\]  

(3.27)

Since $x^*_{i\beta}(t)$ and $x^*_i(t)$ are all $\omega$-periodic, we have

\[
    |x^*_{i\beta}(t) - x^*_i(t)| < \frac{\varepsilon_0}{2} \quad (i = 1, 2), \quad t \geq 0, \quad 0 < \beta < \beta_0.
\]  

(3.28)

Choosing a constant $\beta_1$ ($0 < \beta_1 < \beta_0$, $2\beta_1 < \varepsilon_0$), we have

\[
    x^*_{i\beta_1}(t) \geq x^*_i(t) - \frac{\varepsilon_0}{2} \quad (i = 1, 2), \quad t \geq 0.
\]  

(3.29)
Suppose that Conclusion (3.21) is not true. Then, there exists $F \in R^4$ such that, for the positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.1) with an initial condition $(x_1(0), x_2(0), y_1(0), y_2(0)) = F$, we have

$$\lim_{t \to +\infty} \sup y_i(t) < \beta_i, \quad i = 1, 2. \quad (3.30)$$

So, there exists $T_4 > T_3$ such that

$$y_i(t) < 2\beta_1 < \varepsilon_0, \quad t \geq T_4. \quad (3.31)$$

By applying (3.31), from the first and second equations of system (1.1) it follows that for all $t \geq T_4$,

$$x'_1(t) \geq a(t)x_2(t) - \left( b(t) + 2\beta_1 \frac{p_1(t)}{k_1(t)} \right) x_1(t) - d(t)x_1^2(t),$$

$$x'_2(t) \geq c(t)x_1(t) - \left( f(t) + 2\beta_1 \frac{p_2(t)}{k_2(t)} \right) x_2^2(t). \quad (3.32)$$

Let $(u_1(t), u_2(t))$ be the solution of (3.24) with $\beta = \beta_1$ and $u_i(T_4) = x_i(T_4), i = 1, 2$, then

$$x_i(t) \geq u_i(t) \quad (i = 1, 2), \quad t \geq T_4. \quad (3.33)$$

By the global asymptotic stability of $(x_{1\beta_1}^*(t), x_{2\beta_1}^*(t))$, for the given $\varepsilon = \varepsilon_0/2$, there exists $T_5 \geq T_4$, such that

$$\left| u_i(t) - x_{i\beta_1}^*(t) \right| < \frac{\varepsilon_0}{2} \quad (i = 1, 2), \quad t \geq T_5. \quad (3.34)$$

So,

$$x_i(t) \geq u_i(t) > x_{i\beta_1}^*(t) - \frac{\varepsilon_0}{2} \quad (i = 1, 2), \quad t \geq T_5, \quad (3.35)$$

and hence, by using (3.29), we get

$$x_i(t) > x_i^*(t) - \varepsilon_0 \quad (i = 1, 2), \quad t \geq T_5. \quad (3.36)$$

Therefore, by (3.31) and (3.36), we have

$$y'_1(t) \geq y_1(t) \left(-g_1(t) + \frac{h_1(t)(x_1^*(t) - \varepsilon_0)}{k_1(t) + m(t)\varepsilon_0 + n(t)(x_1^*(t) - \varepsilon_0)} - q_1(t)e_0 \right) = q_{\varepsilon_0}(t)y_1(t),$$

$$y'_2(t) \geq y_2(t) \left(-g_2(t) + \frac{h_2(t)(x_2^*(t) - \varepsilon_0)}{k_2(t) + (x_2^*(t) - \varepsilon_0)} - q_2(t)e_0 \right) = q_{\varepsilon_0}(t)y_2(t). \quad (3.37)$$
for \( t \geq T_3 \). Integrating (3.37) from \( T_3 \) to \( t \) yields
\[
y_1(t) \geq y_1(T_3) \exp \left\{ \int_{T_3}^{t} \varphi_{\alpha_0}(t)dt \right\},
\]
\[
y_2(t) \geq y_2(T_3) \exp \left\{ \int_{T_3}^{t} \varphi_{\alpha_0}(t)dt \right\}.
\] (3.38)

Thus, from (3.22) we know that \( \varphi_{\alpha_0}(t) > 0, \varphi_{\alpha_0}(t) > 0 \). It follows that \( y_1(t) \to +\infty, y_2(t) \to +\infty \) as \( t \to +\infty \). It is a contradiction. This completes the proof.

Proposition 3.4. Suppose that (2.3) holds, then there exist positive constants \( \eta_{iy}, i = 1, 2 \), such that any solution \((x_1(t), x_2(t), y_1(t), y_2(t))\) of system (1.1) with positive initial value satisfies
\[
\lim_{t \to +\infty} \inf y_i(t) > \eta_{iy}, \quad i = 1, 2.
\] (3.39)

Proof. Suppose that (3.39) is not true, then there exists a sequence \( \{\xi_m\} \in R_+^1 \), such that
\[
\lim_{t \to +\infty} \inf y_i(t, \xi_m) < \frac{\delta_{iy}}{(m+1)^2}, \quad m = 1, 2, \ldots.
\] (3.40)

On the other hand, by Proposition 3.3, we have
\[
\lim_{t \to +\infty} \sup y_i(t, \xi_m) > \delta_{iy}, \quad m = 1, 2, \ldots.
\] (3.41)

Hence, there are time sequences \( \{s_q^{(m)}\} \) and \( \{t_q^{(m)}\} \) satisfying
\[
0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots,
\]
\[
s_q^{(m)} \to +\infty, \quad t_q^{(m)} \to +\infty \quad \text{as} \quad q \to +\infty,
\]
\[
y_i(s_q^{(m)}, \xi_m) = \frac{\delta_{iy}}{m+1}, \quad y_i(t_q^{(m)}, \xi_m) = \frac{\delta_{iy}}{(m+1)^2}, \quad \frac{\delta_{iy}}{(m+1)^2} < y_i(t, \xi_m) < \frac{\delta_{iy}}{m+1}, \quad t \in (s_q^{(m)}, t_q^{(m)}).
\] (3.42)

By Proposition 3.1, for a given positive integer \( m \), there is a \( T_1^{(m)} > 0 \), such that for all \( t > T_1^{(m)} \)
\[
x_i(t, \xi_m) < M_x, \quad y_i(t, \xi_m) < M_y, \quad i = 1, 2.
\] (3.43)

Because of \( s_q^{(m)} \to +\infty \) as \( q \to +\infty \), there is a positive integer \( Z^{(m)} \), such that \( s_q^{(m)} > T_1^{(m)} \) as \( q \geq Z^{(m)} \), hence
\[
y_i(t, \xi_m) \geq y_i(t, \xi_m) ( - g_i(t) - q_i(t) M_y )
\] (3.44)
for \( t \in [s_q^{(m)}, t_q^{(m)}] \), \( q \geq Z^{(m)} \). Integrating (3.44) from \( s_q^{(m)} \) to \( t_q^{(m)} \) yields

\[
y_i(t_q^{(m)}, \xi_m) \geq y_i(s_q^{(m)}, \xi_m) \exp \left\{ \int_{s_q^{(m)}}^{t_q^{(m)}} (-g_i(t) - q_i(t)M_y) dt \right\},
\]  

(3.45)

or

\[
\int_{s_q^{(m)}}^{t_q^{(m)}} (g_i(t) + q_i(t)M_y) dt \geq \ln(m + 1) \quad \text{for} \quad q \geq Z^{(m)}.
\]  

(3.46)

Thus, from the boundedness of \( g_i(t) + q_i(t)M_y \), we have

\[
t_q^{(m)} - s_q^{(m)} \longrightarrow +\infty \quad \text{as} \quad m \longrightarrow +\infty, \quad q \geq Z^{(m)}.
\]  

(3.47)

By (3.22) and (3.47), there are constants \( P > 0 \) and \( N_0 > 0 \), such that

\[
\frac{\delta_{hy}}{m + 1} < \beta_1 < \varepsilon_0, \quad t_q^{(m)} - s_q^{(m)} > 2P,
\]  

(3.48)

\[
\int_0^a \psi_{e_i}(t) dt > 0, \quad \int_0^a \psi_{e_i}(t) dt > 0,
\]  

(3.49)

for \( m \geq N_0, \ q \geq Z^{(m)} \), and \( a \geq P \). Inequality (3.48) implies that

\[
y_i(t, \xi_m) < \beta_1 < \varepsilon_0, \quad t \in [s_q^{(m)}, t_q^{(m)}],
\]  

(3.50)

for \( m \geq N_0, \ q \geq Z^{(m)} \). In addition, from (3.43) and (3.50) we have

\[
x_1'(t, \xi_m) \geq a(t)x_2(t, \xi_m) - \left( b(t) + \frac{2p_1(t)\beta_1}{k_1(t)} \right)x_1(t, \xi_m) - d(t)x_1^2(t, \xi_m),
\]  

\[
x_2'(t, \xi_m) \geq c(t)x_1(t, \xi_m) - \left( f(t) + \frac{2p_2(t)\beta_1}{k_2(t)} \right)x_2^2(t, \xi_m),
\]  

(3.51)

for \( t \in [s_q^{(m)}, t_q^{(m)}] \). Let \( (u_1(t), u_2(t)) \) be the solution of (3.24) with \( \beta = \beta_1 \) and \( u_i(s_q^{(m)}) = x_i(s_q^{(m)}, \xi_m) \), then by applying comparison theorem, we have

\[
x_i(t, \xi_m) \geq u_i(t), \quad t \in [s_q^{(m)}, t_q^{(m)}].
\]  

(3.52)

Further, by using Propositions 3.1 and 3.2, there exists an enough large \( Z_i^{(m)} > Z^{(m)} \) such that

\[
\eta_{ix} < x_i(s_q^{(m)}, \xi_m) < M_x,
\]  

(3.53)
for \( q \geq Z_1^{(m)} \). For \( \beta = \beta_1 \), (3.24) has a unique positive \( \omega \)-periodic solution \((x_{1\beta_1}^*(t), x_{2\beta_1}^*(t))\) which is globally asymptotically stable. In addition, by the periodicity of (3.24), the periodic solution \((x_{1\beta_1}^*(t), x_{2\beta_1}^*(t))\) is uniformly asymptotically stable with respect to the compact set \( \Omega = \{ x \mid \eta_{ix} < x < M_x \} \). Hence, for given \( \varepsilon_0 \) in Proposition 3.3, there exists \( T_0 > P \), which is independent of \( m \) and \( q \), such that

\[
u_i(t) > x_{ij_1}^*(t) - \frac{\varepsilon_0}{2}, \quad i = 1, 2 \text{ as } t > T_0 + s_q^{(m)}.
\] (3.54)

Thus, by using (3.29), we get

\[
u_i(t) > x_i^*(t) - \varepsilon_0, \quad i = 1, 2 \text{ as } t > T_0 + s_q^{(m)}.
\] (3.55)

By (3.47), there exists a positive integer \( N_1 \geq N_0 \) such that \( t_{q_1}^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2P \) for \( m \geq N_1 \) and \( q \geq Z_1^{(m)} \). So, we have

\[
x_i(t, \xi_m) \geq x_i^*(t) - \varepsilon_0, \quad i = 1, 2 \text{ as } t \in [T_0 + s_q^{(m)}, t_{q}^{(m)}],
\] (3.56)

where \( m \geq N_1 \) and \( q \geq Z_1^{(m)} \). Hence, by using (3.50) and (3.56), from the third and fourth equations of system (1.1), we have

\[
y_1'(t, \xi_m) \geq \varphi_{\varepsilon_0}(t)y_1(t, \xi_m), \quad y_2'(t, \xi_m) \geq \varphi_{\varepsilon_0}(t)y_2(t, \xi_m), \quad t \in [T_0 + s_q^{(m)}, t_{q}^{(m)}].
\] (3.57)

Integrating the above inequalities from \( T_0 + s_q^{(m)} \) to \( t_{q}^{(m)} \), we have

\[
y_1(t_{q}^{(m)}, \xi_m) \geq y_1(T_0 + s_q^{(m)}, \xi_m) \exp \left\{ \int_{T_0 + s_q^{(m)}}^{t_{q}^{(m)}} \varphi_{\varepsilon_0}(t) dt \right\},
\] (3.58)

\[
y_2(t_{q}^{(m)}, \xi_m) \geq y_2(T_0 + s_q^{(m)}, \xi_m) \exp \left\{ \int_{T_0 + s_q^{(m)}}^{t_{q}^{(m)}} \varphi_{\varepsilon_0}(t) dt \right\},
\]

that is

\[
\frac{\delta_{1y}}{(m + 1)^2} \geq \frac{\delta_{1y}}{(m + 1)^2} \exp \left\{ \int_{T_0 + s_q^{(m)}}^{t_{q}^{(m)}} \varphi_{\varepsilon_0}(t) dt \right\} > \frac{\delta_{1y}}{(m + 1)^2},
\]

\[
\frac{\delta_{2y}}{(m + 1)^2} \geq \frac{\delta_{2y}}{(m + 1)^2} \exp \left\{ \int_{T_0 + s_q^{(m)}}^{t_{q}^{(m)}} \varphi_{\varepsilon_0}(t) dt \right\} > \frac{\delta_{2y}}{(m + 1)^2}.
\] (3.59)

These are contradictions. This completes the proof of Proposition 3.4. \( \Box \)
Proof of Theorem 2.4. The sufficiency of Theorem 2.4 now follows from Propositions 3.1–3.4. We thus only need to prove the necessity of Theorem 2.4. Suppose that

$$A_\omega \left( - g_1(t) + \frac{h_1(t)x_1^*(t)}{k_1(t) + n(t)x_1^*(t)} \right) \leq 0, \quad A_\omega \left( - g_2(t) + \frac{h_2(t)x_2^*(t)}{k_2(t) + x_2^*(t)} \right) \leq 0. \tag{3.60}$$

We will show that

$$\lim_{t \to +\infty} y_i(t) = 0, \quad i = 1, 2. \tag{3.61}$$

In fact, by (3.60), we know that, for any given positive constant $0 < \varepsilon < 1$, there exist $\varepsilon_1 > 0$, $(0 < \varepsilon_1 < \varepsilon)$, $\varepsilon_0 > 0$ such that

$$A_\omega \left( - g_1(t) + \frac{h_1(t)(x_1^*(t) + \varepsilon_1) - q_1(t)\varepsilon}{k_1(t) + n(t)(x_1^*(t) + \varepsilon_1)} \right) \leq -\frac{\varepsilon}{2} A_\omega(q_1(t)) < -\varepsilon_0,$$

$$A_\omega \left( - g_2(t) + \frac{h_2(t)(x_2^*(t) + \varepsilon_1) - q_2(t)\varepsilon}{k_2(t) + (x_2^*(t) + \varepsilon_1)} \right) \leq -\frac{\varepsilon}{2} A_\omega(q_2(t)) < -\varepsilon_0. \tag{3.62}$$

Since

$$x'_1(t) \leq a(t)x_1(t) - b(t)x_1^2(t) - d(t)x_2^2(t),$$

$$x'_2(t) \leq c(t)x_1(t) - f(t)x_2^2(t). \tag{3.63}$$

We know that, for above $\varepsilon_1$ there exists a $T^{(1)} > 0$ such that

$$x_1(t) < x_1^*(t) + \varepsilon, \quad t \geq T^{(1)}. \tag{3.64}$$

It follows from (3.62) and (3.64) that for $t \geq T^{(1)}$,

$$A_\omega \left( - g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t) + n(t)x_1(t)} - q_1(t)\varepsilon \right) < -\varepsilon_0,$$

$$A_\omega \left( - g_2(t) + \frac{h_2(t)x_2(t)}{k_2(t) + x_2(t)} - q_2(t)\varepsilon \right) < -\varepsilon_0. \tag{3.65}$$

First, we show that there exists a $T^{(2)} > T^{(1)}$ such that $y_i(T^{(2)}) < \varepsilon$, $i = 1, 2$. Otherwise, by (3.65), we have

$$\varepsilon \leq y_1(t)$$

$$\leq y_1(T^{(1)}) \exp \left\{ \int_{T^{(1)}}^t \left( - g_1(s) + \frac{h_1(s)x_1(s)}{k_1(s) + n(s)x_1(s)} - q_1(s)\varepsilon \right) ds \right\} \tag{3.66}$$

$$\leq y_1(T^{(1)}) \exp \{-\varepsilon_0(t - T^{(1)}) \rightarrow 0$$
as \( t \to +\infty \). Similarly, we have

\[
e \leq y_2(t) \leq y_2(T^{(1)}) \exp \{ -\varepsilon_0 (t - T^{(1)}) \} \to 0, \quad t \to +\infty,
\]

which are contradictions.

Second, we now show that

\[
y_i(t) \leq \varepsilon \exp \{ M(\varepsilon)\omega \}, \quad i = 1, 2, \text{ for } t \geq T^{(2)},
\]

where

\[
M(\varepsilon) = \max_{0 \leq i \leq \omega} \left\{ g_1(t) + \frac{h_1(t)(x_i^*(t) + \varepsilon)}{k_1(t) + n(t)(x_i^*(t) + \varepsilon)} + q_1(t)\varepsilon, \ g_2(t) + \frac{h_2(t)(x_i^*(t) + \varepsilon)}{k_2(t) + (x_i^*(t) + \varepsilon)} + q_2(t)\varepsilon \right\}
\]

is a bounded constant for \( 0 < \varepsilon < 1 \). Otherwise, there exists a \( T^{(3)} > T^{(2)} \) such that

\[
y_i(T^{(3)}) > \varepsilon \exp \{ M(\varepsilon)\omega \}, \quad i = 1, 2.
\]

By the continuity of \( y_i(t) \), there must exist \( T^{(4)} \in (T^{(2)}, T^{(3)}) \) such that \( y_i(T^{(4)}) = \varepsilon \) and \( y_i(t) > \varepsilon \) for \( t \in (T^{(4)}, T^{(3)}) \). Let \( P_1 \) be the nonnegative integer such that \( T^{(3)} \in (T^{(4)} + P_1\omega, T^{(4)} + (P_1 + 1)\omega] \).

By the first inequality of (3.65), we have

\[
\varepsilon \exp \{ M(\varepsilon)\omega \} < y_1(T^{(3)})
\]

\[
< y_1(T^{(4)}) \exp \left\{ \int_{T^{(4)}}^{T^{(3)}} \left( -g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t) + n(t)x_1(t)} - q_1(t)\varepsilon \right) dt \right\}
\]

\[
= \varepsilon \exp \left\{ \int_{T^{(4)} + P_1\omega}^{T^{(3)}} + \int_{T^{(4)}}^{T^{(3)}} \left( -g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t) + n(t)x_1(t)} - q_1(t)\varepsilon \right) dt \right\}
\]

\[
< \varepsilon \exp \left\{ \int_{T^{(4)} + P_1\omega}^{T^{(3)}} \left( g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t) + n(t)x_1(t)} + q_1(t)\varepsilon \right) dt \right\}
\]

\[
< \varepsilon \exp \left\{ \int_{T^{(4)} + P_1\omega}^{T^{(3)}} \left( g_1(t) + \frac{h_1(t)(x_1^*(t) + \varepsilon)}{k_1(t) + n(t)(x_1^*(t) + \varepsilon)} + q_1(t)\varepsilon \right) dt \right\}
\]

\[
\leq \varepsilon \exp \{ M(\varepsilon)\omega \}.
\]

Similarly, by the second inequality of (3.65), we have

\[
\varepsilon \exp \{ M(\varepsilon)\omega \} < y_2(T^{(3)}) \leq \varepsilon \exp \{ M(\varepsilon)\omega \},
\]
which are contradictions. These imply that (3.68) holds. By the arbitrariness of \( \varepsilon \), it immediately follows that \( y_i(t) \to 0 \) as \( t \to +\infty \). This completes the proof of Theorem 2.4. \( \square \)

4. Example

Consider the following predator-prey system:

\[
\begin{align*}
    x_1'(t) &= 5x_2(t) - 2x_1(t) - x_1^2(t) - \frac{(2 + \sin(t)/200)x_1(t)}{5 + y_1(t) + x_1(t)} y_1(t), \\
    x_2'(t) &= 3x_1(t) - x_2^2(t) - \frac{(2 + \sin(t)/100)x_2(t)}{4 + x_2(t)}, \\
    y_1'(t) &= y_1(t) \left[ -\frac{1}{3} \frac{\sin(t)}{100} + \frac{(2 + \sin(t)/200)x_1(t)}{5 + y_1(t) + x_1(t)} - (4 + \cos(t)) y_1(t) \right], \\
    y_2'(t) &= y_2(t) \left[ -\frac{1}{2} \frac{\sin(t)}{100} + \frac{(2 + \sin(t)/100)x_2(t)}{4 + x_2(t)} - (3 + \cos(t)) y_2(t) \right].
\end{align*}
\]  
(4.1)

In this case, corresponding to system (1.1), one has \( a(t) = 5, b(t) = 2, c(t) = 3, d(t) = 1, f(t) = 1, g_1(t) = 1/3 + \sin(t)/100, g_2(t) = 1/2 + \sin(t)/100, h_1(t) = p_1(t) = 2 + \sin(t)/200, h_2(t) = p_2(t) = 2 + \sin(t)/100, k_1(t) = 5, k_2(t) = 4, m(t) = n(t) = 1, q_1(t) = 4 + \cos(t), q_2(t) = 3 + \cos(t). \)

One could easily see that

\[
\begin{align*}
    x_1'(t) &= 5x_2(t) - 2x_1(t) - x_1^2(t), \\
    x_2'(t) &= 3x_1(t) - x_2^2(t)
\end{align*}
\]  
(4.2)

has a unique positive periodic solution \((x_1^*(t), x_2^*(t)) = (3, 3)\), that is, in this case, the positive periodic solution is the positive equilibrium. By simple computation, one has

\[
\begin{align*}
    A_{\omega} \left( -g_1(t) + \frac{h_1(t)x_1^*(t)}{k_1(t) + n(t)x_1^*(t)} \right) &= \frac{5}{12} > 0, \\
    A_{\omega} \left( -g_2(t) + \frac{h_2(t)x_2^*(t)}{k_2(t) + x_2^*(t)} \right) &= \frac{5}{14} > 0.
\end{align*}
\]  
(4.3)

Hence, corresponding to Theorem 2.4, we know that system (4.1) is permanent.

5. Conclusion

In this paper, a model which describes the nonautonomous periodic predator-prey system with Beddington-DeAngelis and Holling II functional response and stage structure for prey is proposed. Under Assumption (2.3), sufficient and necessary conditions which guarantee the predator and the prey species to be permanent are obtained.
The results of this paper suggest the following biological implication. Note that $(x_1^*(t), x_2^*(t))$ is the globally asymptotically stable periodic solution of system (1.1) without predation, which, as showed by Lemma 2.2, always exists. Hence, condition (2.3) implies that if the death rate of the two predator species is all small enough and the growth by foraging minus the death for the second predator is sufficiently high, the system is permanent.

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**References**